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# More Classes of Stuck Unknotted Hexagons

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**Abstract.** Consider a hexagonal unknot with edges of fixed length, for which we allow universal joint motions but do not allow edge crossings. We consider the maximum number of embedding classes that any such unknot may have. Until now, five was a lower bound for this number. Here we show that there exists a hexagonal unknot with at least nine embedding classes.

## 1. Introduction

Consider a hexagon in $\mathbb{R}^3$ with fixed edge lengths[1] $\ell_1, \ldots, \ell_6$, and for which we allow universal joint motions but no edge crossings. In other words, we allow the vertices $v_1, \ldots, v_6$ of the hexagon to move freely as long as the edges do not cross or deform. Following the notation of Cantarella and Johnston [3], we denote the space of this hexagon as $Pol_6(\ell_1, \ldots, \ell_6)$. The general question that we study here is: *how many connected components can $Pol_6$ have, for a suitable choice of $\ell_1, \ldots, \ell_6$?* It is then natural to ask whether each connected component of $Pol_6$ corresponds to a separate knot type, or if some knot type (say the unknot, or trivial knot) exists in separate components of space. Millet [4] showed that for regular hexagonal unknots there is only one embedding class. The question remained open until Cantarella and Johnston proved that there exist three connected components in $Pol_6(100, 63.5, 22.7, 5.6, 22.7, 63.5)$, each belonging to the unknot[2]. In other words, they showed that there exist "stuck" unknots which cannot be reconfigured into planar convex embeddings, if the edge lengths are chosen

[1] We use $\ell_i$ to denote both the edge and its length.
[2] The numbers given are approximate.

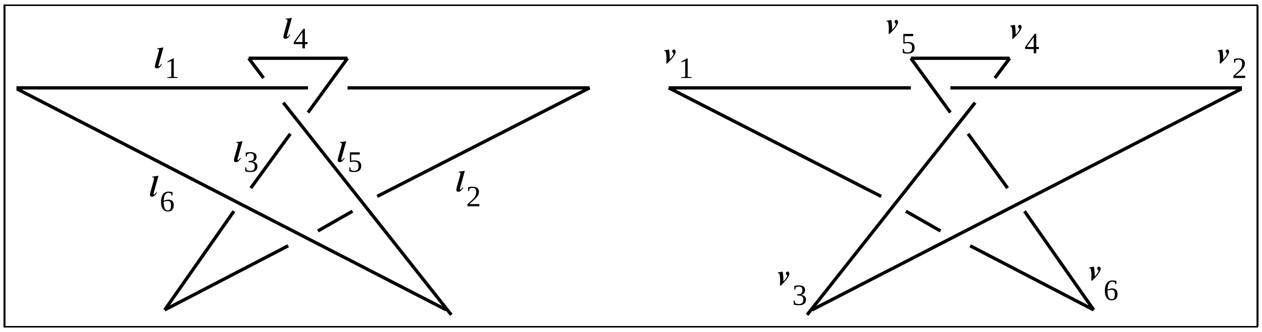


Figure 1. Classes 2 and 3, by Cantarella and Johnston (not drawn to scale)

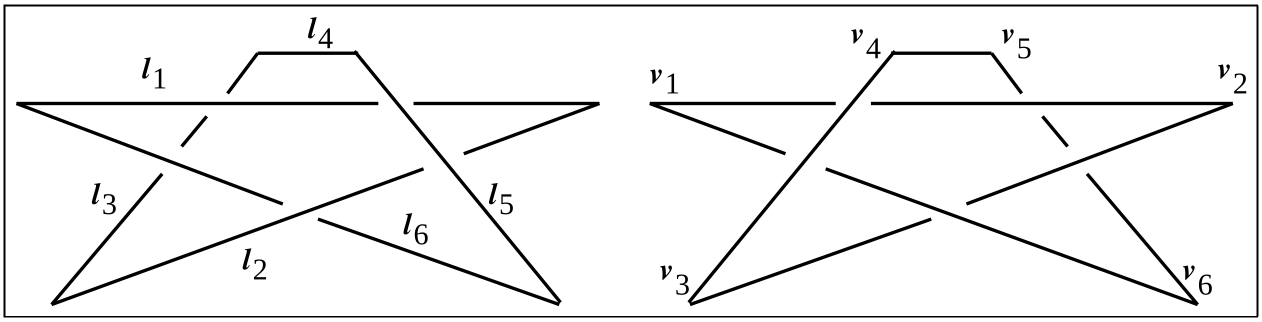


Figure 2. Classes 4 and 5, by Toussaint (not drawn to scale)

