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# Rokhlin's Formula for Dividing T-Curves

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**Abstract.** A nonsingular real algebraic plane projective curve is called a dividing curve or a curve of type I if its real point set divides its complex point set. Rokhlin's formula, which holds for such curves, is an important step in order to classify nonsingular real algebraic plane projective curves. It gives prohibitions on the complex orientations of a curve of type I and also on its real scheme. The concept of type has been defined also for T-curves which are PL-curves constructed using a combinatorial method called T-construction. From the point of view of real algebraic geometry, this construction is very interesting because, under a condition of "convexity" of the triangulation used in the T-construction, the resulting T-curve has the isotopy type of a nonsingular real algebraic plane projective curve. In this work we prove that Rokhlin's formula holds for dividing primitive T-curves constructed with arbitrary (not necessary convex) triangulations.

## Introduction

A real algebraic plane projective curve of degree $m$ is a real homogeneous polynomial in three variables $C(x_0, x_1, x_2)$ of degree $m$ considered up to multiplication by a non zero real number. The equation $C(x_0, x_1, x_2) = 0$ defines a subset of $\mathbb{R}P^2$ (resp. of $\mathbb{C}P^2$) which is called the real (resp. complex) point set of the curve and is denoted by $\mathbb{R}C$ (resp. $\mathbb{C}C$). We suppose the curve to be nonsingular, then $\mathbb{R}C$ is a disjoint union of circles embedded in $\mathbb{R}P^2$. The topological type of the pair $(\mathbb{R}P^2, \mathbb{R}C)$ is given by the description of the mutual disposition of the connected components of $\mathbb{R}C$ and is called the real scheme of the curve. In the following the term "curve" will denote a nonsingular real algebraic plane projective curve. To study classification problems the work proceeds in two directions. The first point

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is to find topological prohibitions on curves due to their algebraic nature, the second is to prove if any scheme which satisfies the prohibitions is realizable by a curve of given degree, i.e. to construct a curve with given scheme and degree.

The classical methods of construction of curves are based on the perturbation of singular curves having "simple" singularities (see for example [12], [13] and [9]). At the beginning of 1980's Viro studied the perturbation of more complicated singularities and this has been the starting point of a new method of construction introduced by Viro himself ([26], [27] and [29]). This work deals with a particular case of Viro's method, called T-construction, which acts as a link between real algebraic geometry and combinatorial geometry. This construction works in any dimension and in any degree. In dimension two, starting from the triangle $T$ in $\mathbb{R}^2$ of vertices $(0,0)$, $(0,m)$, $(m,0)$ equipped with a triangulation and a sign $(+,-)$ at each vertex of the triangulation, we construct a T-curve, i.e. a PL-curve which has, under particular conditions on the triangulation, the isotopy type of a curve of degree $m$ in $\mathbb{R}\mathrm{P}^2$.

Many important prohibitions for topology of algebraic curves are known (see for example [9], [31] and [27]). One of the most classical prohibitions is, for example, Bézout Theorem [30]. From this theorem it follows, for instance, the possibility to relate the existence, in the real point set of an algebraic curve, of a "one-sided" connected component with the degree of the curve. Another important prohibition is Harnack Theorem [12] which gives the sharp upper bound for the number of connected components of a curve. A powerful result is also Rokhlin's formula which is the object of this work. The formula holds for curves whose real point set divides the complex point set.

Working in real algebraic geometry, we usually try to extend properties and prohibitions known for varieties of a certain dimension to higher dimensions. For example Harnack Theorem has been generalized (see for example [9], [31] and [27]): if $\mathbb{R}A$ is a nonsingular real algebraic projective variety and $\mathbb{C}A$ is its complexification, then $dim\, H_*(\mathbb{R}A, \mathbb{Z}_2) \leq dim\ H_*(\mathbb{C}A, \mathbb{Z}_2)$. It is from this point of view that this work can be seen. In fact here we prove, in a combinatorial way, that Rokhlin's formula holds for particular T-curves called "primitive dividing T-curves" or "primitive T-curves of type I". Till now there is no conjecture for a generalization of Rokhlin's formula in higher dimensions. On the other hand, one can expect that the combinatorial proof given here would suggest possible formulation and proof of Rokhlin's formula in higher dimensions at least for T-objects, i.e. for hypersurfaces constructed with T-construction in higher dimensions.

In the last years many prohibitions for real algebraic curves have been proved also for T-curves ([15], [5]). Recently Itenberg and Shustin [17] described a complexification of Viro's construction. The most interesting part of their work is in dimensions higher than two, but they prove, in particular, that all the topological results which are true for algebraic curves are also true for arbitrary T-curves. Then they give a new proof for many prohibitions which were extended to T-curves such as Harnack theorem, Rokhlin's formula and others.

The paper is organized as follows: Section 1 is devoted to introduce Rokhlin's formula for nonsingular real algebraic plane projective curves. In particular we recall the formulation of Rokhlin's formula introduced by Viro [28] in terms of the Euler characteristic of the connected components of the complement of the curve and of the index, with respect to the curve, of the points belonging to these connected components. In Section 2 we recall the fundamental results about T-curves, their type (introduced by Haas in [10]) and we describe briefly necessary and sufficient conditions for a T-curve to be a dividing T-curve [21]. In analogy with the algebraic case, the type of a T-curve is related to the existence of two orientations which are opposite each other and are called symmetric orientations. The characterization of the type of T-curves is related with particular decompositions, called fragmentations, of the triangle $T$ equipped with special distribution of signs. These fragmentations are similar to the decompositions in zones introduced by Haas in [10] in the context of maximal T-curves that is T-curves having maximal number of connected components. In Section 3 we explain how to calculate the index of a point with respect to a T-curve. In Section 4 we introduce new operations, called "modifications", which allow us to pass from a T-curve of type I to others T-curves of type I with controlled topology. Finally in Section 5 we give a proof of Rokhlin's formula for primitive dividing T-curves.

