Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 45 (2004), No. 1, 239-251.

Rings with Indecomposable Modules Local

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Abstract. Every indecomposable module over a generalized uniserial ring is uniserial and hence a local module. This motivates us to study rings R satisfying the following condition: $(*)$ R is a right artinian ring such that every finitely generated right R-module is local. The rings R satisfying $(*)$ were first studied by Tachikawa in 1959, by using duality theory, here they are endeavoured to be studied without using duality. Structure of a local right R-module and in particular of an indecomposable summand of R_R is determined. Matrix representation of such rings is discussed.

MSC 2000: 16G10 (primary), 16P20 (secondary)
Keywords: left serial rings, generalized uniserial rings, exceptional rings, uniserial modules, injective modules, injective cogenerators and quasi-injective modules

Introduction

It is well known that an artinian ring R is generalized uniserial if and only if every indecomposable right R-module is uniserial. Every uniserial module is local. This motivated Tachikawa [8] to study rings R satisfying the condition $(*)$: R is a right artinian ring such that every finitely generated indecomposable right R-module is local. Consider the dual condition $(**)$: R is a left artinian ring such that every finitely generated indecomposable left R-module has unique minimal submodule. If a ring R satifies $(*)$, it admits a finitely generated injective cogenerator Q_R. Let a right artinian ring R admit a finitely generated injective cogenerator Q_R and $B = End(Q_R)$ acting on the left. Then $_BQ_R$ gives a duality between the category $mod - R$ of finitely generated right R-modules and the category $B - module$ of finitely generated left B-modules. Thus if R satisfies $(*)$, then B satisfies $(**)$. In [8] Tachikawa studies $(*)$ through $(**)$, but that does not give enough information about the structure of right ideals of R. In the present paper, the condition $(*)$ is endeavoured to be studied without using duality. Let R satisfy $(*)$. Theorems (2.9), (2.10) give the structure of any local module A_R, in particular of the indecomposable summands of R_R. Theorem (2.12) gives the structure of a local ring satisfying $(*)$. The structure of a right artinian ring R for which $J(R)^2 = 0$, and which satisfies $(*)$ is discussed in Theorem (2.13). In Section 3, the results of Section 2 are applied to some specific situations dealing with some matrix rings. Theorem (3.8) gives a matrix representation of a ring R with $J(R)^2 = 0$, satisfying $(*)$.This theorem shows that a sufficiently large class of such rings can be obtained from certain incidence algebras of some finite partially ordered sets.

1. Preliminaries

All rings considered here are with identity $1 \neq 0$ and all modules are unital right modules unless otherwise stated. Let R be a ring and M be an R-module. $Z(R)$ denotes the *center* of R, $J(M)$, $E(M)$, and $socle(M)$ denote the *radical*, the *injective hull* and the *socle* of M respectively, but $J(R)$ will be generally denoted by J. For any module B, $A < B$ denotes that A is a proper submodule of B. The ring R is called a *local ring* if R/J is a division ring. Given two positive integers n, m, R is called an $(n,m)-ring$, if R is a local ring, $J^2 = 0$, and for $D = R/J$, $dim\ _DJ = n$, $dim\ J_D = m$. Any $(1,2)$ (or $(2,1)$) ring R is called an *exceptional ring* if $E(_RR)$ (respectively $E(R_R)$) is of composition length 3 [2, p 446]. A module in which the lattice of submodules is linearly ordered under inclusion, is called a *uniserial module*, and a module that is a direct sum of uniserial modules is called a *serial module* [3, Chapter V]. If for a ring R, $_RR$ is serial, then R is called a *left serial ring.* A ring R that is artinian on both sides is called an *artinian ring.* An artinian ring that is both sided serial is called a *generalized uniserial ring* [3, Chapter V]. A ring R that is a direct sum of full matrix rings over local, artinian, left and right principal ideal rings is called a *uniserial ring.* If a module M has finite composition length, then $d(M)$ denotes the composition length of M. Let D be a division ring, and $D^{'}$ be a division subring of D. Then $[D : D^{'}]_r$ ($[D : D^{'}]_l$) denotes the dimension of $D_{D'}$ (respectively $_{D'}D$). In case F is a subfield of D contained in $Z(D)$, then $[D : F]$ denotes the dimension of D_F.

2. Local modules

Consider the following condition.

$(*)$: R is a right artinian ring such that any finitely generated, indecomposable right R-module is local.

Throughout all the lemmas, the ring R satisfies $(*)$. Then for any module M_R, $J(M) = MJ$. The main purpose of this section is to determine the structure of local *right* modules over such a ring.

Lemma 2.1. *Any uniform R-module is uniserial. Any uniform R-module is quasi-injective.*

Proof. Consider a uniform R-module M. If M is not uniserial it has two submodules A, B of finite composition lengths such that $A \not\subseteq B$ and $B \not\subseteq A$. Then $A + B$ is a finitely generated

R-module which is indecomposable and is not local. This is a contradiction. Hence M is uniserial. As $E(M)$ is uniserial, M is invariant under every R-endomorphism of $E(M)$. Hence M is quasi-injective. □

Proposition 2.2. *Let R be any right artinian ring. Then R satisfies $(*)$ if and only if it satisfies the following condition:*

Let A_R, B_R be two local, non-simple modules. Let $C < A$, $D < B$ be simple submodules, and $\sigma : C \to D$ be an R-isomorphism. There exists an R-homomorphism $\eta : A \to B$ or $\eta : B \to A$ extending σ or σ^{-1} respectively.