carefully. The first class includes the planar convex embedding. In fact, Aichholzer et al. [1] proved that all planar convex embeddings belong to the same connected component. The remaining two classes, which are left and right hand versions of the same polygon, are shown in Figure 1.

Cantarella and Johnston suspected that these three classes were the only ones for the unknot in $Pol_6$, but Toussaint [5] showed that there exist two more classes, for $Pol_6(20, 13, 4, 1, 4, 13)$. The two new classes are left and right hand versions of each other, and are shown in Figure 2. In the following section, we show that four more classes can exist for appropriate edge lengths.

## 2. A hexagonal unknot with nine embedding classes

We consider the space $Pol_6(1, 1, 0.55, \epsilon, 0.55 - \epsilon, 1)$, where $\epsilon \leq 0.01$ (i.e. sufficiently small). Here, we show that this space has at least nine connected components corresponding to the unknot. In other words there are at least nine embedding classes of the unknot for our given edge lengths.

The first class contains all planar convex embeddings. The next four are similar in shape to those of Cantarella and Johnston and those of Toussaint. However, since we are using different edge lengths it is necessary to verify that our classes of hexagons are still in different connected components of space. The four new classes are shown in Figures 3 and 4.

Cantarella and Johnston showed that a sufficient condition for their classes to be in

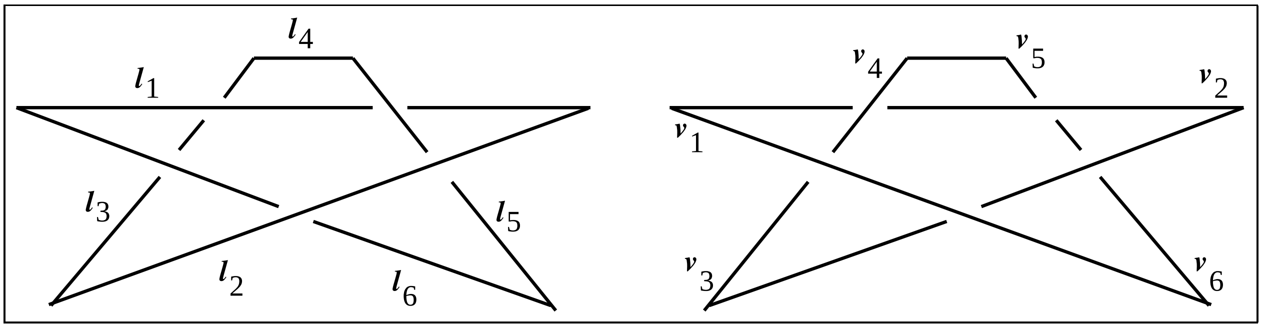


Figure 3. Classes 6 and 7 (not drawn to scale)