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## 1. Rokhlin's formula

An important step in order to classify nonsingular real algebraic plane projective curves is to study how the real point set $\mathbb{R}C$ is situated in the complex point set $\mathbb{C}C$.

An algebraic curve $C$ is a *dividing* curve or a curve of *type* I if $\mathbb{R}C$ divides $\mathbb{C}C$, otherwise it is of *type* II.

In the case of a dividing curve $C$, the real point set divides the complex point set in two halves each of them inducing an orientation on the real curve $\mathbb{R}C$. These two orientations are opposite to each other and are called *complex orientations* of the real curve. The complex orientations have been introduced in the study of the topology of real algebraic curves by Rokhlin in 1974 [23].

Rokhlin's formula gives prohibitions on complex orientations of a dividing curve and also on its real scheme. Its classical formulation, which is given in two different ways for odd and even degrees, can be expressed in terms of the degree, the mutual position of the connected components and their orientations ([23], [19], [24]).

In 1988 Viro [28] proposed a new formulation of Rokhlin's formula which summarizes in a single expression the two classical formulations:

**Theorem 1.1.** (Viro formulation of Rokhlin's Formula) *For any nonsingular real algebraic plane projective curve $C$ of type* I *and degree $m$, one has*

$$\sum_{F\in(\mathbb{R}\mathrm{P}^2\setminus\mathbb{R}\mathrm{C})} ind^2_{\mathbb{R}C}(x_F)\ \chi(F)=\frac{m^2}{4},$$

*where $x_F$ is a point of the connected component $F$ of $\mathbb{R}\mathrm{P}^2\setminus\mathbb{R}\mathrm{C}$, $ind_{\mathbb{R}C}(x_F)$ is the index of $x_F$ with respect to $\mathbb{R}C$ equipped with a complex orientation and $\chi(F)$ is the Euler characteristic of $F$.*

The sum in the left hand side can be regarded as a sort of unusual integral with respect to the Euler characteristic. In fact, even if the Euler characteristic is not a measure, it can be considered as a finitely-additive measure and for this type of functions it is possible to develop an integration theory. As $ind^2_{\mathbb{R}C}(x)$ is a linear combination of characteristic functions of subset of $\mathbb{R}\mathrm{P}^2 \setminus \mathbb{R}\mathrm{C}$ (for a method to calculate the index see Section 3), we can set:

$$\sum_{F\in(\mathbb{R}\mathrm{P}^2\setminus\mathbb{R}\mathrm{C})} ind^2_{\mathbb{R}C}(x_F)\ \chi(F) = \int_{\mathbb{R}\mathrm{P}^2\setminus\mathbb{R}C} ind^2_{\mathbb{R}C}(x)\ d\chi(x)$$

## 2. T-curves

### 2.1. Construction

We use the following notions. An *integer segment* is a segment of $\mathbb{R}^2$ containing only two points with integer coordinates: its endpoints. An *integer polygon* is a closed subset of $\mathbb{R}^2$ homeomorphic to a disc and bounded by a closed path of integer segments. A *boundary-segment* of an integer polygon $P$ is an integer segment of the boundary of $P$, and a *boundary-point* of $P$ is an endpoint of a boundary-segment of $P$.

Take a convex integer polygon $P$ whose vertices have non-negative coordinates. Consider a triangulation $\Gamma$ of $P$ having vertices with integer coordinates and a distribution of signs $\varepsilon$ at the vertices of $\Gamma$, i.e. choose a sign at each vertex of the triangulation.

Construct the symmetric copies $\sigma_x(P)$, $\sigma_y(P)$ and $\sigma_{xy}(P)$ where $\sigma_x$, $\sigma_y$ and $\sigma_{xy}$ are reflections with respect to the $x$-axes, $y$-axes and the origin and denote by $P^{\#}$ the union of the symmetric copies of $P$.

By symmetry we extend the triangulation $\Gamma$ of $P$ to a triangulation of $P^{\#}$. We extend also the distribution of signs on $P$ to a distribution $\bar{\varepsilon}$ on $P^{\#}$ by the following rule: let $(i,j)$ be a vertex of $\Gamma$ having sign $\varepsilon_{i,j}$, then the vertex $((-1)^a i, (-1)^b j)$ where $a$ and $b$ are integers has sign $\varepsilon_{i,j}(-1)^{ai+bj}$. A simplex of a triangulation equipped with a distribution of signs is called *empty* if its vertices have same sign, *non-empty* otherwise. For any non-empty triangle of the triangulation of $P^{\#}$, join the middle points of its two non-empty edges with a segment. Let K be the union of these segments in $P^{\#}$, then the curve K is called the *PL-curve* associated to the triple $(P,\Gamma,\varepsilon)$.

We glue the disjoint union of the four copies of $P$ by their boundary: we identify each point $(x,y)$ on an edge $l$ of $P^{\#}$ with its symmetric copy $\sigma(x,y) = ((-1)^{\alpha_1}(x), (-1)^{\alpha_2}(y))$ where $(\alpha_1,\alpha_2)$ is any vector with integer relatively prime coordinates and orthogonal to $l$. We denote the resulting space by $\hat{P}$. It is well known that we can associate to the polygon $P$ a complex toric surface $X(P)$ (see, for example [1], [2], [4], [6], [7], [8], [20] for a definition and for the principal properties of a toric variety) and that the real part $X_{\mathbb{R}}(P)$ of $X(P)$ is homeomorphic to $\hat{P}$ . Let $A$ be the image of the PL-curve K in $\hat{P}$. The curve A is a closed PL-submanifold called the *T-curve* associated to the triple $(P,\Gamma,\varepsilon)$.