Proof. Let R satisfy $(*)$. Let A_R, B_R be two local, non-simple modules. Let $C < A$, $D < B$ be simple submodules, and $\sigma : C \to D$ be an R-isomorphism. Set $L = \{(c, -\sigma(c)) : c \in C\}$, and $M = A \times B/L$. Then $M = A_1 \oplus A_2$ for some local submodules A_i. Let η_A and η_B be the natural embeddings in M of A and B respectively, and $\pi_i : M \to A_i$ be the projections. Either $\pi_1(\eta_A(A)) = A_1$ or $\pi_1(\eta_B(B)) = A_1$. Suppose $\pi_1(\eta_A(A)) = A_1$. Then $d(A_1) \leq d(A)$. If $d(A_1) = d(A)$, then $\eta_A(A) \cong A_1$ and it is a summand of M, we get an R-epimorphism $\lambda : M \to A$ such that $\lambda\eta_A = 1_A$. Then $\eta = \lambda\eta_B : B \to A$ extends σ^{-1}. Let $d(A_1) < d(A)$. Then $d(A_2) \geq d(B)$. If $\pi_2(\eta_B(B)) = A_2$, then $d(A_2) = d(B)$, as seen above there exists an R-homomorphism $\eta : A \to B$ that extends σ. Suppose $\pi_2(\eta_B(B)) \neq A_2$. Then $\pi_2(\eta_A(A)) = A_2$. As $\eta_B(B) \nsubseteq MJ$, $\pi_1(\eta_B(B)) = A_1$. Then either $d(A) = d(A_2)$ or $d(B) = d(A_1)$. This gives the desired η.

Conversely, let the given condition be satisfied by R. On the contrary suppose that R does not satisfy $(*)$. There exists an indecomposable R-module K of smallest composition length that is not local. Then $socle(K) \subseteq KJ$. Consider any simple submodule S of K. Then K/S is a direct sum of local modules, so $K = A + B$ for some submodules A, B with A a local, and $A \cap B = S$. Then $B = \oplus\sum_{i=1}^{t} B_i$ for some local submodules B_i. Now $S = xR$ and $x = \sum x_i$, $x_i \in B_i$. If for some i, say for $i = 1$, $x_1 = 0$, then $K = (A + \sum_{i=2}^{t} B_i) \oplus B_1$. Hence $x_i \neq 0$ for every i. Suppose $t \geq 2$. Now $S_i = x_iR$ is a simple submodule of B_i. We have an R-isomorphism $\sigma : S_1 \to S_2$ such that $\sigma(x_1) = x_2$. By the hypothesis, σ or σ^{-1} extends to an R-homomorphism $\eta : B_1 \to B_2$ or $\eta : B_2 \to B_1$ respectively. To be definite, let $\eta : B_1 \to B_2$ extend σ. Consider $C_1 = \{(b, \eta(b), 0, \ldots, 0) : b \in B_1\} \subseteq B$. Then $B = C_1 \oplus B_2 \oplus B_3 \oplus \cdots \oplus B_t$ and $S \subseteq C_1 \oplus B_3 \oplus \cdots \oplus B_t$. This is a contradiction. Hence $t = 1$. Thus B is local. So there exists an R-homomorphism η say from B to A that is identity on S. Then for $C = \{b - \eta(b) : b \in B\}$, $K = A \oplus C$. This is a contradiction. Hence R satisfies $(*)$. □

Lemma 2.3. *Let A_R, B_R be two local, non-simple modules such that $d(A) = d(B)$, $AJ^2 = BJ^2 = 0$.*

(i) *Suppose that for some simple submodule C of A, $\sigma : C \to B$ is an embedding. Then there exists an R-isomorphism $\eta : A \to B$ extending σ.*

(ii) *A and B are isomorphic if and only if there exists a simple submodule C of A that embeds in B.*

(iii) *If $socle(A) = AJ$ contains more that one homogeneous components, then each homogeneous component of $socle(A)$ is simple and the number of homogeneous components is two.*

Proof. (i) The hypothesis gives that B does not have any local, non-simple proper submodule. Suppose an R-homomorphism $\eta : A \to B$ extends σ. As $ker\ \sigma \cap C = 0$ and $ker\ \sigma \subseteq AJ$, $d(\eta(A)) \geq 2$. Hence $\eta(A) = B$ and η is an R-isomorphism. If an R-isomorphism $\lambda : B \to A$ extends $\sigma^{-1} : \sigma(C) \to C$, then $\eta = \lambda^{-1}$ extends σ. After this (2.2) completes the proof of (i). Now (ii) is an immediate consequence of (i).

(iii) Suppose $socle(A)$ has more than one homogeneous components. Suppose the contrary. Without loss of generality, we take $AJ = C_1 \oplus C_2 \oplus D$, where C_1 and C_2 are isomorphic simple modules and D is a simple module not isomorphic to C_1. Then $A_1 = A/C_1$ and $A_2 = A/D$ are not isomorphic but C_2 embeds in both of them. This contradicts (ii). Hence each homogeneous component of $socle(A)$ is simple. Suppose there are more than two homogeneous components of $socle(A)$. We can take $socle(A) = C_1 \oplus C_2 \oplus C_3$, where C_i are pairwise non-isomorphic simple modules. Then modules $A_1 = A/C_1$, $A_2 = A/C_2$ contradict (ii). This completes the proof. □

Theorem 2.4. *Let R satisfy $(*)$.*

(i) *Let e, f be two indecomposable idempotents in R such that $eJ \neq 0 \neq fJ$. Then $eR \cong fR$ if and only if eJ/eJ^2 and fJ/fJ^2 have some isomorphic simple submodules.*

(ii) *R is a left serial ring.*

Proof. (i) Let eJ/eJ^2 and fJ/fJ^2 have some isomorphic simple submodules. We can find appropriate images of eR/eJ^2 and fR/fJ^2 which are of same composition length but are not simple, and have some isomorphic simple submodules. By (2.3), these homomorphic images are isomorphic, so $eR/eJ \cong fR/fJ$. Hence $eR \cong fR$.

(ii) Firstly, suppose that $J^2 = 0$. Let $e \in R$ be an indecomposable idempotent such that $Je \neq 0$. By (i), to within isomorphism there exists unique indecomposable idempotent $f \in R$ such that $fJe \neq 0$. Consider any minimal left ideals S and S' contained in Je. Then $S = Rfxe$ and $S' = Rfye$ for some $fxe, fye \in fJe$. Set $T = fxeR$. We have an R-monomorphism $\omega : T \to fJ$ such that $\omega(fxe) = fye$. By (2.3), ω extends to an R-automorphim η of fR. Thus there exists an $fcf \in fRf$ such that $\omega(x) = fcfx$ for any $x \in T$, so $fye = fcfxe \in S$, $S' = S$. It follows that R/J^2 is left serial. From this it is obvious that R is left serial. □

Lemma 2.5.

