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Dihedral f-Tilings of the Sphere by Spherical Triangles and Equiangular well-centered Quadrangles

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Abstract. The classification of f-tilings was inspired in Stewart Robertson's work [5], "Isometric Foldings of Riemannian Manifolds" and was initiated by Ana Breda [3], where a complete classification of all monohedral f-tilings of the Riemannian sphere S^2 was done. Here we shall classify, up to a spherical isometry, the class of all dihedral f-tilings of S^2 whose prototiles are spherical triangles and well centered spherical quadrangles with all congruent internal angles. Table 1 and Figure 19 give a summary of the families involved in this classification.

Keywords: *WCSQ*, dihedral tilings, isometric foldings, spherical trigonometry

1. Introduction

Let S^2 be the Riemannian sphere of radius 1. A spherical moon, L is *well centered* if its vertices belong to the great circle $S^2 \cap \{ (x,y,z) \in \mathbb{R}^3 : x = 0 \}$ and the semi-great circle bisecting L contains the point $(1,0,0)$. If L_1 and L_2 are two spherical moons with orthogonal vertices then L_1 and L_2 are said to be *orthogonal*. By a *well centered spherical quadrangle*

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(*WCSQ*) we mean a spherical quadrangle which is the intersection of two well centered spherical moons with distinct vertices. In [2] was established that any spherical quadrangle with congruent internal angles is congruent to a well centered spherical quadrangle, which is the intersection of two orthogonal moons.

In this paper we shall discuss dihedral f-tilings by spherical triangles and well centered spherical quadrangles (*WCSQ*) with congruent internal angles.

By a *dihedral folding-tiling* or *dihedral f-tiling* for short, of the euclidean sphere $S^2 = \{ x \in \mathbb{R}^3 : \|x\| = 1 \}$, we mean a polygonal decomposition of the sphere by polygons congruent to a two fixed non-congruent polygons, where all vertices satisfy the *angle-folding relation* i.e., each vertex is of even valency and the sums of alternating angles at each vertex are π.

Two dihedral f-tilings of S^2, S_1 and S_2, are isomorphic iff there is an isometry ψ of S^2 such that $\psi(S_1) = S_2$. By "unique f-tiling" we mean unique up to an isomorphism.

We shall denote by $\Omega(Q,T)$ the set, up to an isomorphism, of all dihedral f-tilings of S^2, whose prototiles are a *WCSQ* Q and a spherical triangle T.

Relations between faces, edges, vertices and angles of any dihedral f-tiling of S^2, with prototiles Q and T are stated in proposition 1.1.

Proposition 1.1. *Let $\tau \in \Omega(Q,T)$. If $M > 0$ and $N > 0$ denote, respectively, the number of spherical quadrangles congruent to Q and the number of spherical triangles congruent to T of τ, and E and V denote, respectively, the number of edges and vertices of τ, then:*

i) $N + 2M = 2V - 4$;

ii) $3V = 6 + M + E$;

iii) *there are at least $6 + M$ vertices of valency four;*

iv) $\beta + \gamma + \delta < \frac{3\pi}{2}$, *where β, γ and δ denote the internal angles of T.*

Proof. Let α_1, α_2, α_3, α_4 and β, γ, δ be the internal angles measure of the spherical quadrangle Q and the spherical triangle T, respectively, then:

$$M(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 2\pi) + N(\beta + \gamma + \delta - \pi) = 4\pi. \tag{1.1}$$

Since

$$2\pi V = M(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + N(\beta + \gamma + \delta)$$

we may conclude that

$$N + 2M = 2V - 4.$$

By the Euler's relation we also have

$$M + N - E + V = 2$$

therefore

$$3V = 6 + M + E. \tag{1.2}$$

Denoting by V_k, $k \geq 2$ the number of vertices of valency $2k$, then

$$\sum_{k\geq 2} kV_k = E.$$

Now, using (1.2), one has

$$3\sum_{k\geq 2} V_k = 6 + M + \sum_{k\geq 2} kV_k \Leftrightarrow V_2 = 6 + M + \sum_{k\geq 4}(k-3)V_k$$

and so

$$V_2 \geq 6 + M, \tag{1.3}$$

that is the number of vertices of valency four is at least $6 + M$. Obviously $V \geq V_2$ and by (1.3) we obtain, $N + 2M = 2V - 4 \geq 2(6 + M) - 4$ and so

$$N \geq 8.$$

Finally by (1.1) we may conclude that

$$\beta + \gamma + \delta < \frac{3\pi}{2}. \qquad \square$$

From now on Q denotes a spherical quadrangle with all congruent internal angles, say α, and T denotes a spherical triangle with internal angles β, γ, δ such that $\beta \geq \gamma \geq \delta$, see Figure 1.

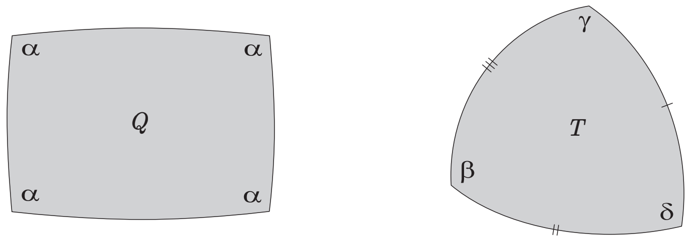

Figure 1

It follows straight away that $\frac{\pi}{2} < \alpha < \pi$ and $\pi < \beta + \gamma + \delta < \frac{3\pi}{2}$.

Next, we shall describe the set $\Omega(Q, T)$, considering separately different cases depending on the nature of Q and T.

In order to get the dihedral f-tilings of $\Omega(Q, T)$, we always start by considering one of its *planar representation* (PR) beginning with a common vertex to a spherical quadrangle and a spherical triangle in adjacent positions (it must be pointed out that there is always a vertex satisfying this condition).