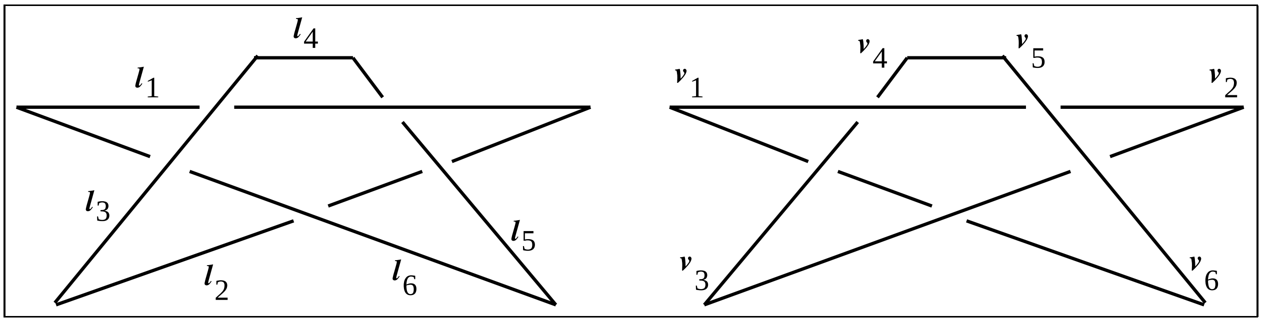


Figure 4. Classes 8 and 9 (not drawn to scale)

different connected components than the convex class is that $(\ell_3 + \ell_4)^2 < \ell_1\ell_2 - (\ell_1)^2/2$. Our chosen edge lengths satisfy this condition, so we can say that the proof of Cantarella and Johnston holds for our hexagon. Of course, it is not enough just to show that every class is in a different connected component than that containing the convex class.

Unfortunately, we can not do the same as above and borrow the proof for Toussaint's classes. A sufficient (but not necessary) condition used in this case is that $\ell_3 + \ell_4 + \ell_5 < min\{\ell_2, \ell_6\}$. Our hexagon is modified enough that this condition no longer holds. It seems that we cannot avoid such a modification, in order to create the new classes.

Even though Toussaint's proof cannot be used here, the intuition is quite the same. For each class, there are a couple of motions that must be made in order to convexify the hexagon. We will show that such motions are impossible to make, and conclude that it is impossible not only to convexify a hexagon belonging to each class but also to reconfigure between the non-convex classes.

Let us now fix a coordinate system in order to view all possible motions. Let $\ell_1$ be fixed in the $xy$ plane, and $\ell_6$ be constrained to the plane. Specifically, $v_1$ is at $(0, 0, 0)$, $v_2$ is at $(1, 0, 0)$, and $v_6$ is at $(v_{6x}, v_{6y}, 0)$ where $v_{6x}^2 + v_{6y}^2 = 1$. Our view will be along the normal of the plane. Thus vertex $v_6$ may move only in a circle about $v_1$. We choose to focus on class 9, as shown on the right of Figure 4. Intuitively we can see that to convexify this hexagon we would have to pass $v_3$ over $\ell_6$ and into the plane, or we would have to pass $\ell_5$ over $v_2$ so that $v_5$ could be placed in the plane.

We now proceed to describe certain constraints in the configuration of our polygon. The

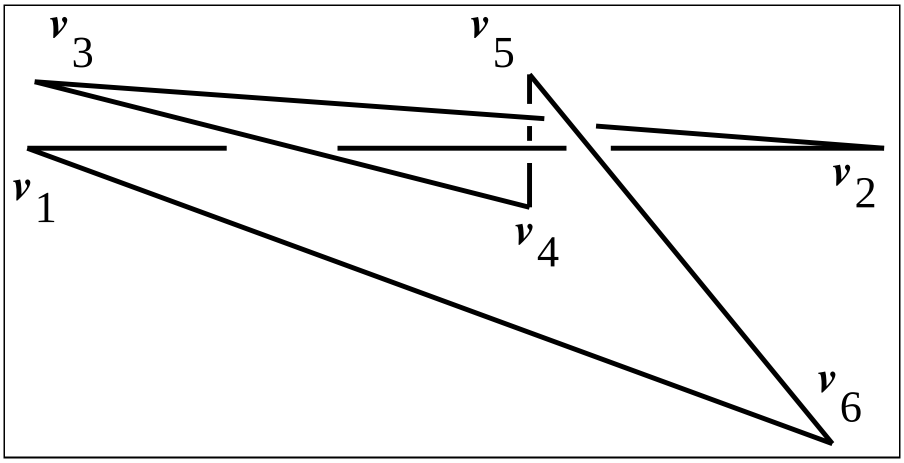


Figure 5. Alternate configuration of class 9 (not drawn to scale)

polygon shown in Figure 4 possesses certain properties, listed below:

1. $v_{5z} > 0$, and $v_{4z} < 0$. Thus we can define the point $P = P_{(x,y)} = (P_x, P_y)$ to be the intersection of $\ell_4$ with the plane. This also means that $v_4$ and $v_5$ are at a distance of at most $\epsilon$ from the plane.
2. $P_y > 0$, and $\epsilon < P_x < 1 - \epsilon$.
3. $v_{3z} > 0$.

We will show that, while reconfiguring the polygon, it is impossible to change any single property without causing an intersection or contradicting the validity of at least one other. We will also show that we cannot change more than one of these properties simultaneously. This implies that the properties are always true. Note that *changing* a property is done via a continuous motion of the polygon. Thus for example, property 1 changes the moment that $v_{5z} = 0$ and/or $v_{4z} = 0$. In case we attempt to set both of these vertices into the plane simultaneously, the point $P$ will be defined to be the last unique point of intersection of $\ell_4$ with the plane.

First let's see if we can change only the third property. Consider the case where $v_{3y} < 0$. $v_3$ can be placed in the plane only if the angle at $v_2$ or the angle at $v_1$ opens to more than $\pi/3$. This would mean that the distance from $v_3$ (or $v_6$ respectively) to $\ell_1$ becomes greater than $\frac{\sqrt{3}}{2}$. The distance to $P$ is at least as great, since $P_y > 0$. However, the distance from $v_3$ ($v_6$) to $P$ can be at most the sum of the lengths $\ell_3 + \ell_4$ (or $\ell_5 + \ell_4$ respectively). These sums are less than $\frac{\sqrt{3}}{2}$. Thus, by contradiction, we conclude that $v_3$ cannot be placed in the plane if $v_{3y} < 0$. Allowing properties 1 and/or 2 to change simultaneously with property 3 does not affect the arguments given above. At the "critical" moment that we attempt to place $v_3$ into the plane, we still have $P_y \geq 0$, for example.

It remains to be seen if we can change only the third property, when $v_{3y} > 0$. Even reaching a configuration where $v_{3y} > 0$ requires a subtle motion. It can be done by bringing $P$ sufficiently close to $\ell_1$, which allows us to obtain $v_{4y} < 0$. Then $\ell_2$ can move in a clockwise motion as viewed from the normal to the plane. The final configuration is shown in Figure 5. During the motion described above, we have $0 \leq v_{3x} \leq 1$. Thus the x-coordinates of both endpoints of $\ell_3$ are within this range. This means that if $\ell_3$ is to avoid intersection with $\ell_1$ it is in fact necessary to have $v_{4y} < 0$. Keeping in mind that $v_{4z} < 0$ (property 1), we see that it is impossible to place $v_3$ in the plane without causing an intersection between $\ell_3$ and $\ell_1$. We conclude that $v_3$ always remains above the plane, as long as properties 1 and 2 are true. Again, allowing properties 1 and/or 2 to change at the same time (i.e. allowing $P_x = \epsilon$ and/or $v_{4z} = 0$) doesn't affect our arguments. Thus property 3 cannot change, either on its own, or at the same time as properties 1-2.

We now focus on the second property: To obtain $P_y < 0$, we must first have $P_x < 0$ or $P_x > 1$. This is because of the position of $\ell_1$. So we would be requiring that $v_{4x} < \epsilon$ or $v_{4x} > 1 - \epsilon$. The same holds for $v_{5x}$. In other words, $v_4$ and $v_5$ would have to approach to within a distance $\epsilon$ from the halfline $x = 0, y > 0, z = 0$ or the halfline $x = 1, y > 0, z = 0$. By examining the triangle $v_1v_5v_6$ (which can be made to approach arbitrarily close to the plane if $\epsilon$ is small enough) and using elementary trigonometry, we can see that the former case is impossible. On the other hand, for $v_4$ to approach the halfline $x = 1, y > 0, z = 0$, we would need the angle $v_1v_2v_3$ to reach $\pi/3$. As shown previously, this cannot be done when $v_{3y} < 0$. We also established that if $v_{3y} > 0$, as shown in Figure 5, then $v_{4y} < 0$. This means that the angle $v_1v_2v_3$ cannot open to $\pi/3$, by the same logic used when dealing with changing property 3. Thus we must always have $0 < P_x < 1$ and $P_y > 0$. Once again, the argument given is not affected by allowing property 1 to change simultaneously.