Consider the following additional condition on the triangulation $\Gamma$:

**Definition 2.1.** *A triangulation $\Gamma$ of $P$ is convex if there exists a convex piecewise-linear function $\nu : T \to \mathbb{R}$ such that $\nu$ is linear on each triangle of $\Gamma$, but $\nu$ is not linear on the union of any two triangles of $\Gamma$.*

Let us recall that non convex triangulations exist [3].

The theory developed by Viro (for more details see for example [25], [26], [27], [29], [18], [22], [7]) assures that:

**Theorem 2.2.** (O. Ya. Viro) *Under the assumption of convexity of the triangulation $\Gamma$ of $P$, there exists a nonsingular real algebraic curve $C$ in $X(P)$ with Newton polygon $P$, and a homeomorphism $X_{\mathbb{R}}(P) \longrightarrow \hat{P}$ mapping the real point set of $C$ onto $A$.*

The pair $(\hat{P}, A)$ is called a *chart* of the algebraic curve $C$ and the T-curve $A$ is called an *algebraic T-curve.*

If the polygon $P$ is the triangle $T$ in $\mathbb{R}^2$ of vertices $(0,0)$, $(0,m)$, $(m,0)$, T-construction allows us to construct a PL-curve K in $T^{\#}$ and a T-curve $A$ in $\hat{T}$. The space $\hat{T}$ is homeomorphic to $\mathbb{RP}^2$ and in this case K (resp. $A$) is simply called the PL-curve (resp. the T-curve) associated to the pair $(\Gamma, \varepsilon)$. We say that $A$ is a T-curve of degree $m$ as it is constructed starting from the integer $m$. Viro's theorem assures that, under the assumption of convexity of the triangulation $\Gamma$ of $T$, there exists a nonsingular real algebraic plane projective curve $C$ of degree $m$ and a homeomorphism $\mathbb{RP}^2 \longrightarrow \hat{\mathrm{T}}$ mapping $\mathbb{R}C$ onto $A$.

A triangulation is called *primitive* if it has, as vertices, all the integer points of $T$. In the following, if not otherwise specified, we consider only primitive triangulations. In general given a polygon $P$ equipped with a triangulation $\Gamma$ and a distribution of signs, we call an integer point $v$ of $\hat{P}$ *isolated* if each edge of the extended triangulation containing $v$ is nonempty.

The *parity* of an integer point $(i,j) \in \mathbb{Z}^2$ is the pair $([i]_2, [j]_2) \in (\mathbb{Z}_2)^2$ where $[i]_2$ (resp. $[j]_2$) is the reduction, modulo 2, of the integer $i$ (resp. the integer $j$).

We denote by $\delta_i$ with $i = 1,2,3,4$ the four different parities of vertices; an integer segment connecting two vertices of parities $\delta_i$ and $\delta_j$ is called of *type* $\delta_{i,j}$.

Let us consider a boundary-edge $l$ of type $\delta_{i,j}$ of an integer polygon $P$; let $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$ be the endpoints of $l$, then any point $(x,y)$ of $l$ is identified in $\hat{P}$ with its symmetric copy $\left((-1)^{\beta_1+\beta_2}(x), (-1)^{\alpha_1+\alpha_2}(y)\right)$, this is why we will denote by $\sigma_{i,j}$, with $i,j \in \{1,2,3,4\}$, the symmetry $\left((-1)^{\beta_1+\beta_2}, (-1)^{\alpha_1+\alpha_2}\right)$ of $\mathbb{R}^2$ where $(\alpha_1, \beta_1)$ (resp. $(\alpha_2, \beta_2)$) is an integer point of $\mathbb{Z}^2$ having parity $\delta_i$ (resp. $\delta_j$).

In 1993 Itenberg [14] introduced special distributions of signs in T-construction.

**Definition 2.3.** *The distributions $H^{\mu}_{\delta_i}$ with $i = 1,2,3,4$ and $\mu = \pm$ defined as follows:*

$$\begin{cases} H^{\mu}_{\delta_i}(a,b) = \mu & \forall\ (a,b) \in (P \cap \mathbb{Z}^2) \text{ having parity } \delta_i \\ H^{\mu}_{\delta_i}(a,b) = -\mu & \forall\ (a,b) \in (P \cap \mathbb{Z}^2) \text{ having parity } \delta_s \text{ with } s \neq i \end{cases}$$

*are called Harnack distributions.*

From the definition it follows that two Harnack distributions which coincide on three pairwise different parities of vertices are equal.

Harnack distributions are very special. In fact, for instance, if a convex integer polygon $P$ is equipped with a triangulation and a Harnack distribution, then the isotopy type of the T-curve obtained in $\hat{P}$ by Viro method is independent of the choice of the triangulation and of the choice of the Harnack distribution (see for example [16], [10], [11]). We recall here only a description of the PL-curve associated to a triple $(P, \Gamma, H^{\mu}_{\delta_i})$.

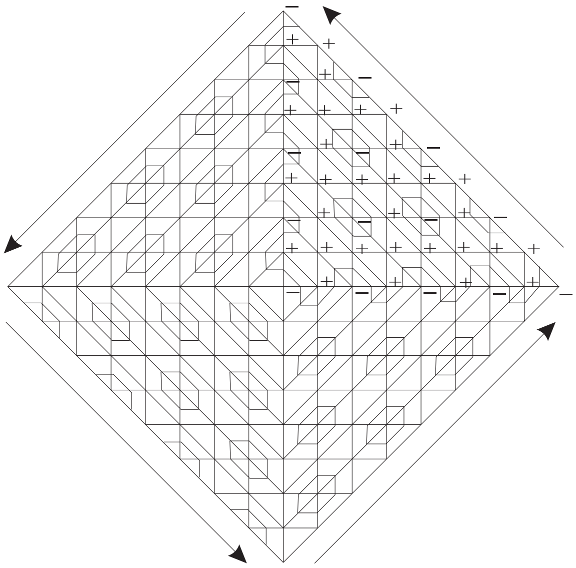

Figure 1. A chart of a T-curve of degree 8 constructed starting from the triangle $T$ equipped with a primitive triangulation and the distribution $H^{-}_{([0],[0])}$.