In following sections we shall denote by $WCSQ^*$ a $WCSQ$ with all congruent internal angles.

2. Dihedral f-tilings by equilateral spherical triangles and $WCSQ^*$

Proposition 2.1. *If T is an equilateral spherical triangle with internal angle measure, say β, then $\Omega(Q,T)$ is the empty set.*

Proof. Let us suppose that $\Omega(Q,T)$ is non empty. Let v be a vertex of Q. Considering that $\alpha > \frac{\pi}{2}$ and $\beta > \frac{\pi}{3}$ then v is a vertex of valency four and the cells surrounding v have, in cyclic order, angles measure $(\alpha, \alpha, \beta, \beta)$, with $\alpha + \beta = \pi$, see Figure 2. Since $2\beta < \pi < 3\beta$ and $2\beta + \alpha > \pi$, then around w there is no way to position the other cells to have the angle-folding relation full filled, i.e, $\Omega(Q,T) = \emptyset$. □

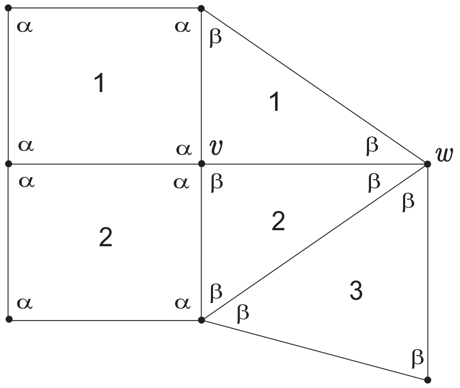

Figure 2

To go on with the description of $\Omega(Q,T)$ we find it useful to label any dihedral f-tiling $\tau \in \Omega(Q,T)$ according to the following procedures,

1. Label 1 the tiles by which we begin the planar representation of the dihedral f-tiling τ.
2. Label a tile j, if the knowledge of the *PR* of τ by polygons labelled $(1, 2, \ldots, j-1)$ leads, in a unique way, to the extended planar representation $(1, 2, \ldots, j)$.

Observe that we have used the above labelling procedures in the diagrams illustrated in Figure 2.

The arguments used in next results follow a fairly rigid pattern and we decided to give, in detail, only the proofs we consider more representative. The proofs of all propositions can be obtained, in full detail, from the authors.

3. Dihedral f-tilings by isosceles spherical triangles and $WCSQ^*$ with all congruent sides

In this section Q and T denote, respectively, a spherical quadrangle and a spherical triangle, where Q has congruent internal angles, say $\alpha \in]\frac{\pi}{2}, \pi[$, and all congruent sides, and T has internal angles β, γ, γ, $\beta \neq \gamma$.

We shall consider, separately, the cases $\gamma > \frac{\pi}{2}$, $\gamma = \frac{\pi}{2}$ and $\gamma < \frac{\pi}{2}$.

Proposition 3.1. *If $\gamma > \frac{\pi}{2}$, then $\Omega(Q,T) = \emptyset$.* □

Proposition 3.2. *If $\gamma = \frac{\pi}{2}$, then $\Omega(Q,T)$ consists of a single tiling given by an antiprism denoted by $\mathcal{A}_\alpha$, with $\alpha = \arccos(1-\sqrt{2})$ and $\beta = \pi - \alpha$.*

Sketch of the proof. Consider a vertex of Q in which α and γ are adjacent angles. We may prove that the extension of the planar f-tiling is uniquely determined, with $\alpha+\beta = \pi = \gamma+\gamma$, Figure 3-I. Now, if b is the length side of Q (opposite to β), then

$$\cos b = \frac{1+\cos\alpha}{1-\cos\alpha} = -\cos\alpha, \text{ see [2].}$$

Therefore

$$\cos^2\alpha - 2\cos\alpha - 1 = 0 \text{ and so } \cos\alpha = 1-\sqrt{2}.$$

Hence $\alpha = \arccos(1-\sqrt{2}) = \alpha_0$ is the internal angle of Q, since $\alpha \in]\frac{\pi}{2}, \pi[$. Similarly the length side of T opposite to γ is $\frac{\pi}{2}$. The 3D representation of $\tau = \mathcal{A}_{\alpha_0}$, is illustrated in Figure 3-II. $\square$

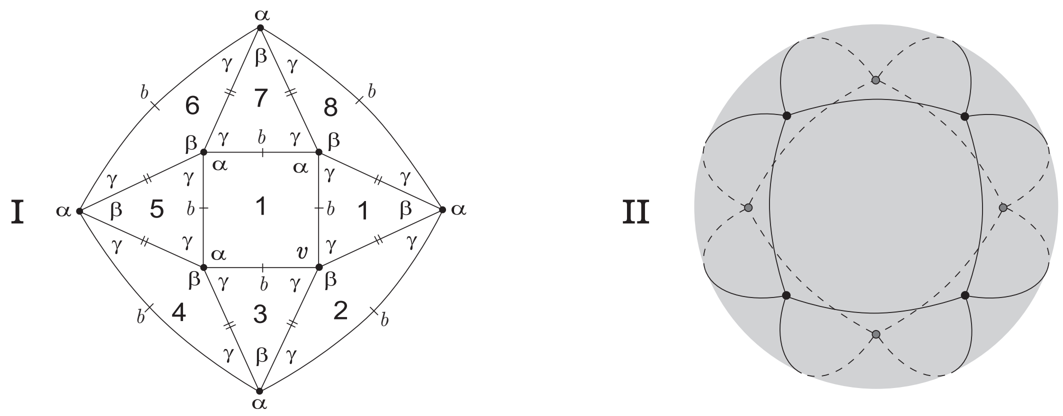

Figure 3

Next, we shall consider that $\gamma < \frac{\pi}{2}$. Let b and c be, respectively, the length sides of T opposite to β and γ. Necessarily, the length side of Q, say a, has to be b or c.

