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Two-dimensional Real Division Algebras Revisited

Dedicated to Issai Kantor

Marion Hübner **Holger P. Petersson**

Fachbereich Wirtschaftswissenschaft, FernUniversität in Hagen
D-58084 Hagen
e-mail: marion.huebner@fernuni-hagen.de
e-mail: holger.petersson@fernuni-hagen.de

Abstract. A new classification of two-dimensional real division algebras is given. We also obtain a new classification of commutative real division algebras.

1. Introduction

A finite-dimensional real vector space V equipped with a bilinear product xy is said to be a *real division algebra* if there are no zero divisors: $xy = 0$ implies $x = 0$ or $y = 0$. By the celebrated Bott-Milnor-Kervaire Theorem [4], real division algebras exist only in dimensions $1, 2, 4, 8$. Standard examples are the reals $\mathbb{R}$, the complexes $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$, an excellent up-to-date reference to the latter being Baez [2]. Writing $\mathrm{Alg}(V)$ for the totality of bilinear products on V, which is a finite-dimensional real vector space in its own right, the division algebra structures on V form a subset of $\mathrm{Alg}(V)$ which, by a theorem of Kuzmin [8] (see also Petersson [9, **5.**]), is open in the natural topology. Hence, if n-dimensional real division algebras exist at all, they exist in abundance, and classifying them up to isomorphism becomes a nontrivial problem (unless $n = 1$). In the present paper, we take up this problem for $n = 2$, which was done before by Althoen and Kugler [1], Burdujan [3] and, more recently, by Gottschling [5]. Our reason for doing so again is that we adopt a completely different point of view. Rather than working with structure constants and multiplication tables as in [1], [3], [5], we prefer a more intrinsic approach that is based on the second author's general classification theory [10] for two-dimensional nonassociative

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algebras over arbitrary base fields. Indeed, the classification of two-dimensional real division algebras follows from this almost immediately and implies the classification of commutative real division algebras as an instant corollary. We also show how the original solution to the latter problem due to Kantor-Solodovnikov [7] fits canonically into this picture. The paper concludes with a few comments on the methodology of the Althoen-Kugler approach [1] to two-dimensional real division algebras. In particular, the key ingredient of this approach, a theorem of Segre [13] which says that the number of nonzero idempotents in a two-dimensional real division algebra is at least one and at most three, will be recast here in a purely algebraic setting.

2. The unital heart

We begin by recalling a few facts from [10]. Specializing $V = \mathbb{R}^2$ throughout, we continue to write $\mathrm{Alg}(V)$ for the set of nonassociative (possibly nonunital) algebra structures on V. The product of $x, y \in V$ relative to $A \in \mathrm{Alg}(V)$ will be denoted by $xAy = L_A(x)y = R_A(y)x$, where $L_A(x), R_A(y)$ stand for the left, right multiplication of x, y, respectively, in A. The group $G = \mathrm{GL}(V) \times \mathrm{GL}(V)$ acts on $\mathrm{Alg}(V)$ exponentially from the right according to the rule

$$xA^{(f,g)}y := f(x)Ag(y) \qquad (x, y \in V) \tag{1}$$

for $A \in \mathrm{Alg}(V), (f, g) \in G$. This action obviously preserves the property of being a division algebra and is compatible with passing to the opposite multiplication:

$$A^{(f,g)\mathrm{op}} = A^{\mathrm{op}(g,f)}. \tag{2}$$

Notice that (1), (2) make sense also when f, g are not invertible. $A \in \mathrm{Alg}(V)$ is said to be *regular* if $L_A(u)$ and $R_A(v)$ are invertible for some $u, v \in V$. In this case, the orbit of A under G contains a unital algebra which is unique up to isomorphism [10, 1.10]; we call it the *unital heart* of A. If A is a division algebra, its unital heart, being a *unital* real division algebra of dimension two, must be the complex numbers [4, Kap. 7, §3 **5.**], forcing $A \cong \mathbb{C}^{(f,g)}$ for some $(f, g) \in G$. Conversely, every algebra of this form is a division algebra.