**Proposition 2.4.** *Let $P$ be an integer polygon and $n_s$, $p_s$ (for $s = 1, 2, 3, 4$) be the number of integer points of $P$ having parity $\delta_s$ and belonging respectively to the interior part of $P$ and to the boundary of $P$. Let* K *be the PL-curve associated to the triple $(P, \Gamma, H^{\mu}_{\delta_i})$ where $H^{\mu}_{\delta_i}$ is a Harnack distribution and $\Gamma$ is a primitive triangulation. Then the extended distribution of signs on the symmetric copy $\sigma_{i,s}(P)$ of $P$ is the Harnack distribution $H^{\mu'}_{\delta_s}$ for an appropriate $\mu' \in \{+, -\}$ and $K \cap \sigma_{i,s}(P)$ can be described as follows:*

- *There exist $n_s$ circles (called ovals) such that each of them splits $\sigma_{i,s}(P)$ in two connected components. One of these two components contains only one integer point and this point is of parity $\delta_s$.*
- *There exist $p_s$ arcs such that each of them splits $\sigma_{i,s}(P)$ in two connected components. One of these two components contains only one integer point of $P$ and this point is a vertex of $P$ with parity $\delta_s$.*

**Example 2.5.** Figure 1 represents a T-curve of degree 8 constructed starting from the triangle $T$ equipped with a primitive triangulation and the distribution $H^{-}_{([0],[0])}$.

### 2.2. The type of a T-curve

In this section we briefly recall the principal concepts related to the type of a T-curve referring to [21] for more details and proofs.

Take a T-curve $A$ associated to a pair $(\Gamma, \varepsilon)$ and let $\tau$ be a triangle of $\Gamma$. The T-curve $A_\tau$ obtained applying T-construction to the triangle $\tau$ is a circle and then it admits only two orientations, opposite each other, each of them inducing, in a natural way, an orientation on the PL-curve $K_\tau$ associated to $\tau$. Given an orientation of $A_\tau$, for each $\tau \in \Gamma$, it is not true in general that these orientations can be glued together obtaining an orientation of the T-curve $A$, i.e. it is not true in general that the induced orientations on $K_\tau$, for each $\tau \in \Gamma$, give an orientation of $A$. An orientation of a T-curve $A$ associated to a pair $(\Gamma, \varepsilon)$ is called *symmetric* if it is obtained as the gluing of an orientation of each of the T-curves in $\hat{\tau}$, for any $\tau \in \Gamma$.

It is easy to verify that if a T-curve $A$ associated to a pair $(\Gamma, \varepsilon)$ admits a symmetric orientation, then $A$ admits exactly two symmetric orientations and one is opposite to the other one.

**Definition 2.6.** *A T-curve is a dividing T-curve or a T-curve of type* I *if it admits a symmetric orientation, it is of type* II *otherwise.*

In [21] it is proved that if an algebraic curve is associated to a T-curve by Viro's method, then the type of the T-curve coincides with the type of the algebraic curve and its symmetric orientations are complex orientations as defined by Rokhlin.

The type of a T-curve can be expressed in terms of the triangulation and the distribution of signs. Two different triangles of $\Gamma$ with a common edge have *same orientation* if they induce opposite orientations on their common edge, they have *opposite orientations* otherwise.

**Definition 2.7.** *A symmetric orientation for the pair $(\Gamma, \varepsilon)$ is a collection of pairs $(\tau, \theta)$, where $\tau \in \Gamma$ and $\theta$ is an orientation of $\tau$, satisfying the following condition: each pair of triangles $\tau$, $\tau'$ with a common edge, have same orientation if and only if the distribution $\varepsilon|_{(\tau \cup \tau')}$ is a Harnack distribution.*

It is easy to show [21] that a T-curve $A$ associated to a pair $(\Gamma, \varepsilon)$ admits a symmetric orientation if and only if the pair $(\Gamma, \varepsilon)$ admits a symmetric orientation.

### 2.3. Cycles, rays and fragmentation of $T$

The classification theorem of dividing T-curves, given in combinatorics terms, is based on the study of a special decomposition of the triangle $T$.

**Definition 2.8.** *A cycle of $T$ is a closed path of integer segments $l_1, \ldots, l_r$ contained in $T$, having the same type and such that for $s = 1, \ldots, r$, $l_s$ is not a boundary-segment of $T$, $l_s \cap l_{s-1}$ and $l_s \cap l_{s+1}$ are exactly the endpoints of the segment $l_s$ and $l_s \cap l_j = \emptyset$ if $j \notin \{s-1, s, s+1\}$ (where $(s-1)$ and $(s+1)$ are reduced modulo $r$).*

We call *zone* of a cycle $\mathcal{L}$ the integer polygon of $T \setminus \mathcal{L}$ having $\mathcal{L}$ as boundary.