Proposition 3.3. *With the above terminology if $a = c$, then $\Omega(Q,T)$ consists of a single tiling given by an antiprism denoted by $\mathcal{A}_\alpha$, with $\alpha = \frac{2\pi}{3}$.*

The relation between angles is given by $\alpha+\gamma = \pi = \beta+\gamma$ and so $\alpha = \beta$ and $\gamma = \pi - \alpha$, see Figure 4-I. Now, as c is the side of Q which is common to the side of T opposite to γ, then

$$\cos c = -\frac{\cos\alpha(1+\cos\alpha)}{\sin^2\alpha} = \frac{1+\cos\alpha}{1-\cos\alpha}.$$

Hence $\cos\alpha = \frac{1}{2}$ and taking in account that $\frac{\pi}{2} < \alpha < \pi$, then $\alpha = \frac{2\pi}{3} = \beta$ and $\gamma = \frac{\pi}{3}$. In Figure 4-II is illustrate a 3D representation of $\tau = \mathcal{A}_\alpha,\ \alpha = \frac{2\pi}{3}$. $\square$

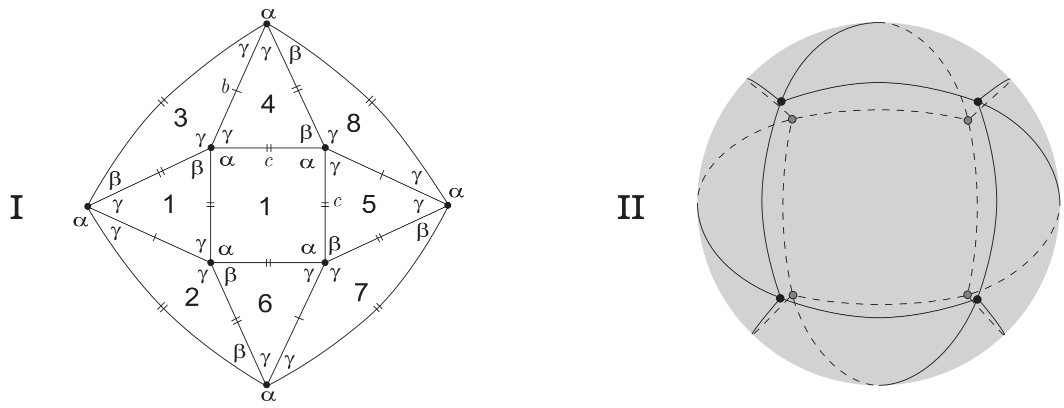

Figure 4

Proposition 3.4. *If $a = b$, then $\Omega(Q,T) = \{\mathcal{R}^k \mid k \geq 2\}$, where $\mathcal{R}^k$ is a dihedral f-tiling satisfying, $\alpha + \gamma = \pi$ and $\beta = \frac{\pi}{k}$, $k \geq 2$.*

In the presence of this situation, Figure 5-I illustrates the unique planar representation, that we may obtain obeying the angle folding relation.

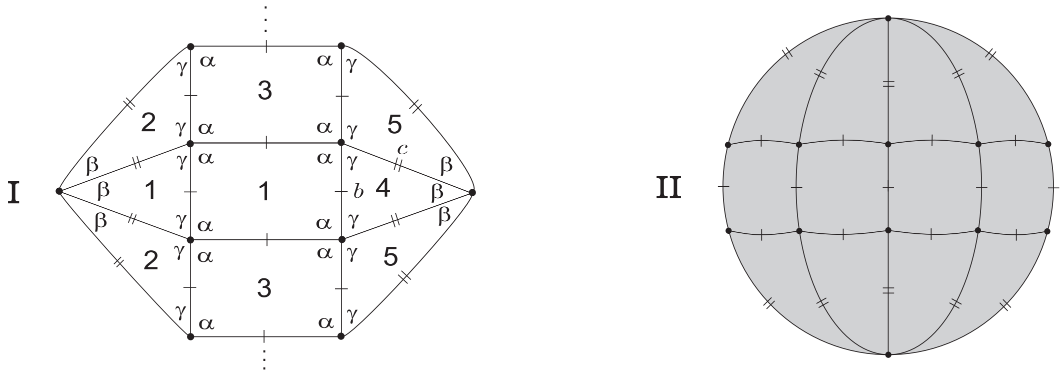

Figure 5

For any $k \geq 2$, one has

$$\cos\alpha = \frac{-1+\cos\frac{\pi}{k}}{2}, \ \cos b = \frac{1+\cos\frac{\pi}{k}}{3-\cos\frac{\pi}{k}} \ \text{ and } \ c = \frac{\pi - b}{2}.$$

In Figure 5-II is illustrated a 3D representation of $\mathcal{R}^4$. □

4. Dihedral f-tilings by scalene spherical triangles and $WCSQ^*$ with all congruent sides

Consider here a spherical quadrangle Q with congruent internal angles, say α, $\frac{\pi}{2} < \alpha < \pi$, and congruent sides, and a spherical triangle T with internal angles β, γ, δ such that $\beta > \gamma > \delta$. As seen before, $\pi < \beta + \gamma + \delta < \frac{3\pi}{2}$. Denote by b, c and d the length sides of T opposite to β, γ and δ, respectively. Suppose that the length side of Q is a. Then a is either b or c or d. We shall consider separately each one of these cases.