Finally, we can examine the first property: Given property 2, we cannot place $v_5$ into the plane because this would cause an intersection between the polygonal arc $Pv_5v_6$ and $\ell_1$. Also, we cannot place $v_4$ into the plane because the arc $v_3v_4P$ would intersect $\ell_1$. More specifically, if $v_{3y} \geq 0$, bringing $v_4$ to the plane would cause an intersection between $\ell_4$ and $\ell_1$, since $P_y > 0$. If $v_{3y} < 0$, the intersection would be between $\ell_1$ and either $\ell_3$ or $\ell_4$ depending on the value of $v_{4y}$. We conclude that $v_4$ ($v_5$) must always be below (above) the plane.

Having established property 1, we know that class 9 cannot be reconfigured to classes 2,5,7,8. Now consider class 6: $\ell_2$ passes above the polygonal arc $Pv_5v_6$, whereas in class 9 it passes beneath the arc. In both cases, $\ell_2$ is above the plane. We know that starting from class 9 we cannot position $v_3$ into the plane. This implies that if we are to reconfigure between these classes, $v_3$ must pass under $\ell_5$. This cannot be done, due to the angle restrictions mentioned while handling property 1. In fact, if this were possible, then $v_3$ could be placed in the plane. The same applies for class 3. Finally, the only difference between classes 4 and 9 is in the crossing of $\ell_3$ and $\ell_6$, as we view along the normal to the plane. To change this crossing, the point $Q$, defined earlier, would have to move to an intermediate position such that $Q_y < v_{6y}$. This involves passing $v_3$ under $\ell_5$, which cannot be done, as we have just seen.

We can now say that class 9 cannot be reconfigured to any of the other classes shown. If we ignore the small difference in the lengths of $\ell_3$ and $\ell_5$, which does not play a role in the proofs given above, then the four new classes are either left/right hand versions or rigid transformations of each other. Thus the arguments given for class 9 hold for classes 6–8. As for our modified examples of Toussaint's classes, the proofs are similar. This leads us to the following theorem:

**Theorem 2.1.** *For suitable choices of edge lengths, there are at least nine connected components belonging to the unknot in $Pol_6$.*

## 3. Remarks

Although we show that there are at least four more embedding classes for $Pol_6$ we note that our examples are not very "stable", in the sense that the slightest change in the length of either $\ell_2$ or $\ell_6$ would result in the loss of four classes: those of Toussaint, and two new ones. If we made both lengths larger, then we would only have the three classes of Cantarella and Johnston. An open problem is therefore to find suitable edge lengths so that $Pol_6$ has more than five "stable" embedding classes.

As was mentioned by Cantarella and Johnston, these results can be extended to $Pol_n$, by replacing the shortest edge with a chain. Alternatively, a link or a knot can be placed to obtain similar results.

Calvo showed that even if edge lengths can vary, there exist four components of trefoil knots in $Pol_6$ (two left-handed and two right-handed). In fact, no other type of knot can be formed with only six edges [2]. It was tempting to see whether we could modify our classes, by changing some crossings, to obtain more than four components corresponding to the trefoil knot. The examples of Cantarella and Johnston cannot be transformed into trefoils, and the remaining six non-planar unknots produce only four trefoil classes, due to symmetry conditions. It remains an interesting open problem to determine if more than four trefoil embeddings exist in $Pol_6$ when edge lengths are fixed.

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