**Definition 2.9.** *A ray of $T$ is a path of integer segments $l_1, \ldots, l_r$ contained in $T$, having the same type and such that the path $l_1 \cup \ldots \cup l_r$ has exactly two different points on the boundary of $T$: its initial and final points. Moreover, for $s = 2, \ldots, r-1$, $l_s \cap l_{s-1}$ and $l_s \cap l_{s+1}$ are exactly the endpoints of the segment $l_s$ and $l_s \cap l_j = \emptyset$ if $j \notin \{s-1, s, s+1\}$.*

A ray $\mathcal{R}$ divides $T$ in two parts. We call *zone* of $\mathcal{R}$ one of these two parts: if they contain different number of vertices of the triangle $T$ then the zone of $\mathcal{R}$ is the one containing less number of vertices, otherwise it is the part of $T \setminus \mathcal{R}$ containing the integer segment whose endpoints are $(0,0), (1,0)$.

As cycles and rays are paths of integer segments having same type, it makes sense to speak about their *biparity* referring to the type of their integer segments.

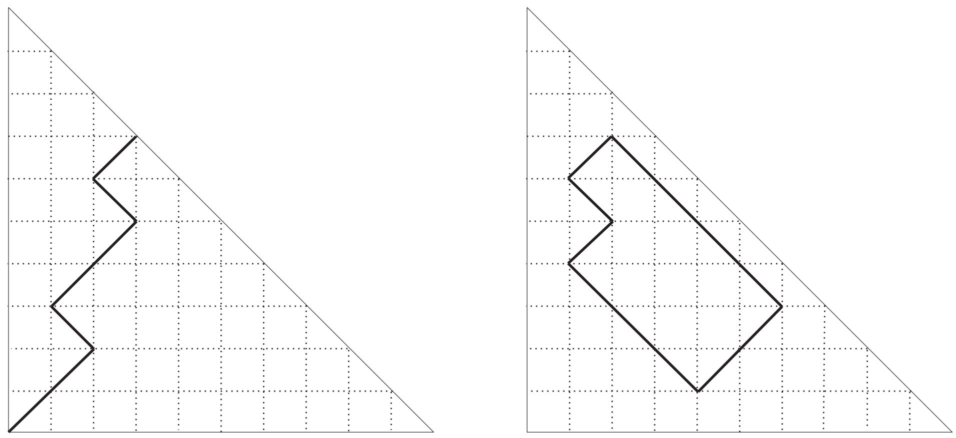

Figure 2. A ray and a cycle of $T$.

**Example 2.10.** Figure 2 represents a ray and a cycle of $T$.

If $\mathcal{L}$ is a cycle of $T$ having biparity $\delta_{i,j}$, denote by $n_s$ the number of the integer points of parity $\delta_s$ contained in the interior part of its zone $\mathcal{Z}$ and by $p$ the number of the boundary-points of $\mathcal{Z}$ that is the number of integer points of $\mathcal{L}$.

**Proposition 2.11.** *The zone $\mathcal{Z}$ of a cycle $\mathcal{L}$ of biparity $\delta_{i,j}$ verifies the following relation:*

$$n_k + n_l = n_i + n_j + \frac{p}{2} - 1$$

*Proof.* Use Pick's formula to calculate the area $\Omega$ of $\mathcal{Z}$ in $\mathbb{Z}^2$ and in the sublattice generated by the vertices of parity $\delta_j$, $\delta_i$.

$$\begin{aligned} \Omega &= n_k + n_l + n_i + n_j + \frac{p}{2} - 1 = \\ &= 2\left(n_i + n_j + \frac{p}{2} - 1\right) \end{aligned}$$

This relation implies the statement. □

**Remark.** The property of the zone of a cycle stated by Proposition 2.11 is invariant under translation of the cycle.

Let $\mathcal{L}_1, \ldots, \mathcal{L}_h$ be a finite number of cycles and rays of $T$ such that, if $i \neq j$ and $\mathcal{L}_i \cap \mathcal{L}_j \neq \emptyset$ then $\mathcal{L}_i \cap \mathcal{L}_j$ is a finite number of integer points. These cycles and rays subdivide $T$ into finitely many connected components, this decomposition of $T$ is called *fragmentation.* In particular $\mathcal{L}_1 \cup \ldots \cup \mathcal{L}_h$ is the boundary of the fragmentation, the vertices of the fragmentation are the integer points of $\mathcal{L}_1 \cup \ldots \cup \mathcal{L}_h$ which are not boundary-points of $T$ and the closure of a connected component $P$ of $T \setminus \{\mathcal{L}_1 \cup \ldots \cup \mathcal{L}_h\}$ is called a *fragment* of $T$.

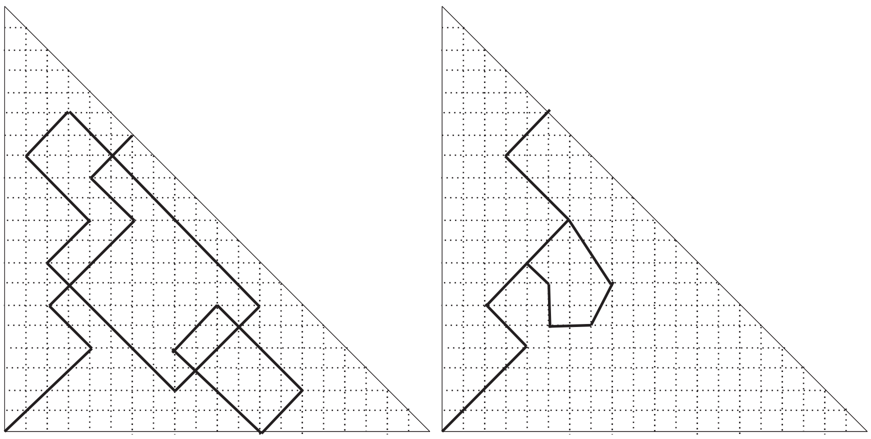


Figure 3. A fragmentation of $T$ (case (A)) and a decomposition of $T$ which is not a fragmentation of $T$ (case (B)).