(i) *There does not exist a local module A_R such that A/AJ^k is uniserial, $AJ^{k+1} = 0$, AJ^k is non-zero but not simple for some $k \geq 2$.*

(ii) *Let B_R be a local module such that $BJ \neq 0$. Then B is uniserial if and only if BJ is local.*

Proof. (i) Suppose the contrary, so an A_R satisfying the given hypothesis exists. Without loss of generality we take $AJ^k = C \oplus D$ for some simple submodules C, D. Consider $B = AJ/D$. Clearly $d(A) = k + 2$, $d(B) = k$. Consider the natural isomorphism $\sigma : C \to C \oplus D/D$. Suppose an R-homomorphism $\eta : A \to B$ extends σ. As ker $\eta \cap C = 0$, $d(ker\ \eta) \leq 1$, so $d(\eta(A)) > d(B)$. This is a contradiction. Hence, by (2.2), there exists an R-homomorphism $\eta : B \to A$ extending σ^{-1}. Then η is an R-monomorphism. This contradicts the fact that A

does not contain any uniserial submodule of composition length more than one. Finally, (ii) follows from (i). □

Lemma 2.6. *Let A_1, A_2 be two uniserial R-modules such that $d(A_i) \geq 3$. Then $M = A_1 \oplus A_2$ does not contain any local, non-uniserial submodule of composition length 3.*

Proof. Suppose the contrary. Let A be a local, non-uniserial submodule of M with $d(A) = 3$. Then $AJ = socle(M)$. Let $\pi_i : M \to A_i$ be the projections. Then $A = (a_1, a_2)R$. For $B_i = a_iR$, $d(B_i) = 2$, $A/AJ \cong B_i/B_iJ$, $B_iJ = socle(A_i)$ and we have an R-isomorphism $\sigma : B_1/B_1J \to B_2/B_2J$ such that $\sigma(\overline{a_1}) = \overline{a_2}$. There exist submodules $C_i \subseteq A_i$ with $d(C_i) = 3$. By using (2.1), we get an R-isomorphism $\eta : C_1/B_1J \to C_2/B_2J$ extending σ. We can find $c_i \in C_i$ such that $C_i = c_iR$ and $\eta(\overline{c_1}) = \overline{c_2}$. Consider $B = (c_1, c_2)R$. Now $a_1 = c_1r$ for some $r \in J$. Then $a_2 = c_2r + x$ for some $x \in B_2J$. As $B_1J \subseteq A$, there exists an $s \in J$ such that $a_1s \neq 0$ but $a_2s = 0$. Then $(c_1, c_2)rs = (a_1s, 0)$. Hence $B_1J \subseteq B$. Similarly, $B_2J \subseteq B$. Then $(a_1, a_2) = (c_1, c_2)r + (0, x)$ gives $A \subseteq BJ$. Also $BJ^2 = socle(M)$. Now $C_1/B_1J \cong B/BJ^2$. So $d(B) = 4$ and $BJ = A$. Hence B is local. This contradicts (2.5)(i). This proves the result.□

Lemma 2.7. *Let R satisfy (∗). For any local R-module A the following hold:*

(i) *AJ is a direct sum of uniserial submodules.*

(ii) *Any local submodule of AJ is uniserial.*

Proof. (i). Suppose the contrary. As AJ is a direct sum of local modules, without loss of generality, we take $AJ = C$, a local module that is not uniserial. For some $k \geq 1$, C/CJ^k is uniserial but CJ^k is not local. We can find a submodule B of CJ^k such that CJ^k/B is a direct sum of two minimal submodules. Then A/B contradicts (2.5)(i).

(ii) Suppose the result is true for all local modules of composition length less than $d(A)$, but the result is not true for A. There exists a local submodule B of AJ that is not uniserial. Let S be a minimal submodule of B. By the induction hypothesis B/S is uniserial. Thus $d(socle(B)) = 2$. Let C be a complement of $socle(B)$ in A. As B embeds in A/C, the induction hypothesis gives $C = 0$. Thus $socle(A) = socle(B) = C_1 \oplus C_2$ for some simple submodules C_i. Then $A \subseteq E(C_1) \oplus E(C_2)$. Now $d(E(C_i)) \geq 3$ and by (2.5)(i), $d(B) = 3$. This contradicts (2.6). Hence the result follows. □

Lemma 2.8. *Let C_1, C_2 be two uniserial right R-modules such that for some $k \geq 2$, $C_1/C_1J^k \cong C_2/C_2J^k$, $C_1J^k \neq 0 \neq C_2J^k$. Then $C_1/C_1J^{k+1} \cong C_2/C_2J^{k+1}$.*

Proof. We take $C_iJ^{k+1} = 0$. Set $B_i = C_iJ^k$. In view of 2.1, it is enough to prove that B_i are isomorphic. Suppose the contrary. As $socle(C_1/B_1) \cong socle(C_2/B_2)$, there exists an indecomposable idempotent $e \in R$ and a right ideal $A \subseteq eJ$ such that $socle(eR/A) \cong B_1 \oplus B_2$. Then eR/A is embeddable in $C_1 \oplus C_2$. This contradicts (2.6). □

Theorem 2.9. *Let R satisfy (∗) and A_R be a local module such that $AJ = C_1 \oplus C_2 \oplus D$ for some non-zero uniserial submodules C_i. If for some $k \geq 1$, $C_1/C_1J^k \cong C_2/C_2J^k$, $C_1J^k \neq 0 \neq C_2J^k$, then C_i/C_iJ^{k+1} are isomorphic.*

Proof. Without loss of generality, we take $AJ = C_1 \oplus C_2$. To prove the result, we take $socle(C_i) = C_i J^k \neq 0$. Consider $D_i = A/C_i$. Then each D_i is uniserial with $d(D_i) = k+2$, further, (2.1) and the hypothesis give that $D_1/D_1J^{k+1} \cong D_2/D_2J^{k+1}$. As $k+1 \geq 2$, (2.8) completes the proof. □