3. Complex numbers

In dealing with complex numbers, we dispense ourselves from the previous notations to replace them by more conventional ones, writing $L(z), z' \mapsto zz'$, for the left multiplication by $z \in \mathbb{C}$ and $\tau, z \mapsto \overline{z}$, for complex conjugation. Specializing [10, 2.2] to $K = \mathbb{C}$, we obtain

Proposition 1. *Every $f \in \mathrm{End}_{\mathbb{R}}(V)$ can be written uniquely in the form*

$$f = L(z) + L(w)\tau \qquad (z, w \in \mathbb{C}).$$

Moreover,

$$\det f = |z|^2 - |w|^2. \tag{3}$$ □

Similarly, writing $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ for the unit circle in the complex plane, $\mathbf{1} = \mathrm{id}_V$ for the identity transformation on V, and putting $\mathbb{C}^\times = \mathbb{C} - \{0\}$, [10, 2.8] for $K = \mathbb{C},\ d = 1$ specializes to

Lemma 1. *For $v, v' \in \mathbb{C} - S^1,\ g, g' \in \mathrm{GL}(V)$, the following statements are equivalent.*
(i) $\mathbb{C}^{(\mathbf{1}+L(v)\tau, g)} \cong \mathbb{C}^{(\mathbf{1}+L(v')\tau, g')}$.
(ii) *There exist $u \in \mathbb{C}^\times,\ \sigma \in \{\mathbf{1}, \tau\}$ satisfying*

$$v' = \frac{\overline{u}^2}{|u|^2}\sigma(v),\ g' = \sigma g \sigma L(u).$$ □

In applications, we use Proposition 1 to decompose $g \in \mathrm{GL}(V)$ as

$$g = L(z) + L(w)\tau \qquad (z, w \in \mathbb{C}, |z| \neq |w|) \tag{4}$$

and observe

$$gL(u) = L(zu) + L(w\overline{u})\tau \qquad (u \in \mathbb{C}), \tag{5}$$

$$\tau g \tau = L(\overline{z}) + L(\overline{w})\tau. \tag{6}$$

4. Main results

We write $H = \{z \in \mathbb{C} \mid \mathrm{Im}(z) \geq 0\}$ for the closed upper half-plane.

Theorem 1. (Classification of two-dimensional real division algebras) *The two-dimensional real division algebras are isomorphic to precisely one of the following.*
a) $\mathbb{C}^{(\tau,\tau)}$,
b) $\mathbb{C}^{(\mathbf{1}+L(v)\tau,\tau)}, v \in (\mathbb{C} - S^1) \cap H$,
c) $\mathbb{C}^{(\tau,\mathbf{1}+L(w)\tau)}, w \in (\mathbb{C} - S^1) \cap H$,
d) $\mathbb{C}^{(\mathbf{1}+L(v)\tau,\mathbf{1}+L(w)\tau)}, v \in (\mathbb{C} - S^1) \cap H, w \in \mathbb{C} - S^1$, *and $v \in \mathbb{R}$ implies $w \in H$.*
Conversely, all algebras listed in a) – d) *are two-dimensional real division algebras.*

Proof. The final statement follows from (3). Now suppose D is a two-dimensional real division algebra. Specializing [10, 2.3 and 2.12] to $K = \mathbb{C},\ M = \mathbb{R}_+^\times = \{r \in \mathbb{R} \mid r > 0\}$, D is isomorphic to precisely one of the following types of algebras.
α) $\mathbb{C}^{(\mathbf{1}+r\tau,g)},\ r \in \mathbb{R}_+^\times - \{1\},\ g \in \mathrm{GL}(V)$,
β) $\mathbb{C}^{(\mathbf{1},\tau)}$,
γ) $\mathbb{C}^{(\tau,L(w)\tau)},\ w \in S^1$,

δ) $\mathbb{C}^{(\mathbf{1},\mathbf{1}+L(w)\tau)}$, $w \in \mathbb{C} - S^1$,

ε) $\mathbb{C}^{(\tau,\mathbf{1}+L(w)\tau)}$, $w \in \mathbb{C} - S^1$.

These five types will now be discussed separately.

α) Decomposing g as in (4), we consider the following cases.

Case 1. $z \neq 0$.

By Lemma 1 (for $u = 1$, $\sigma = \tau$) and (6) we may assume $\overline{z}^{-2} \in H$. Applying Lemma 1 again (for $u = z^{-1}$, $\sigma = \mathbf{1}$), (5) yields $D \cong \mathbb{C}^{(\mathbf{1}+L(v')\tau,g')}$, where

$$v' = r|z|^2\overline{z}^{-2} \in H,\ g' = \mathbf{1} + L(w')\tau,\ w' = w\overline{z}^{-1} \in \mathbb{C} - S^1.$$

If $v' \in \mathbb{R}$, we may invoke (6) once more to assume $w' \in H$. Thus D is of type d) with $v \neq 0$. Conversely, reading this argument backwards, every algebra of type d) with $v \neq 0$ is of type α).