Proposition 4.1. *If $a = b$, then $\Omega(Q,T) = \emptyset$.* $\square$

Consider, next that $a = c$. The planar representation near a vertex v belonging to adjacent tiles congruent to Q and T, respectively, is illustrated in Figure 6. With the labelling of this figure $\theta = \beta$ or $\theta = \gamma$.

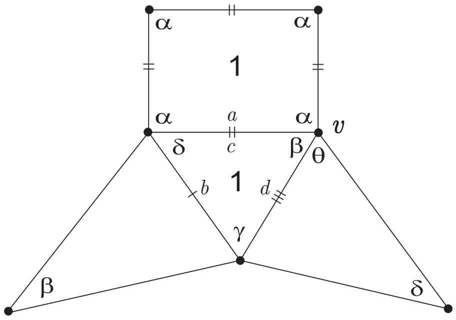

Figure 6

Proposition 4.2. *If $\theta = \beta$, then $\Omega(Q,T) = \emptyset$.*

Proof. Suppose that $\theta = \beta$, then the PR near v can be extended as follows:

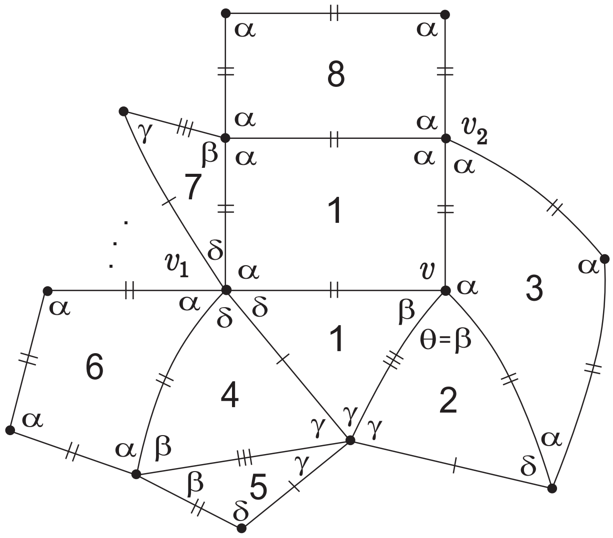

Figure 7

Taking in account that there are necessarily vertices of valency four, then $\alpha + \beta = \pi$. Since v_1 is surrounded by δ, v_1 cannot be of valency four, and so $\alpha + t\delta = \pi$, $t \geq 2$ (note that $\alpha + \gamma + \delta > \beta + \gamma + \delta > \pi$), see Figure 7.

The tile labelled 5 is completely identified ($\gamma + \beta < \alpha + \beta = \pi < \beta + \gamma + \delta \leq \beta + \gamma + \rho$, $\rho \in \{\alpha, \beta, \gamma, \delta\}$), therefore the tile labelled 6 is a quadrangle and the tile labelled 7 is a triangle as indicated above. The vertex v_2 of tile 8 must be surrounded by three angles α, leading to a contradiction. $\square$

We shall consider now that $\theta = \gamma$, see Figure 6.

Proposition 4.3. *If $\theta = \gamma$ and $\alpha + \gamma = \pi$, then $\Omega(Q,T)$ consists of a family of antiprismatic tilings $(\mathcal{A}_\alpha)_{\alpha \in]\frac{2\pi}{3}, \pi[}$.*

Sketch of the proof. In Figure 8-I is illustrated a PR of such f-tiling, where $\alpha+\gamma=\pi=\beta+\delta$, $\beta>\alpha>\gamma>\delta$. A 3D representation is presented in Figure 8-II.

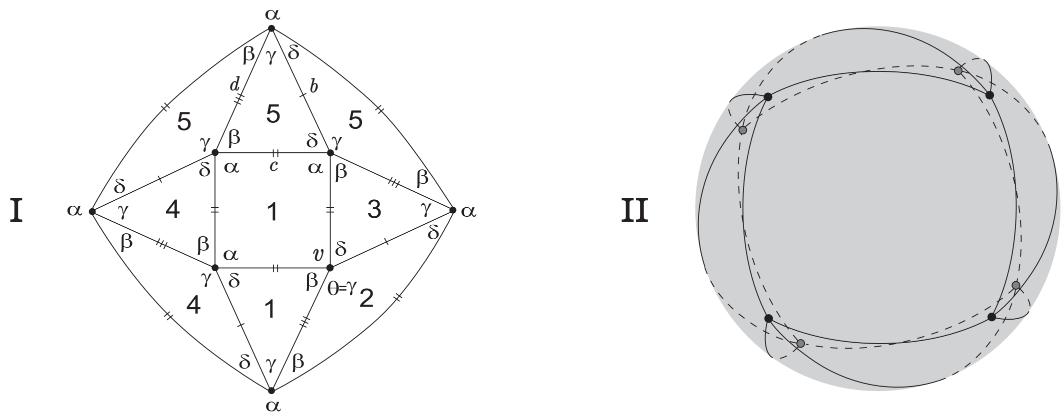

Figure 8

One has,

$$\cos c = \frac{1+\cos\alpha}{1-\cos\alpha} = -\frac{\cos\alpha+\cos^2\beta}{\sin^2\beta}, \quad \frac{\pi}{2}<\alpha<\pi, \quad \frac{\pi}{2}<\beta<\pi,$$

where c is the length side of T opposite to γ. Therefore,

$$\sin\beta = \frac{\sin\alpha}{\sqrt{-2\cos\alpha}}. \tag{4.1}$$

On the other hand $\beta>\alpha$, and so $\alpha\in]\frac{2\pi}{3},\pi[$. By the relation (4.1) we may also observe that $\frac{2\pi}{3}<\beta<\pi$. □