Theorem 2.10. *Let R satisfy $(*)$ and A_R be a local module with $AJ \neq 0$. Then $AJ = C_1 \oplus C_2 \oplus \cdots \oplus C_t$ for some uniserial submodules C_i and the following hold:*

(a) *Either all C_i/C_iJ are isomorphic or $t \leq 2$.*

(b) *Any local submodule of AJ is uniserial.*

(c) *If $d(C_1) \geq 2$, then either $t \leq 2$ or any C_i is simple for $i \geq 2$.*

Proof. That AJ is a direct sum of uniserial modules follows from (2.7), (a) follows from (2.3)(iii) by considering A/AJ^2, and (b) follows from (2.7). Finally, suppose $d(C_1) \geq 2$, $t \geq 3$, but for some $i \geq 2$, C_i is not simple. We can take $AJ = C_1 \oplus C_2 \oplus C_3$ such that $d(C_1) = 2$, $d(C_2) = 2$ and $d(C_3) = 1$. Set $B_2 = socle(C_2)$. Consider $A_2 = A/B_2$, $A_3 = A/C_3$. Then A_2, A_3 are non-isomorphic, they have same composition length and neither of them has a uniserial submodule of composition length three. For $S = socle(C_1)$, we have the natural R-isomorphism $\sigma : S + B_2/B_2 \to S + C_3/C_3$. There exists an R-homomorphism $\eta : A_2 \to A_3$ or $\eta : A_3 \to A_2$ extending σ or σ^{-1} respectively. In any case, by (b), the image of η is a uniserial module of composition length at least three. This is a contradiction. This proves (c). □

Corollary 2.11. *Let R satisfy $(*)$. Then for any idempotent $e \in R$, every finitely generated indecomposable eRe-module is local.*

Proof. Let M be a finitely generated indecomposable eRe-module. Then $N = M \otimes_{\mathrm{eRe}} eR$ is a finitely generated R-module. Thus $N = \oplus \sum_{i=1}^{m} A_i$ for some local R-submodules A_i. As $M = Ne$, $M = A_ie$ for some i. But $A_i = xfR$ for some indecomposable idempotent $f \in R$. If f is isomorphic to an indecomposable idempotent in eRe, trivially, A_ie is a local module. If f is not isomorphic to any indecomposable idempotent in eRe, then $A_ieR = xfReR \subseteq xfJ$. By (2.10)(b), A_ieR is a direct sum of uniserial R-modules. Consequently, $M = A_ieRe$ is a uniserial eRe-module. □

Any (1,2) exceptional ring R satisfies $(*)$ and has $J^2 = 0$. We now study a ring R with $J^2 = 0$.

Theorem 2.12. *Let R be a local ring satisfying $(*)$. Then either $J^2 = 0$ or R is a uniserial ring.*

Proof. By (2.4), R is left serial. Suppose, R is not right serial and $J^2 \neq 0$. By (2.7), $J_R = C_1 \oplus C_2 \oplus D$ with C_1, C_2 uniserial submodules such that $d(C_1) \geq 2$, and $C_2 \neq 0$. Let $A = C_2 \oplus D$. As R/A is a uniserial module of composition length at least three, for $E = E(R/J)$, $d(E) \geq 3$. We have a local module M such that J(M) is a direct sum of two minimal submodules. Clearly M embeds in $E \oplus E$. This contradicts (2.6). Hence R is uniserial. □

Theorem 2.13. *Let R be a right artinian ring such that $J^2 = 0$. If R satisfies $(*)$, then R satisfies the following conditions.*

(a) *Every uniform right R-module is either simple or injective with composition length 2.*

(b) *R is a left serial ring.*

(c) *For any indecomposable idempotent $e \in R$ either eJ is homogeneous or $d(eJ) \leqslant 2$.*

Conversely, if R satisfies (a), (b), and $d(eJ) \leqslant 2$ for any indecomposable idempotent $e \in R$, then R satisfies $(*)$.

Proof. Let every finitely generated indecomposable right R-module be local. Then (2.1) gives (a), (2.4) gives (b) and (2.10) gives (c). Conversely, let R satisfy (a), (b) and for any indecomposable idempotent $e \in R$, let $d(eJ) \leq 2$. Let A, B be two local R-modules that are not simple. Then $d(A) \leq 3$, $d(B) \leq 3$. Let C be a minimal submodule of A, and $\sigma : C \to B$ be an embedding. If $d(B) = 2$, B is uniserial and hence injective by (a), so there exists an R-homomorphism $\eta : A \to B$ extending σ. If $d(A) = 2$, similarly we get an extension $\eta : B \to A$ of $\sigma^{-1} : \sigma(C) \to C$. Thus we take $d(A) = 3 = d(B)$. There exist indecomposable idempotents e, $f \in R$, such that $A \cong eR$, $B \cong fR$. We take $A = eR$, $B = fR$. Then $C = exgR$, where for indecomposable idempotent $g \in R$, $exg \in eJg$. Further, $\sigma(exg) = fyg \in fJg$. By (b) Jg is a simple left R-module. So, $fyg = fvexg$ for some $fve \in fRe$. Then $\eta : eR \to fR$ given by left multiplication by fve extends σ. Hence, by (2.2), R satisfies $(*)$. □

3. Matrix representations

Lemma 3.1. *Let M_R be a quasi-injective module and K be a maximal submodule of M. If K is not indecomposable, then K contains a summand of M different from K.*

Proof. Let $K = A \oplus B$ with $A \neq 0$, $B \neq 0$. As M is quasi-injective, by using the fact that M is invariant under the endomorphism ring of its injective hull, $M = C \oplus D \oplus E$ with $A \subset' C$, $B \subset' D$ [3, Proposition 19.2]. As K is maximal, if $E \neq 0$, we get $K = C \oplus D$, so K contains a summand of M different from K. If $E = 0$, once again the maximality of K gives $A = C$ or $B = D$. Hence K contains a summand of M different from K. □

Let A_R be a local module of finite composition length, $D = End(A/J(A))$ and $T = End(A_R)$. T is a local ring and the division ring $D' = T/J(T)$ has natural embedding into D. The pair of division ring (D, D') is called a *dual division ring pair associate* (in short a *ddpa*) of A. This concept is dual of the concept of a division ring pair associate of a uniform module of a finite composition length as given in [6, p 296].