**Example 2.12.** Figure 3 represents two decompositions of $T$: the first one (case (A)) is a fragmentation of $T$ while the second one (case (B)) is not a fragmentation of $T$.

**Definition 2.13.** *A distribution of signs $\varepsilon$ on a fragmentation $\mathcal{F}$ is a fragmented Harnack distribution if:*

1. *The distribution $\varepsilon_s = \varepsilon|_{P_s}$ is a Harnack distribution for each fragment $P_s$ of $\mathcal{F}$.*
2. *If $P_r$ and $P_s$ are fragments of $\mathcal{F}$ such that $P_r \cap P_s \neq \emptyset$ then $\varepsilon_r = \varepsilon_s$ on $P_r \cap P_s$.*
3. *If $P_r$ and $P_s$ are fragments of $\mathcal{F}$ such that $P_r \cap P_s$ contains an integer segment, then $\varepsilon_s$ and $\varepsilon_r$ are two different Harnack distributions.*

The classification of dividing T-curves is based on fragmentations and fragmented Harnack distributions. In [21] it is proved that, given a fragmentation $\mathcal{F}$ it is always possible to choose a fragmented Harnack distribution on $\mathcal{F}$. Moreover the procedure used to assign a fragmented Harnack distribution to $\mathcal{F}$ is unique in the sense that even if we assign to $\mathcal{F}$ two different fragmented Harnack distributions the resulting T-curves are obtained one from the other by a symmetry. Now we are able to state the classification theorem:

**Theorem 2.14.** (Classification Theorem) [21] *A primitive T-curve is of type* I *if and only if it is constructed starting from a fragmentation $\mathcal{F}$ of $T$ equipped with a fragmented Harnack distribution and a primitive triangulation on each fragment of the fragmentation.*

## 3. The index of a point

Given a curve $C$ in $\mathbb{RP}^2$ equipped with an orientation, we can consider for each point $x \in \mathbb{RP}^2 \setminus \mathrm{C}$, the index $ind_C(x)$ of this point with respect to the curve. We can calculate this number in the following way: consider a line through the point $x$ and transversal to the curve $C$. Choose a normal vector field to the line which vanishes in $x$ (Figure 4). Assign

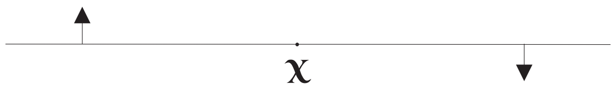


Figure 4. A normal vector field to a line.

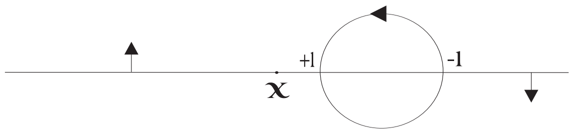


Figure 5. Algebraic value of the intersection points.

to each intersection point between the curve and the line the value $+1$ if, in that point, the local orientation of the curve agrees with the normal vector to the line, assign the value $-1$ otherwise (Figure 5). Let $\alpha$ be the sum of the values assigned to each intersection point. Then the index in $x$ is defined as

$$ind_C(x) = \left|\frac{\alpha}{2}\right|$$

It is easy to check that the index does not depend on the choices of the line and of the normal vector field. Besides it immediately follows that the index is constant on each connected component of $\mathbb{RP}^2 \setminus \mathrm{C}$ and that the index of a point does not change if we reverse the orientation of $C$. Thus for any real algebraic plane projective curve $C$ of type I equipped with one of its complex orientations $ind_C$ is a well defined function on $\mathbb{RP}^2 \setminus \mathrm{C}$ which takes half-integer values if the degree of the curve is odd, integer values otherwise.

Consider now a T-curve $A$ of type I associated to a pair $(\Gamma, \varepsilon)$ and one of its symmetric orientations. We want to study the index of a connected component of $\hat{T} \setminus A$, that is the index with respect to the T-curve $A$ of a point belonging to that connected component.
We observe that each integer point of the extended triangulation is a point of $\hat{T} \setminus A$, and that each connected component of $\hat{T} \setminus A$ contains at least one integer point. To calculate the index of an integer point $E$, we construct, along the edges of the extended triangulation, a pseudo-line $J$ (i.e. $J$ is homeomorphic to $S^1$ and $\hat{T} \setminus J$ is connected). We require that $E \in J$ and $J$ is symmetric with respect to the origin.
Because of the construction, the T-curve intersects $J$ transversally and $J \cap A$ does not contain integer points. Consider an intersection point B between an edge $l$ of $J$ and $A$, and let $\tau$ and $\tau'$ be the triangles of the extended triangulation such that $\tau \cap \tau' = l$. As $A$ is of type I, the orientation of $A$ in $\tau \cup \tau'$ is obtained gluing an orientation of the piece of $A$ contained in $\tau$ with an orientation of the piece of $A$ contained in $\tau'$. This fact implies that the vectors of the orientation of $A$ at the point $B$ in $\tau$ and $\tau'$ point outward for one of the two triangles and inward for the other one, i.e. they coincide if we consider the two vectors applied in $l$ (see for example Figure 6).
Given a pseudo-line $J$, we associate to each integer segment $l$ of $J$ a non-zero normal vector $\vec{v}$ of $l$. Such a collection is called a *normal vector system* for $J$ around the point $E$ if each couple of pairs $(l_1, \vec{v}_1)$, $(l_2, \vec{v}_2)$ satisfies the following condition:
($\vec{v}_1$ and $\vec{v}_2$ point to the same connected component of $T^\# \setminus J$ ) $\Leftrightarrow$ ($l_1$ and $l_2$ belong to the

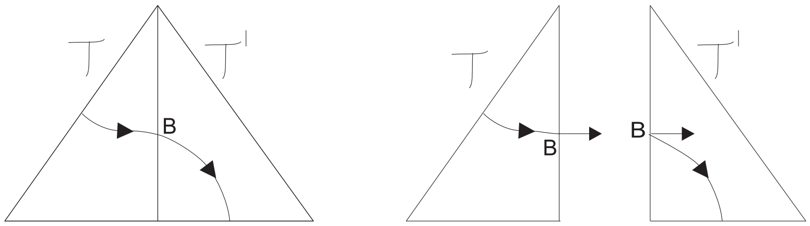


Figure 6. The vectors in $B$ of the orientation of the T-curve in $\tau \cup \tau'$ point outward for $\tau$ and inward for $\tau'$.