Proposition 4.4. *If $\theta=\gamma$ and $\alpha+\gamma<\pi$, then $\Omega(Q,T)=\{I^k \mid k=2,3,\dots\}$, where for each $k\geq 2$, I^k is a dihedral f-tiling satisfying,*

$$\alpha+\gamma+(k-1)\delta=\beta+k\delta=\pi \quad \wedge \quad \beta+\gamma=\pi.$$

In Figure 9 is illustrated a complete PR of I^2, that can be obtained from the PR illustrated in Figure 6.
The current conditions between angles allows write α, β and γ as functions of δ, namely,

$$\alpha=\pi-(2k-1)\delta, \quad \beta=\pi-k\delta, \quad \gamma=k\delta, \ k\geq 2.$$

Since

$$\frac{1+\cos\alpha}{1-\cos\alpha}=\frac{\cos\gamma+\cos\beta\cos\delta}{\sin\beta\sin\delta},$$

we conclude that

$$\frac{1-\cos((2k-1)\delta)}{1+\cos((2k-1)\delta)}=\frac{\cos k\delta(1-\cos\delta)}{\sin k\delta\sin\delta}. \tag{4.2}$$

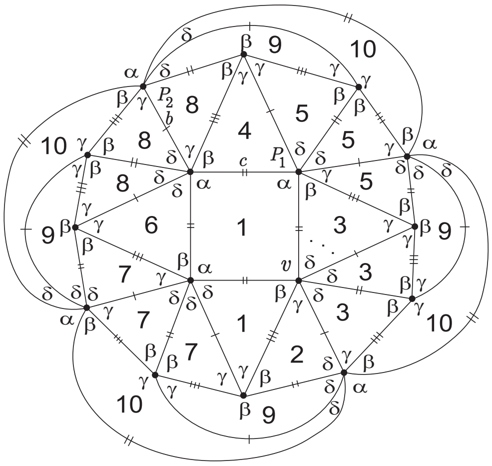

Figure 9

Given $0 < k\delta < \frac{\pi}{2}$, $k \geq 2$, δ is completely determined for each $k \geq 2$, by equation (4.2). For instance if $k = 2, 3$ and 4, then

$$\delta \approx 16,8^\circ, \quad \delta \approx 8,6^\circ \quad \text{and} \quad \delta \approx 5,4^\circ, \quad \text{respectively.}$$

As vertices P_1 and P_2 (Figure 9) are in antipodal positions, observe that $b + c = \pi$, where b and c are the length sides of T opposite to β and γ, respectively.

In Figure 10 we show a 3D representation of I^2 and I^3. $\square$

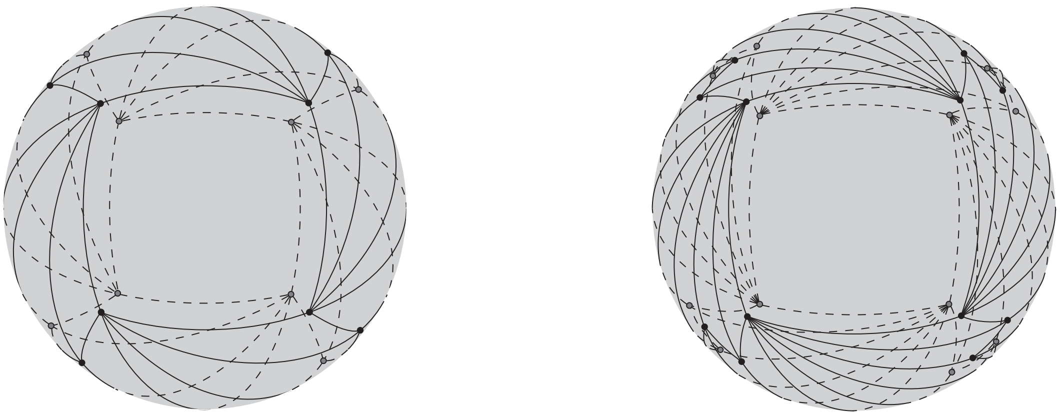

Figure 10

Proposition 4.5. *If $a = d$, then $\Omega(Q, T)$ consists of a family of antiprismatic tilings*

$$(\mathcal{A}_\alpha)_{\alpha \in]\alpha_0, \frac{2\pi}{3}[}, \quad \alpha_0 = \arccos(1 - \sqrt{2}).$$

In Figure 11 is illustrated a PR and a 3D representation of $\mathcal{A}_\alpha$, $\alpha \in]\alpha_0, \frac{2\pi}{3}[$.

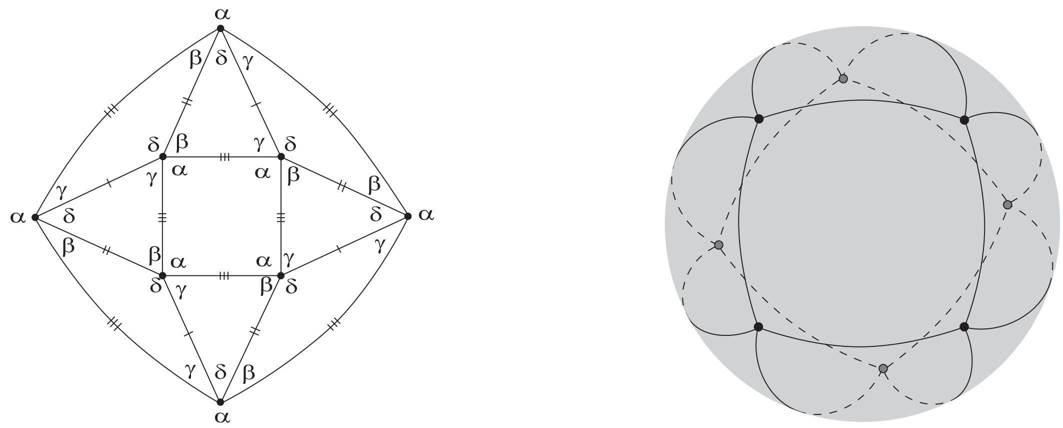

Figure 11

As seen before if d is the side length of T opposite to δ, then

$$\sin\beta = \frac{\sin\alpha}{\sqrt{-2\cos\alpha}}.$$

Since $\alpha \in]\frac{\pi}{2}, \pi[\,,\ \beta \in]\frac{\pi}{2}, \pi[$, then by previous relation $\beta < \alpha$ iff $\alpha < \frac{2\pi}{3}$. On the other hand $\sin\alpha \leq \sqrt{-2\cos\alpha}$, and so