Proposition 3.2. *Let R satisfy $(*)$ and $e \in R$ be an indecomposable idempotent such that $eJ \neq 0$. Let $X < eJ$ be such that $A = eR/X$ is uniserial. If (D, D') is the ddpa of A, then $[D : D']_r \leq 2$.*

Proof. Suppose the contrary. There exist ω_1, ω_2, ω_3 right linearly independent over D'. Consider $M = \{(a_1, a_2, a_3) \in A^{(3)} : \omega_1\overline{a_1} + \omega_2\overline{a_2} + \omega_3\overline{a_3} = \overline{0}\}$. Then M is a maximal submodule of $A^{(3)}$. Suppose M is not indecomposable. By (2.1), A is quasi-injective, so $A^{(3)}$ is also quasi-injective. By using (3.1) and the Krull-Schmidt Theorem, we get a summand B of M isomorphic to A. Then for some $\eta_i \in End(A)$, $i = 1, 2, 3$, with at least one of them an automorphism, $B = \{(\eta_1(a), \eta_2(a), \eta_3(a)) : a \in A\}$. Then $(\omega_1\overline{\eta_1} + \omega_2\overline{\eta_2} + \omega_3\overline{\eta_3})(\overline{a}) = \overline{0}$, for every $a \in A$. Thus ω_1, ω_2, ω_3 are right linearly dependent over D'. This is a contradiction. Hence M is indecomposable. However $d(M/A^{(3)}J) = 2$, gives that M is not local. This is a contradiction. This proves the result. □

Proposition 3.3. *Let D be a division ring with center F, and D' be a division subring of D with center F' such that $[D : D']_r < \infty$. Then $[D : F]$ is finite if and only if $[D' : F'] < \infty$.*

Proof. Let $S = D'F$ and $K = F'F$. Clearly $K \subseteq Z(S)$. Let $[D : F]$ be finite. Then S is a division subring, K is a subfield and S is finite dimensional over K. Now $D' \otimes_{F'} K$ is central simple K-algebra [5, Proposition b, p 226] isomorphic to S, $[D' : F'] = [S : K]$, so $[D' : F'] < \infty$. Conversely, let $[D' : F'] < \infty$. This gives that S is a division ring finite dimensional over the field K and $[D : K]_r = n < \infty$. This gives an embedding $\phi : D \to M_n(K)$ such that for any $x \in F$, $\phi(x)$ is the scalar matrix xI. This induces an embedding $\mu : D \otimes_F K \to M_n(K)$, so $[D \otimes_F K : K] < \infty$ and hence $[D : F] < \infty$. □

Proposition 3.4. *Let D and D' be two division rings, V a (D, D')-bivector space such that $dim_D V = 1$ and $dim V_{D'} = n > 1$. Let $V = Dv$, $R = \begin{bmatrix} D & V \\ 0 & D' \end{bmatrix}$. Let L be any proper D'-subspace of V and $A_L = \begin{bmatrix} 0 & L \\ 0 & 0 \end{bmatrix}$. For $e_1 = e_{11}$, set $M = e_1R/A_L$.*

(I) *There exists an embedding $\sigma : D' \to D$ such that $va = \sigma(a)v$ for any $a \in D'$; this embedding makes D a right D'-vector space such that $d.c' = d\sigma(c')$ for any $d \in D$, $c' \in D'$, and $[D : \sigma(D)]_r .= n$.*

(II) *M is a faithful right R-module.*

(III) *$D_L = \{c \in D\text{: } cL \subseteq L\}$ is a division subring of D, $F_L = \{a \in D : av \in L\}$ is a (D_L, D')-subspace of D such that dim $(F_L)_{D'}$ = dim $L_{D'}$. Further, $L \leftrightarrow F_L$ is a lattice isomorphism between D'-subspaces of V and D'-subspaces of D.*

(IV) *Let L be a maximal D'-subspace of V.*

  (i) *M is quasi-injective if and only if for any $a \in D\backslash F_L$, $D = a\sigma(D') \oplus F_L = D_L a \oplus F_L$.*

  (ii) *M is injective if and only if M is quasi-injective, and for any maximal D'-subspace L' of V, there exists an $a \in D$ such that $aL = L'$.*

(V) *Let dim $V_{D'} = 2$ and L be a maximal D'-subspace of V. Then M is injective if and only if $[D : \sigma(D')]_l = 2$.*

(VI) *Let dim $V_{D'} = 2$. Then every finitely generated indecomposable right R-module is local if and only if $[D : \sigma(D')]_l = 2$.*

Proof. (I), (II) and (III) are obvious. Let L be a maximal D'-subspace of V. Then $d(M) = 2$ and M is uniserial. Consider any $a \in D\backslash F_L$. Then $w = av \notin L$, for $\overline{we_{12}} = we_{12} + A_L$, $socle(M) = e_1J/A = \overline{we_{12}}R$ and $End(socle(M)) \cong D'$. Consider $0 \neq c \in D'$. This gives $\lambda_c \in End(socle(M))$ such that $\lambda_c(\overline{we_{12}}) = \overline{wce_{12}}$. Suppose M is quasi-injective. Then λ_c

extends to an endomorphism of M, this when lifted to an endomorphism of e_1R gives an element $d \in D_L$ such that $d\overline{we_{12}}r = \lambda_c(\overline{we_{12}}r)$ for any $r \in R$, so $dw - wc \in L$. As $d^{'}L = L$ for any non-zero $d^{'} \in D_L$, it is immediate that d is uniquely determined by c. Conversely, given a $d \in D_L$, the left multiplication by d inducedsan endomorphism of $socle(M)$, so there exists a $c \in D^{'}$ such that $dw - wc \in L$. Thus $dav - avc \in L$, $da - a\sigma(c) \in F_L$, $d \in a\sigma(D^{'})a^{-1} + F_La^{-1}$, $D_L + F_La^{-1} \subseteq a\sigma(D^{'})a^{-1} + F_La^{-1}$. Similarly a$\sigma(D^{'})a^{-1} + F_La^{-1} \subseteq D_L + F_La^{-1}$. Hence $D_L + F_La^{-1} = a\sigma(D^{'})a^{-1} + F_L$. But $a\sigma(D^{'}) \cap F_L = 0 = D_La \cap F_L$ and F_L is a maximal $D^{'}$-subspace of D, so $D = a\sigma(D^{'}) \oplus F_L$ as $D^{'}$-vector spaces. This also gives $D_La \oplus F_L = D$ as left D_L-vector spaces. Conversely, if $D = D_La \oplus F_L = a\sigma(D^{'}) \oplus F_L$, $c \in D^{'}$ there exists a $d \in D_L$ such that $da - a\sigma(c) \in F_L$, so the endomorphism of $socle(M)$ induced by c can be realized by left multplication by d, hence M is quasi-injective. This proves (IV)(i).