Case 2. $z = 0$.

First Lemma 1 (for $u = 1$, $\sigma = \tau$) and (6) allow us to assume $w^{-2} \in H$, then Lemma 1 (for $u = \overline{w}^{\,-1}$, $\sigma = \mathbf{1}$) and (5) imply $D \cong \mathbb{C}^{(\mathbf{1}+L(v')\tau,\tau)}$, where $v' = r|w|^2w^{-2} \in (\mathbb{C} - S^1) \cap H$. Thus D is type b) with $v \neq 0$. Again we have the converse, so every algebra of type b) with $v \neq 0$ is of type α).

β) D is of type b) with $v = 0$.

γ) Since [10, 2.12 c)] allows us to multiply w by a third power in S^1 without changing the isomorphism class of D , we may assume $w = 1$, forcing D to be of type a).

δ), ε) We have $D \cong \mathbb{C}^{(\sigma,\mathbf{1}+L(w)\tau)}$ for some $\sigma \in \{\mathbf{1}, \tau\}$. Since [10, 2.12 d)] allows us to replace w by $\overline{w}$ if necessary, we may assume $w \in H$. Thus D is of type d) with $v = 0$ for $\sigma = \mathbf{1}$ and of Type c) for $\sigma = \tau$.

It remains to prove uniqueness. Lemma 1 combined with the preceding discussion shows that, since types α) – ε) are disjoint, so are types a) – d). Uniqueness of parameters for each type except c) again follows from Lemma 1. But since passing to the opposite algebra interchanges types b) and c) by (2), it follows for type c) as well. □

Theorem 2. (Classification of commutative real division algebras) *The commutative real division algebras are isomorphic to precisely one of the following.*

a) $\mathbb{R}$.

b) $\mathbb{C}^{(\tau,\tau)}$.

c) $\mathbb{C}^{(\mathbf{1}+L(w)\tau,\mathbf{1}+L(w)\tau)}$, $w \in (\mathbb{C} - S^1) \cap H$.

Conversely, all algebras listed in a), b), c) *are commutative real division algebras.*

Proof. By a theorem of Hopf [4, Kap. 7, §3 **3.**], a commutative real division algebra has dimension at most two. Hence Theorem 2 follows from Theorem 1 and the following lemma.

Lemma 2. *Given* $f = L(u) + L(v)\tau$, $g = L(z) + L(w)\tau \in \mathrm{End}_{\mathbb{R}}(V)$ $(u, v, z, w \in \mathbb{C})$, *the algebra* $\mathbb{C}^{(f,g)}$ *is commutative if and only if* $uw = vz$.

Proof. A straightforward computation gives

$$x\mathbb{C}^{(f,g)}y - y\mathbb{C}^{(f,g)}x = \Big(L(vz - uw) + L(uw - vz)\tau\Big)(\overline{x}y)$$

for all $x, y \in V$. Applying Proposition 1, the assertion follows. $\square$

5. The Kantor-Solodovnikov classification

In order to match the Kantor-Solodovnikov classification of commutative real division algebras [7, Theorem 20.1] with our own, we first note that the group $\mathrm{GL}(V)$ acts on $\mathrm{Alg}(V)$ exponentially from the left according to the rule $x({}^fA)y = f(xAy)$ for $x, y \in V,\ f \in \mathrm{GL}(V),\ A \in \mathrm{Alg}(V)$. This action obviously preserves the property of being a commutative algebra and is compatible with the right action of G on $\mathrm{Alg}(V)$ as defined in (1). Secondly, we canonically identify $\mathrm{End}_{\mathbb{R}}(V)$ with the algebra of 2-by-2 real matrices through the basis of unit vectors and write

$$\vartheta : V \times V \longrightarrow \mathbb{R},\ \ (x, y) \longmapsto \vartheta(x, y) := \mathrm{Re}(x\overline{y}),$$

for the canonical scalar product on V. Then $\tau = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and the transpose (i.e., the adjoint relative to ϑ) of $f = L(z) + L(w)\tau \in \mathrm{End}_{\mathbb{R}}(V)$ $(z, w \in \mathbb{C})$ is given by

$$ {}^tf = L(\overline{z}) + L(w)\tau. \tag{7}$$

Linearizing (3) at $\mathbf{1}$ in the direction f, we also get

$$\mathrm{trace}(f) = 2\ \mathrm{Re}(z). \tag{8}$$

Lemma 3. *Let $A \in \mathrm{Alg}(V)$ and $f \in \mathrm{GL}(V)$. Then*

a) *$f : A^{(f,f)} \xrightarrow{\sim} {}^fA$ is an isomorphism.*

b) *${}^fA \cong {}^{\alpha f}A$ for all $\alpha \in \mathbb{R},\ \alpha \neq 0$.*