$$\alpha \geq \alpha_0 = \arccos(1-\sqrt{2}) \approx 114,47^\circ.$$

The case $\alpha = \alpha_0$ implies that T is an isosceles triangle (proposition 3.2), consequently $\alpha_0 < \alpha < \frac{2\pi}{3}$. □

5. Dihedral f-tilings by isosceles spherical triangles and *WCSQ** with incongruent adjacent sides

In this section Q represents a spherical quadrangle with congruent internal angles, say α, $\alpha \in]\frac{\pi}{2}, \pi[$ and distinct congruent opposite pairs of sides; T is an isosceles spherical triangle with internal angles measure, say β, γ, γ such that $\pi < 2\gamma + \beta < \frac{3\pi}{2}$.

Proposition 5.1. *With the above terminology if $\gamma \geq \frac{\pi}{2}$, then $\Omega(Q,T) = \emptyset$.* □

Next, we shall assume that $\gamma < \frac{\pi}{2}$.

If $\tau \in \Omega(Q,T)$ then there are necessarily two cells of τ congruent to Q and T, respectively, such that they are in adjacent positions and in one of the following situations:

Figure 12

Proposition 5.2. *With the above terminology, if $\gamma < \frac{\pi}{2}$, and Q and T are adjacent tiles as indicated in Figure* 12-I*, then $\Omega(Q,T) = \emptyset$.* □

Proposition 5.3. *If* $\gamma < \frac{\pi}{2}$ *and* Q *and* T *are adjacent tiles as indicated in Figure* 12-II, *then for each integer* $k \geq 2$ *there is a family of dihedral f-tilings, denoted by* $\mathcal{R}_\alpha^k$, *such that:*

$$\alpha + \gamma = \pi, \quad \beta = \frac{\pi}{k}, \quad \alpha \in \left]\frac{\pi}{2}, \frac{(k+1)\pi}{2k}\right[\quad \textit{and} \quad \cos\alpha \neq \frac{-1+\cos\frac{\pi}{k}}{2}.$$

In Figure 13 is shown a 3D representation of $\mathcal{R}_\alpha^4$. Similarly we may draw $\mathcal{R}_\alpha^k$, for any $k \geq 2$.

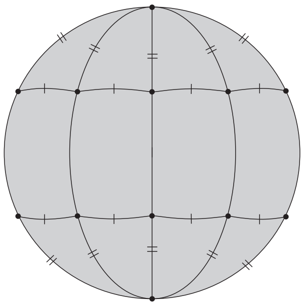

Figure 13

The restriction on $\cos\alpha$ comes from the following facts: Denoting by $A(Q)$ the area of Q then

$$0 < A(Q) = 4\alpha - 2\pi < \frac{2\pi}{k}$$

therefore

$$\frac{\pi}{2} < \alpha < \frac{(k+1)\pi}{2k}.$$

Since Q has distinct pairs of opposite sides then by proposition 3.4

$$\alpha \neq \arccos\frac{-1+\cos\frac{\pi}{k}}{2} = \alpha_k. \qquad \square$$

6. Dihedral f-tilings by scalene spherical triangles and *WCSQ** with incongruent adjacent sides

As in last section, Q stands for a spherical quadrangle with internal angle α, $\alpha \in]\frac{\pi}{2}, \pi[$, and congruent distinct opposite pairs of sides, while T stands for a scalene spherical triangle with internal angles β, γ and δ, such that, $\beta > \gamma > \delta$.

Any element of $\Omega(Q, T)$ has, at least, two cells congruent to Q and T, respectively, such that they are in adjacent positions and in one of the following situations:

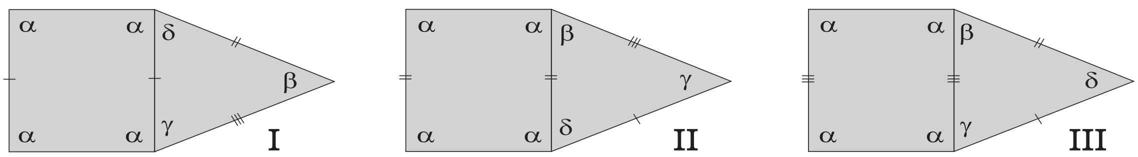

Figure 14

Proposition 6.1. *If Q and T are in adjacent positions as illustrated in Figure* 14-I, *then $\Omega(Q,T)$ is the empty set.* □

Proposition 6.2. *If Q and T are in adjacent positions as in Figure* 14-II, *then $\Omega(Q,T) = \{ J^k \mid k = 2,3,\dots \}$, where J^k, $k \geq 2$ is a dihedral f-tiling, such that the sum of alternating angles around vertices are of the form*

$$\alpha + k\delta = \beta + k\delta = \pi \quad \wedge \quad \alpha + \gamma = \beta + \gamma = \pi \quad \wedge \quad \beta + \gamma = \pi.$$

Sketch of the proof. Suppose that Q and T are adjacent tiles as illustrated in Figure 14-II. Taking in account the side length of Q and T we may extend in a uniquely way the PR, see Figure 15. The main step of the proof is to prove that $\alpha + \theta_1 = \pi$, $\theta_1 = \gamma$, and $\theta_2 = \delta$ that can be done studying all possible cases for θ_1 and θ_2.