(IV)(ii) Let E be the injective hull of M. Then $E/socle(M)$ is homogeneous. Given any other maximal $D^{'}$- subspace $L^{'}$ of V, we get corresponding right ideal $A_{L^{'}}$ and uniserial module $M^{'}$ $= e_1R/A_{L^{'}}$. Now $socle(M^{'}) \cong socle(M)$. So $M^{'}$ embeds in E. If M is injective, $M \cong M^{'}$; this isomorphism is induced by a $c \in D$ such that $cL = L^{'}$. Conversely, if for each $L^{'}$ such a c exists, then $M \cong M^{'}$. If in addition M is quasi-injective, it gives that M is injective.

Let $dim\ V_{D^{'}} = 2$. Now $L = bvD^{'}$ for some $0 \neq b \in D$. Given any other maximal $D^{'}$-subspace $L^{'} = b^{'}vD^{'}$, clearly $L^{'} = cL$ for $c = b^{'}b^{-1}$. So to prove that M is injective it is enough to prove that M is quasi-injective. Let M be quasi-injective. Now $[D : \sigma(D^{'})]_r = 2$, $F_L = b\sigma(D^{'})$ and $D_L = b\sigma(D^{'})b^{-1}$, thus for an $a \in D\backslash F_L$, $D = D_La \oplus F_L$ gives $[D : D_L]_l = 2$, $[D : b\sigma(D^{'})b^{-1}]_l = 2$, hence $[D : \sigma(D^{'})]_l = 2$. Conversely, let $[D : \sigma(D^{'})]_l = 2$. As $L = bvD^{'}$, for some $b \in D$, $F_L = b\sigma(D^{'})$, $D_L = b\sigma(D^{'})b^{-1}$, so $[D : D_L]_l = 2$. But for any $a \in D\backslash F_L$, $a\sigma(D^{'}) \cap F_L = 0 = D_La \cap F_L$. We have $D = a\sigma(D^{'}) \oplus F_L = D_La \oplus F_L$. By (IV) M is injective. The other indecomposable injective right R-module is e_1R/e_1J, which is simple. The ring is left serial. By (2.13), R satisfies $(*)$. □

Corollary 3.5. *Let R be as in the above theorem, such that D or $D^{'}$ is finite dimensional over its center. Then R satisfies $(*)$ if and only if $dim\ V_{D^{'}} = 2$.*

Proof. By (3.3) both D and $D^{'}$ are finite dimensional over their respective centers. Suppose R satisfies $(*)$. Let L be a maximal $D^{'}$-subspace of V. Consider $M = e_1R/A_L$ as in (3.4). By (2.13), M is injective. Now $ddpa$ of M is (D, D_L). By (3.2), $[D : D_L]_r = 2$, thus by (IV)(i) in (3.4), $[F_L : D_L]_l = 1$, $F_L = D_Lb$ for some $b \in D$, $b\sigma(D^{'})b^{-1} \subseteq D_L$. . By [5, Proposition 3, p 158], $[D : \sigma(D^{'}]_l = [D : \sigma(D^{'})]_r = n$. Consequently, $n = 2[D_L : b\sigma(D^{'})b^{-1}]_r$. At the same time, $n - 1 = [F_L : \sigma(D^{'})]_r = [D_L : b\sigma(D^{'})b^{-1}]_r$. Hence $n = 2(n-1)$, $n = 2$. The converse follows from part (VI) of (3.4). □

Proposition 3.6. *Let D be a division ring and $R = \begin{bmatrix} D & D & D \\ 0 & D & 0 \\ 0 & 0 & D \end{bmatrix}$. Then $e_{11}R$ contains only two minimal right ideals, $X = e_{12}D$ and $Y = e_{13}D$. The modules $e_{11}R/X$ and $e_{11}R/Y$ are injective and non-isomorphic and R satisfies $(*)$.*

Proof. Now $e_{11}J = X \oplus Y$, $X \cong e_{22}R$ and $Y \cong e_{33}R$. So X, Y are the only minimal right ideals contained in $e_{11}R$ and they are non-isomophic. Now $ann(e_{11}R/X) = e_{12}D + e_{22}D =$

A, and $R/A \cong \begin{bmatrix} D & D \\ 0 & D \end{bmatrix}$ a generalized uniserial ring. So $M = e_{11}R/X$ is quasi-injective. Consider any non-zero R-homomorphism $\lambda : e_{11}J \to M$, then $\ker \lambda = X$, so λ induces a mapping $\overline{\lambda}$ from $socle(M)$ to M. This extends to an endomorphism $\overline{\mu}$ of M. Then $\overline{\mu}$ gives $\mu : e_1R \to M$ extending λ. Thus M is $e_{11}R$-injective. M is trivially $e_{22}R$ and $e_{33}R$ injective. Hence M is injective. Similarly $e_{11}R/Y$ is injective. Any non-simple uniform right R-module contains a copy of X or Y, so it is going to be isomorphic to M or N. Clearly R is left serial. The last part now follows from (2.13). □

Proposition 3.7. *Let S be a local uniserial ring of composition length 2, $D = S/J(S)$, V a (D, D)-bivector space one dimensional on each side, and $R = \begin{bmatrix} S & V \\ 0 & D \end{bmatrix}$.*

(i) *$e_{11}R$ contains only two minimal right ideals, $X = e_{11}J(S)$ and $Y = e_{12}V$ and they are non-isomorphic.*