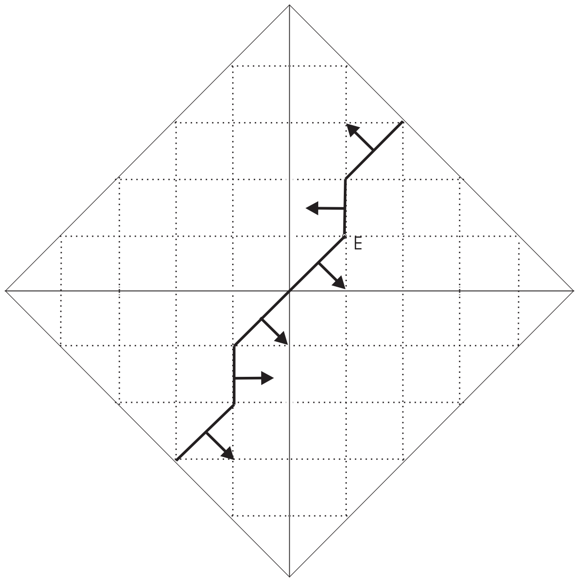


Figure 7. A normal vector system for $J$ around the point $E$.

same connected component of $(J \setminus E) \cap T^{\#}$).
Choose therefore a normal vector system $\nu_E$ for $J$ around the point $E$ (see for example Figure 7).
We can now consider two vectors in B: one is given by $\nu_E$, while the other one is determined by one of the two vectors of the orientation of the T-curve. The *algebraic value* $i_{J,E}(B)$ of the point B is $+1$ if these two vectors agree, it is $-1$ otherwise. The index of the vertex $E$ with respect to the T-curve $A$ is defined in the following way:

$$ind_A(E) = \left| \frac{\sum_{B \in (J \cap A)} i_{J,E}(B)}{2} \right|$$

The value

$$\alpha(E) = \frac{\sum_{B \in (J \cap A)} i_{J,E}(B)}{2}$$

is called the *algebraic index of* $E$ with respect to $(A, J, \nu_E)$. It is easy to verify that the index is independent of the choice of $J$ and that the sign of the algebraic index changes if we reverse the normal vector system.

Let $F$ be another integer point belonging to $J$. It is possible to calculate $ind_A(F)$ using $J$. For each pair of points $E_1$ and $E_2$ of $J$, denote by $\overline{E_1E_2}$ the piece of $J$ which connects in $T^\#$ the points $E_1$ and $E_2$. Given two normal vector systems $\nu_1$ and $\nu_2$ for $J$ respectively around the points $E_1$ and $E_2$, we say that $\nu_1$ and $\nu_2$ are coherent if $\nu_{1|\overline{E_1E_2}}$ coincides with $\nu_{2|\overline{E_1E_2}}$, otherwise they are called opposite.

**Lemma 3.1.** *Let $E$ and $F$ be two integer points of $J$ and $\nu_E$ and $\nu_F$ be two opposite normal vector systems for $J$ around $E$ and $F$ respectively. The algebraic indices $\alpha(E)$ and $\alpha(F)$ of $E$ and $F$ with respect to $(A, J, \nu_E)$ and $(A, J, \nu_F)$ satisfy the following relation:*

$$\alpha(F) = \alpha(E) - \sum_{B \in (\overline{EF} \cap A)} i_{J,E}(B)$$

*Proof.* Consider $B \in J \cap A$. The vector in B determined by the orientation of $A$ is the same when we calculate the index of E or the index of F, while the vectors in B determined by the normal vector systems are opposite in the two cases if and only if B is contained in the piece $\overline{EF}$.
By definition of algebraic value of B, we obtain:

$$i_{J,F}(B) = \begin{cases} -i_{J,E}(B) & \text{if } B \in \overline{EF} \\ \\ i_{J,E}(B) & \text{otherwise} \end{cases}$$

The statement follows therefore from these relations and from the definition of algebraic index. □

**Definition 3.2.** *The number $t_E^F = \sum_{B \in (\overline{EF} \cap A)} i_{J,E}(B)$ is called the relative algebraic index of $F$ with respect to $(E, \alpha(E))$.*

**Lemma 3.3.** *Let $l$ be an integer segment of $J$. The algebraic values of the two intersection points between $A$ and the union $l^\#$ of the four symmetric copies of $l$ are opposite.*

*Proof.* Let $\tau$ be a triangle having $l$ as edge, and $B_1 \in \sigma_1(l)$, $B_2 \in \sigma_2(l)$ be the intersection points between $A$ and $l^\#$. If $B_1$ or $B_2$ does not lie in $J$, we consider also $J' = \sigma_{xy}(J)$. We choose two coherent normal vector systems for $J$ and $J'$ around the origin. The triangles $\sigma_1(\tau)$ and $\sigma_2(\tau)$ are glued in $\hat{\tau}$ along the non-empty copies of $l$, then the vectors of the orientation of the T-curve in $B_1$ and in $B_2$ point outward to a triangle and inward to the other one. The vectors determined by the normal vector systems of the pseudo-lines point inward or outward to $\sigma_1(\tau)$ and $\sigma_2(\tau)$. Then the two points have opposite algebraic values. □