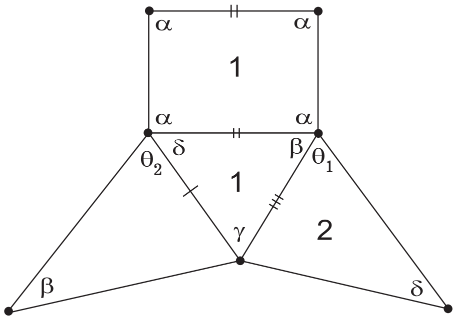

Figure 15

Thus we obtain $\alpha + \gamma = \beta + \gamma = \pi$, $\alpha + k\delta = \beta + k\delta = \pi$, $k \geq 2$ and $\beta + \gamma = \pi$ as the sums of alternating angles around vertices.

Considering $k = 2$ the extended PR, is given in Figure 16-I. We denote such f-tiling by J^2. In Figure 16-II is represented a generic PR of J^k, $k \geq 2$.

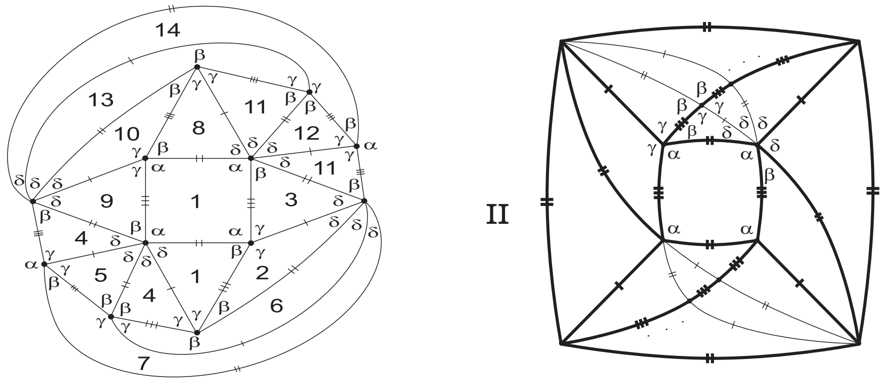

Figure 16

The relations between angles allow us to write α, β and γ as functions of δ, namely,

$$\alpha = \pi - k\delta, \quad \beta = \pi - k\delta \quad \text{and} \quad \gamma = k\delta.$$

Now, the diagonal of Q divides Q into two congruent spherical triangles T_1 and T_2 of internal angles measure α, α_1, α_2, $\alpha_1 + \alpha_2 = \alpha$, see Figure 17.

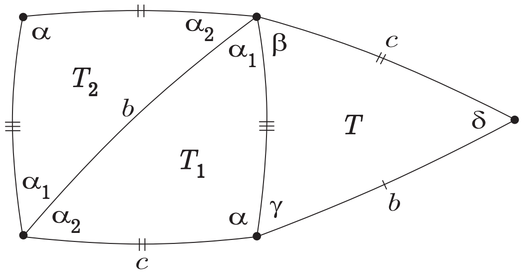

Figure 17

T and T_1 have two congruent sides and considering that $\alpha = \beta$ we conclude that T and T_1 are congruent spherical triangles. Therefore $\alpha_1 = \gamma$ and $\alpha_2 = \delta$.

Denoting by b and c the length sides of T opposite to β and γ, respectively, we obtain $b + c = \pi$ (observe that $\alpha_1 + \beta = \alpha + \gamma = \pi$. And so $T \cup T_1$ is a spherical moon). Now,

$$\alpha = \alpha_1 + \alpha_2 \Leftrightarrow \beta = \gamma + \delta \Leftrightarrow \pi - k\delta = k\delta + \delta.$$

Therefore

$$\delta = \frac{\pi}{2k+1}.$$

Consequently

$$\gamma = \frac{k\pi}{2k+1} \qquad \alpha = \frac{(k+1)\pi}{2k+1} = \beta\ , \qquad k \geq 2.$$

In Figure 18 is illustrated a 3D representation of J^2, where

$$\alpha = \frac{3\pi}{5}, \quad \beta = \frac{3\pi}{5}, \quad \gamma = \frac{2\pi}{5} \quad \text{and} \quad \delta = \frac{\pi}{5}.$$ □

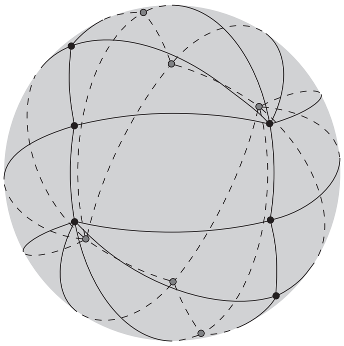

Figure 18

Proposition 6.3. *If Q and T are in adjacent positions as in Figure* 14-III, *then $\Omega(Q,T)$ is the set obtained in proposition* 6.2. □

In Table 1 is shown a complete list of all dihedral f-tilings, whose prototiles are an isosceles spherical triangle of angles β, γ, γ or a scalene triangle of angles β, γ, δ, with $\beta > \gamma > \delta$ and a $WCSQ^*$ with angle α. (There is not any dihedral f-tiling if T is equilateral and Q is a $WCSQ^*$). Our notation is as follows.