(ii) *$e_{11}R/X$ and $e_{11}R/Y$ are non-isomorphic injective modules.*

(iii) *R satisfies $(*)$.*

Proof. That X and Y are the only minimal right ideals contained in $e_{11}R$ is straight forward to prove. Now $ann(e_1R/X) = e_{11}J(S) = A$ and $R/A \cong \begin{bmatrix} D & D \\ 0 & D \end{bmatrix}$ a generalized uniserial ring, so $M = e_{11}R/X$ is quasi-injective. Follow the arguments in (3.6) to conclude that M is injective. Now $e_{11}J = X \oplus Y$. Again, $ann(e_{11}R/Y) = e_{12}V + e_{22}D = B$, and $R/B \cong S$, a uniserial ring. This gives $N = e_{11}R/Y$ is quasi-injective, and as for M, N is injective. Once again any non-simple uniform right R-module is isomorphic to M or N. Also R is left serial. After this, (2.13) completes the proof. □

We now give a matrix representation of R, without of loss of generality, we assume that R is a basic ring.

Theorem 3.8. *Let R be an indecomposable basic right artinian ring with $J^2 = 0$ such that every finitely generated indecomposable right R-module is local. Let $S = \{e_i : 1 \leqslant i \leqslant n\}$ be a complete orthogonal set of indecomposable idempotents. Then either R is a local $(1, n)$ ring for some positive integer n, or the following hold:*

(I) *For any $f \in S$ there does not exist more than one $e \in S$ such that $eJf \neq 0$.*

(II) *For any two e, f in S, $eJfJ = 0$.*

(III) *For any $e \in S$, there do not exist more than two $f \in S$ such that $eJf \neq 0$.*

(IV) *For any $e \in S$, one of the following holds:*

  (i) *eRe is a division ring,*

  (ii) *eRe is a uniserial ring with composition length 2.*

(V) *For any e, $f \in S$ with $eJf \neq 0$, eJf is a simple left eRe-module and either eJf is a simple right fRf-module or there does not exist any $g \in S$ different from f such that $eJg \neq 0$.*

(VI) *Consider any* $e \in S$, *and let* f_1, f_2 *be the only members of* S *such that* $eJf_1 \neq 0$, $eJf_2 \neq 0$. *Let* $D = eRe/eJe$, $D_i = f_iRf_i/f_iJf_i$. *Then the following hold:*

(i) eJf_i *is a* (D, D_i)*-bivector space.*

(ii) *There exists an embedding* $\sigma_i : D_i \to D$ *such that, if* $f_1 \neq f_2$, *then* σ_i *is an isomorphism, and if* $f_1 = f_2$, *then* $[D : \sigma_i(D_1)]_r$ *equals the composition length of the right* f_1Rf_1*-module* eJf_1.

(iii) *If* $f_1 = f_2$, *then for* $V = eJf_1$, $[D : \sigma_1(D_1)]_l = 2$ *whenever dim* $V_{D_1} = 2$.

Conversely, if R *satisfies conditions (I) through (VI) and in addition dim* $(eRf_1)_{D_1} \leqslant 2$ *whenever* $f_1 = f_2$, *then every finitely generated indecomposable right* R*-module is local.*

Proof. If R is a local ring, as R is left serial, it is a $(1, n)$ ring for some positive integer n. Suppose R is not a local ring. By (2.13), R is left serial. This gives (I). As $J^2 = 0$ (II) holds. Consider any $e \in S$ such that $eJ \neq 0$. By (2.3) either eJ is homogeneous, or eJ has only two homogeneous components and each of them is a simple module. So there exist at most two members f_1, f_2 of S satisfying $eJf_j \neq 0$. Then $eJ = eJf_1 + eJf_2$. As R is left serial, each eJf_i is a simple left eRe-module. Suppose $e = f_1 = f_2$. Consider any $g \in S\backslash\{e\}$. Then $eRg = 0$. As $eJe \neq 0$, by (I) $gRe = 0$. this gives that eR is a summand of R as an ideal. However, R is indecomposable, so R is a local ring. This is a contradiction. Hence $e = f_1 = f_2$ is not possible. Let $f_1 \neq f_2$, then $eJ = eJf_1 \oplus eJf_2$ with each eJf_i a simple right f_iRf_i-module. If $e \neq f_1$, f_2, then $eJe = 0$, so eRe is a division ring. If $e = f_1$, then $eJ = eJe \oplus eJf_2$ with eJe a simple right eRe-module. So eRe is a uniserial ring with composition length 2. Let $f_1 = f_2$. Then eJ is homogeneous and $eJg = 0$ for any $g \in S\backslash\{f_1\}$. This proves (III), (IV) and (V). Set $D = eRe/eJe$ and $D_i = f_iRf_i/f_iJf_i$. Now $Jf_i = eJf_i = Dv$ for some $v \in eJf_i$. This gives an embedding $\sigma_i : D_i \to D$ such that $va = \sigma_i(a)v$ for any $a \in D_i$. In case $f_1 \neq f_2$, eJf_i being a simple right f_iRf_i-module, gives that σ_i is an isomorphism. Now D can be made into a right D_i-vector space, by defining $xa = x\sigma_i(a)$ for any $x \in D$ and $a \in D_i$. Then $eJf_i \cong D$ as (D, D_i)-bivector spaces, so $[D : \sigma_i(D_i)] = d(eJf_i)_{D_i}$. This gives parts (i) and (ii) of (VI). We shall prove (VI)(iii) within the proof of partial converse.