Proof. a) is immediate, and b) follows from a) combined with [10, 1.14]. $\square$

Proposition 2. (cf. Kantor-Solodovnikov [7, Theorem 20.1]) *For a real algebra D to be a commutative division algebra it is necessary and sufficient that $D \cong \mathbb{R}$ or there exist*

$$f = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \in \mathrm{End}_{\mathbb{R}}(V) \qquad\qquad (\alpha, \beta, \gamma \in \mathbb{R})$$

satisfying

a) $D \cong {}^f\mathbb{C}$,

b) $\det f \in \{1, -1\}$,

c) $\beta \geq 0$,

d) $\alpha \geq 0$, *and* $\alpha = 0$ *implies* $\gamma \geq 0$,

e) $\alpha + \gamma = 0$ *implies* $\alpha = -\gamma = 1,\ \beta = 0$.

In this case, f is unique.

Proof. Sufficiency being obvious, it is enough to prove necessity and uniqueness. So let D be a commutative real division algebra, without loss of dimension > 1. We first observe that Lemma 1 (for $u = 1,\ \sigma = \tau$), (6) and Lemma 3 a) yield

$$^{(\mathbf{1}+L(w)\tau)}\mathbb{C} \cong {}^{(\mathbf{1}+L(\overline{w})\tau)}\mathbb{C} \qquad (w \in \mathbb{C} - S^1). \tag{9}$$

Hence, by Theorem 2, D is isomorphic to $^f\mathbb{C}$ where either $f = \tau$ or $f = \mathbf{1} + L(w')\tau$ for some $w' = \alpha' + \beta' i$ ($\alpha', \beta' \in \mathbb{R}$) such that β' is nonnegative for $\alpha' = -1$ and has the same sign as $1 + \alpha'$ otherwise. Moreover, $f = \tau$ iff $\mathrm{trace}(f) = 0$ by (8), and an easy computation gives

$$\mathbf{1} + L(w')\tau = \begin{pmatrix} 1+\alpha' & \beta' \\ \beta' & 1-\alpha' \end{pmatrix}. \tag{10}$$

Thus a), e) hold, and scaling, which is justified by Lemma 3 b), allows us to assume b), c), d) as well. Finally, to establish uniqueness, suppose $f, g \in \mathrm{End}_{\mathbb{R}}(V)$ are symmetric and satisfy a) – e). By symmetry and e) we may assume that the trace of f is nonzero. Thus $f = r\mathbf{1} + L(w)\tau$ ($r \in \mathbb{R},\ r \neq 0,\ w \in \mathbb{C}$) by (7), (8), and a), Lemma 3 b) yield $D \cong {}^{f'}\mathbb{C}$ where $f' = \frac{1}{r} f = \mathbf{1} + L(w')\tau,\ w' = \frac{1}{r} w$. Hence the trace of g is nonzero as well, by Theorem 2, (9) and e), so the preceding argument also gives $g = s\mathbf{1} + L(z)\tau$ ($s \in \mathbb{R},\ s \neq 0,\ z \in \mathbb{C}$), $D \cong {}^{g'}\mathbb{C},\ g' = \frac{1}{s} g = \mathbf{1} + L(z')\tau,\ z' = \frac{1}{s} z$. Combining Theorem 2 with (9) again, we conclude $w' = z'$ or $w' = \overline{z'}$. This not only implies $\det f' = \det g'$ by (3), hence $\det f = (\frac{r}{s})^2 \det g$ and then $(\frac{r}{s})^2 = 1$ by b), but also, thanks to the first part of d) and (10), that $r(1 + \alpha')$ and $s(1 + \alpha')$, where $\alpha' = \mathrm{Re}(w') = \mathrm{Re}(z')$, are nonnegative. Therefore r and s have the same sign for $\alpha' \neq -1$, while $\alpha' = -1$ implies $r > 0,\ s > 0$ by the second part of d) and (8), (10). Summing up we obtain $r = s$, whence $\mathrm{Im}(w'), \mathrm{Im}(z')$ by c) and (10) have the same sign. Since w' differs from z' at most by conjugation, this yields $w' = z'$, hence $f = g$. ☐