- β_α and δ_k are the solutions of equation (4.1) and equation (4.2), respectively;
- $\alpha_k = \arccos \frac{-1+\cos\frac{\pi}{2}}{k}$;
- $|V|$ is the number of distinct classes of congruent vertices (we shall say that vertices v_1 and v_2 are congruent iff there is a spherical isometry that maps the faces incidents to v_1 into the faces incidents to v_2);
- M and N are, respectively, the number of quadrangles congruent to T and the number of triangles congruent to Q, used in such dihedral f-tilings.

f-tiling	α	β	γ	δ	$\lvert V\rvert$	M	N
$\mathcal{A}_{\alpha_0}$	α_0	$\pi-\alpha_0$	$\frac{\pi}{2}$	-	1	2	8
$\mathcal{A}_\alpha$	$\alpha_0<\alpha<\frac{2\pi}{3}$	β_α	$\pi-\beta_\alpha$	$\pi-\alpha$	1	2	8
$\mathcal{A}_{\frac{2\pi}{3}}$	$\frac{2\pi}{3}$	$\frac{2\pi}{3}$	$\frac{\pi}{3}$	-	1	2	8
$\mathcal{A}_\alpha$	$\frac{2\pi}{3}<\alpha<\pi$	β_α	$\pi-\alpha$	$\pi-\beta_\alpha$	1	2	8
$\mathcal{R}^k$, $k\geq 2$	α_k	$\frac{\pi}{k}$	$\pi-\alpha_k$	-	2	$2k$	$4k$
$\mathcal{R}^k_\alpha$, $k\geq 2$	$]\frac{\pi}{2},\frac{(k+1)\pi}{2k}[\setminus\{\alpha_k\}$	$\frac{\pi}{k}$	$\pi-\alpha$	-	2	$2k$	$4k$
I^k, $k\geq 2$	$\pi-(2k-1)\delta_k$	$\pi-k\delta_k$	$k\delta_k$	δ_k	2	2	$8(2k-1)$
J^k, $k\geq 2$	$\frac{(k+1)\pi}{2k+1}$	$\frac{(k+1)\pi}{2k+1}$	$\frac{k\pi}{2k+1}$	$\frac{\pi}{2k+1}$	3	2	$8k$

Table 1: Dihedral f-Tilings by Spherical Triangles and $WCSQ^*$

In Figure 19 we present in 3D all the dihedral f-tilings obtained before, whose prototiles are a spherical triangle, T, and a $WCSQ^*$, Q, consisting of

- *A family of square antiprisms* $(\mathcal{A}_\alpha)_{\alpha\in[\alpha_0,\,\pi[}$, in which T is an isosceles triangle iff $\alpha \in \{\,\alpha_0, \frac{2\pi}{3}\,\}$. We have considered $\alpha_0<\alpha_1<\frac{2\pi}{3}<\alpha_2<\pi$. $\alpha_0=\arccos(1-\sqrt{2})$
- For each $k\geq 2$ *a family of* $2k$*-polygonal radially elongated dipyramids*, $\mathcal{R}^k_\alpha$, $\alpha\in\,]\frac{\pi}{2},\frac{(k+1)\pi}{2k}[$ in which T is an isosceles triangle and Q is a spherical square iff $\alpha=\alpha_k$.
- A *class of f-tilings* $I^k,\ k\geq 2$, in which Q is a square and T is a scalene triangle. We illustrate I^2.
- *A class of f-tilings* J^k, $k\geq 2$ in which Q is a spherical quadrangle with distinct pairs of opposite sides and T is a scalene triangle. We have considered $k=2$.

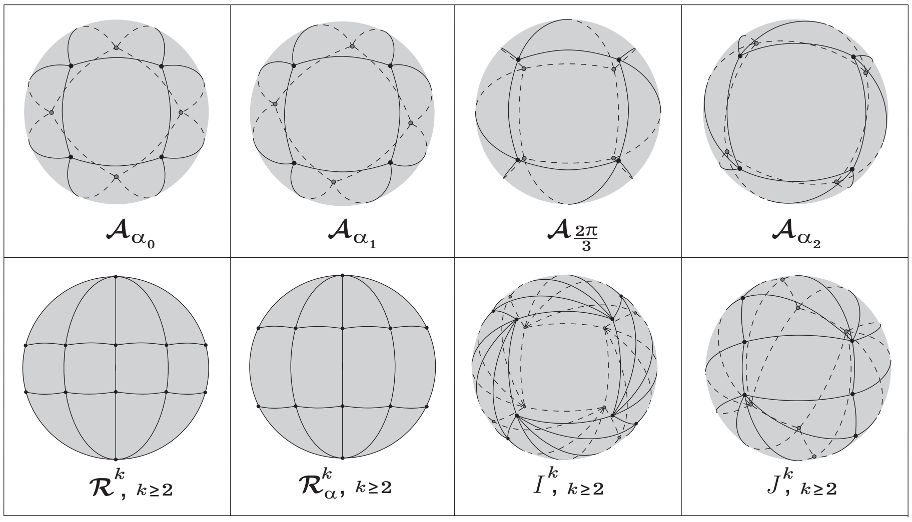

Figure 19

References

[1] Berger, Marcel: *Geometry*. Volume II, Springer-Verlag, New York 1996.

[2] d'Azevedo Breda, Ana M.; Santos, Altino F.: *Well Centred Spherical Quadrangles*. Beitr. Algebra Geom. **44**(2) (2003), 339–349. Zbl pre1973847

[3] d'Azevedo Breda, Ana M.: *A Class of Tilings of* S^2. Geometriae Dedicata **44** (1992), 241–253. Zbl 0770.52010

[4] Cromwell, Peter R.: *Polyhedra*. Cambridge University Press 1997. Zbl 0888.52012

[5] Robertson, Stewart: *Isometric Folding of Riemannian Manifolds*. Proc. Royal Soc. Edinburgh **79** (1977), 275–284. Zbl 0418.53016

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