Let R be not local and let it satisfy the conditions (I) through (V) and parts (i) and (ii) of (VI). Condition (II) shows that $J^2 = 0$. Conditions (I) and (V) show that R is left serial. For any $e \in S$, set $\overline{eRe} = eRe/eJe$. Consider any $e \in S$ such that eR is not simple. There exist at most two members f, $g \in S$ such that $eJf \neq 0 \neq eJg$. Set $C = \sum_h hR + fJ + gJ$ where $h \in S\backslash\{e, f, g\}$. Consider the case when $e \neq f$ and $e \neq g$, then eRe is a division ring. For any $e' \in S\backslash\{e, f, g\}$, $eRe' = 0$, $eRfJ = eJfJ = 0$. This gives $C \subseteq r.ann(eR)$. Let $x = er_1 + fr_2 + gr_3 \in r.ann(eR)$. Then $ex = 0$ gives $er_1 = 0$, $x = fr_2 + gr_3$. Let $f \neq g$, then $eRfR \cap eRgR = 0$. For any $r \in R$, $erx = 0$ gives $eRfr_1 = 0$, $eRgr_2 = 0$, $fr_1 \in fJ$ and $gr_2 \in gJ$. In case $f = g$, $x = fr'$ and once again $fr' \in fJ$. Hence, in any case $C = r.ann(eR)$. Once again suppose that $f \neq g$. Then $R/C \cong \begin{bmatrix} eRe & eJf & eJg \\ 0 & \overline{fRf} & 0 \\ 0 & 0 & \overline{gRg} \end{bmatrix}$; for $D = eRe$, condition (V) and (VI)(ii) give that $R/C \cong T = \begin{bmatrix} D & D & D \\ 0 & D & 0 \\ 0 & 0 & D \end{bmatrix}$. By (3.6) $e_{11}T$ has only two homomorphic images that are uniserial but not simple, and they are injective. So $X = eR/eJf$ and $Y = eR/eJg$ are quasi-injective modules, indeed both are eR-injective. By (I), for any $h \in S$ different from e, $hJg = 0 = hJf$, so $\mathrm{Hom}_R(hJ, X) = 0 = \mathrm{Hom}_R(hJ, Y)$ and hence X, Y are hR-injective. Consequently X, Y are injective. In case $f = g$, $R/C \cong \begin{bmatrix} D & V \\ 0 & D' \end{bmatrix}$ where $V = eRf$ and $D' = fRf/fJf$. In case $dim\ V_{D'} = 2$, by using part (VI) of (3.4) we get eR/L is an injective R-module for every maximal submodule L of eJ if and only if $[D : \sigma(D)]_l = 2$. This gives (VI)(iii). In addition, let R also satisfy (VI)(iii) and that $dim\ V_{D'} \leqslant 2$. In case $dim\ V_{D'} = 1$, R/C is a generalized uniserial ring, and eR itself is uniserial and injective. We now consider the case when e equals one of f and g, say $e = f$, then $e \neq g$. Then $r.ann(eR) = C = \sum_h hR + gJ$, $h \in S\backslash\{e, g\}$. Then $R/C \cong eR \oplus (gR/gJ)$. As $eJg \neq 0$, $gJg = 0$ by (I), so $D' = gRg$ is a division ring. Consequently, $R/T \cong \begin{bmatrix} S & V \\ 0 & D' \end{bmatrix}$, where $S = eRe$ is a local, uniserial ring of composition length 2 by (IV), $V = eRg$ is a $(S/J(S), D')$-bivector space with dimension one on each side. By using (3.7), as before, we get that any uniserial homomorphic image of eR is either simple or injective. Any non-simple uniform R-module M contains a non-simple homomorphic image of some eR, $e \in S$, as the latter is injective and uniserial, we get that M itself is injective and uniserial. By (2.13) R satisfies $(*)$. □

We give an example of a ring R satisfying $(*)$, which is not right serial and in which $J^2 \neq 0$.

Example. Let D be any division ring, and let $R = \begin{bmatrix} D & D & D & D \\ 0 & D & D & 0 \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & D \end{bmatrix}$. Here $J^2 = e_{13}D$. That R/J^2 satisfies $(*)$ can be proved on lines similar to those in (3.6). Set $e_i = e_{ii}$. Now $e_1J = X \oplus Y$, with $X = e_{12}D + e_{13}D \cong e_2R$, $Y = e_{14}D \cong e_4R$, Any R-endomorphism of $e_{13}D$, X or Y is given by multiplication by an element of D, so it can be extended to an R-endomorphism of e_1R. This observation gives that $F = e_1R/X$, $G = e_1R/Y$ are quasi-injective and e_1R is e_2R-injective. Follow the arguments in (3.6) to show that F,G are indeed injective. These are the only non-simple uniserial homomorphic images of e_1R. We now apply (2.2) to prove that R satisfies $(*)$. Let A_R and B_R be two local modules, C a minimal submodule of A, and $\sigma : C \to B$ an embedding. The only minimal right ideals contained in e_1R are $e_{13}D$, Y and they are non-isomorphic; their R-endomorphisms being given by multiplication by elements of D, can be extended to R-endomorphisms of e_1R. Thus if $d(A) = d(B) = 4$, then σ extends to an R-homomorphism $\eta : A \to B$. If one of A, B has composition length 3, then that being isomorphic to G, is injective, so a desired extension of σ or σ^{-1} exists. Observe that any uniserial R-module of composition length 2 is either isomorphic to e_2R or to F. Suppose $d(A) = 4$, $d(B) = 2$. As $socle(F) \ncong socle(A)$, $B \cong e_2R$, so A is B-injective and we finish. If $AJ^2 = 0 = BJ^2$, then we finish by using the fact that R/J^2 satisfies $(*)$.

Remark. Consider R and S as in the above theorem. For any e, $f \in S$ define a directed edge $e \to f$ whenever $eJf \neq 0$. This gives the quiver [5, Chapter 8] of R with the following properties. For any $e \in S$ there do not exist more than two egdes with source e, and there does not exist more than one edge with same sink. Consider a finite partially ordered set X such that no element x of X has more than two covers and no element is a cover of more than one element [7, Definition 1.1.5]. For a division ring D consider the incidence algebra $T = I(X,D)$. Given $\alpha \leqslant \beta$ in X, set $e_{\alpha\beta} \in T$ such that $e_{\alpha\beta}(\gamma,\delta) = 0$ for any $(\gamma,\delta) \neq (\alpha,\beta)$ in $X \times X$ and $e_{\alpha\beta}(\alpha,\beta) = 1$. Consider the ideal A of T generated by all $e_{\alpha\beta}e_{\beta\gamma}$ with $\alpha < \beta < \gamma$. It follows from the above theorem that $R = T/A$ satisfies $(*)$.

Acknowledgement. The authors are grateful to the referee for his valuable suggestions.

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Received May 16, 2001