Remark. Condition e) is missing in [7, Theorem 20.1], which therefore lists the algebras $^{L(w)\tau}\mathbb{C},\ w \in S^1 \cap H$, as being mutually nonisomorphic while in fact they are all isomorphic to $^\tau\mathbb{C}$. Explicitly, choosing any cube root v of $\overline{w},\ L(v) : {}^{L(w)\tau}\mathbb{C} \xrightarrow{\sim} {}^\tau\mathbb{C}$ is an isomorphism. ☐

6. The Althoen-Kugler Classification

We won't even try to match the Althoen-Kugler classification [1] (nor Burdujan's [3] or Gottschling's [5]) of two-dimensional real division algebras with our own since this would be a daunting and not particularly rewarding task. Instead, we will focus on two points of a more methodological nature. We begin by recasting Segre's Theorem [13] in a purely algebraic setting. To do so, we pick up ideas going back to Walcher [14], [15] and Röhrl-Walcher [12], who work with commutative algebras of characteristic not 2 but often allow m-ary (rather than just binary) ones and get more detailed information over the reals and complexes, cf. [14, 3.3].

Proposition 3. *Let A be a two-dimensional nonassociative algebra over an arbitrary base field k. Then one of the following holds.*

a) *A has rank 2, i.e., there exists a linear form λ on A such that $x^2 = \lambda(x)x$ for all $x \in A$.*

b) *The number of one-dimensional subalgebras of A is at most 3. Moreover, it is at least 1 if there are no cubic field extensions of k.*

Proof. For completeness, we give the proof in full. Referring to Roby [11] for polynomial maps over commutative rings (including, e.g., finite fields), we follow Walcher (loc. cit.) to consider the cubic form $N : A \to \bigwedge^2 A = k$ given by $N(x) = x \wedge x^2$ for all $x \in A \otimes R$ and all commutative associative k-algebras R. If N is zero, x, x^2 are linearly dependent for all x in any base field extension of A. Hence there is a rational function ρ on A, homogeneous of degree 1, such that every base change of A satisfies the relation $x^2 = \rho(x)x$ whenever it makes sense. For $\rho = 0$ we are done. Otherwise, clearing denominators, we find an integer $m \geq 0$ and relatively prime homogeneous polynomial functions f, g of degree $m, m+1$, respectively, on A such that $f(x)x^2 = g(x)x$ for all x in any base field extension of A. Thus $f(x) = 0$ implies $g(x) = 0$ provided x is not zero, and the homogeneous form of Hilbert's Nullstellensatz (Hartshorne [6, I Ex. 2.1]) shows that f divides some power of g. This forces $m = 0$, and we obtain a). If N is not zero, the equation $N = 0$ defines a closed subscheme of $\mathbb{P}^1_k$ (the projective line over k) whose irreducible components correspond to the irreducible factors of N and have all codimension 1, i.e., consist of single points. Furthermore, the k-rational points of this subscheme are precisely the one-dimensional subalgebras of A. Hence b) holds. □

Corollary. (Segre's Theorem [13]) *The number of nonzero idempotents in a two-dimensional real division algebra is at least* 1 *and at most* 3.

Proof. Since nonzero elements that square to zero do not exist, Proposition 3 a) does not hold. Hence b) does, and the one-dimensional subalgebras correspond exactly to the non-zero idempotents. □

The second point we wish to make concerns the Althoen-Kugler classification itself. Classifying algebraic objects up to isomorphism in the naive sense of the word amounts to writing down a list of examples that represents each isomorphism class exactly once. While Theorem 1 above produces such a list for two-dimensional real division algebras, the Althoen-Kugler classification does not (but Burdujan [3] and Gottschling [5] do, though [3] contains no proof). More specifically, it is the two-dimensional real division algebras containing three nonzero idempotents (one of the two "generic" cases, the other one being comprised by those algebras that contain a single nonzero idempotent) where a classification list fails to materialize. Instead, the authors merely construct classifying invariants by attaching three multiplication tables $T_i(D)$ $(i = 0, 1, 2)$ to any two-dimensional real division algebra D containing three nonzero idempotents in such a way that, given another algebra D' of this kind, D and D' are isomorphic if and only if $T_i(D) = T_j(D')$ for some $i, j = 0, 1, 2$ [1, Theorem 6]. Therefore the approach to two-dimensional real division algebras by means of idempotents seems to be less natural than the one adopted here.

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