**Theorem 3.4.** *Let $\mathcal{F}$ be a fragmentation of $T$ and $P$ be a fragment of $\mathcal{F}$. If $E$ and $F$ are two vertices of $\Gamma|_P$ having same parity, then for each symmetry $\sigma$ of $\mathbb{R}^2$*

$$ind_A(\sigma(E)) = ind_A(\sigma(F))$$

*Proof.* Let $P^s$ be a copy of $P$ and $H^{\mu'}_{\delta_s}$ be the Harnack distribution on $P^s$. Consider the symmetric copies $E^s$ and $F^s$ of $E$ and $F$ in $P^s$, a pseudo line $J$ symmetric with respect to the origin through O, $E^s$ and $F^s$ and opposite normal vector systems for $J$ around $E^s$ and $F^s$.

By Lemma 3.1 the following equality holds:

$$\alpha(E^s) = \alpha(F^s) - \sum_{B \in (\overline{E^s F^s} \cap A)} i_{J,F^s}(B)$$

Let us verify that $\sum_{B \in (\overline{E^s F^s} \cap A)} i_{J,F^s}(B) = 0$. An integer segment of $J \cap P^s$ is non-empty if and only if it contains an integer point of parity $\delta_s$, thus there is exactly an even number of intersection points between $\overline{E^s F^s}$ and $A$. Let $B_1$ and $B_2$ be two intersection points between $\overline{E^s F^s}$ and $A$ such that the piece $\overline{B_1 B_2}$ of $J$ contains no other intersection points between $J$ and $A$. For such points one has:

$$i_{J,F^s}(B_1) = -i_{J,F^s}(B_2)$$

In fact let $\tau_1 = Conv\{v_1, v_2, v_3\}$ (resp. $\tau_2 = Conv\{v_4, v_5, v_6\}$) be a triangle such that the point $B_1$ belongs to the edge $Conv\{v_1, v_2\}$ (resp. $B_2 \in Conv\{v_4, v_5\}$) and let $v_1$ (resp. $v_4$) be the vertex of type $\delta_s$. As $P$ is a fragment we can construct a path of triangles in $P^s$ which connects $\tau_1$ with $\tau_2$ and such that all the triangles of the path have same orientation. We can have two different situations:

- $\tau_1$ and $\tau_2$ face on the same side of $J$. In this case if the orientation of $\tau_1$ is given by $v_2$, $v_1$, $v_3$, the orientation of $\tau_2$ is $v_4$, $v_5$, $v_6$. It is simple to verify that the vectors determined by the orientation of the T-curve in $B_1$ and $B_2$ point in opposite directions with respect to $J$.
- $\tau_1$ and $\tau_2$ face on opposite sides of $J$. In this case if the orientation of $\tau_1$ is given by $v_2$, $v_1$, $v_3$, the orientation of $\tau_2$ is $v_4$, $v_6$, $v_5$ and therefore it is simple to verify that again the vectors determined by the orientation of the T-curve in $B_1$ and $B_2$ point in opposite directions with respect to $J$. □

Consider a dividing T-curve $A$ of degree $m$ associated to a pair $(\Gamma, \varepsilon)$ and let $\mathcal{F}$ be the fragmentation of $T$, equipped with a fragmented Harnack distribution, to which $A$ is associated. Let $\mathcal{L}$ be a cycle (resp. a ray) of the boundary of $\mathcal{F}$ having biparity $\delta_{i,j}$ and zone $\mathcal{Z}$. Denote by $P_1, \ldots, P_n$ the fragments contained in $\mathcal{Z}$. Let $P_1$ be such that its intersection with $\mathcal{L}$ contains at least two integer points. As $P_1$ is equipped with a Harnack distribution, in each of its copies there exists a parity of isolated vertices, i.e. all the integer points of a certain parity are isolated (Proposition 2.4). In the following we use these notions: for $s \in \{i, j, k, l\}$, $P_1^s$ is the copy of $P_1$ in which the vertices of parity $\delta_s$ are isolated, $Q_s$ is the quadrant containing $P_1^s$, if $\mathcal{S}$ is a subset of $T$, $\mathcal{S}^\#$ is the union of $\mathcal{S}$ and its symmetric copies and $\mathcal{S}^s = \mathcal{S}^\# \cap Q_s$.

**Lemma 3.5.** *There exist $\alpha_i$ with $i = 1, 2, 3, 4$ such that:*

- $\sum_{s=1}^{4} \alpha_s = 0$
- *the index situation in ${P_1}^\#$ is the following:*

| copy of $P_1$ | Type of isolated vertex | Index of isolated vertices | Index of non isolated vertices |
|---|---|---|---|
| ${P_1}^j$ | $\delta_j$ | $\mid \alpha_j \mp 1 \mid$ | $\mid \alpha_j \mid$ |
| ${P_1}^i$ | $\delta_i$ | $\mid \alpha_i \mid$ | $\mid \alpha_i \pm 1 \mid$ |
| ${P_1}^k$ | $\delta_k$ | $\mid \alpha_k \mp 1 \mid$ | $\mid \alpha_k \mid$ |
| ${P_1}^l$ | $\delta_l$ | $\mid \alpha_l \mp 1 \mid$ | $\mid \alpha_l \mid$ |

Tab. 3.5. Index situation in ${P_1}^{\#}$

*Proof.* Consider an integer point $E \in (P_1 \cap \mathcal{L})$ of parity $\delta_i$. Construct a pseudo-line $J_1$, symmetric with respect to the origin, along the edges of the triangulation and such that $E \in J_1$. Construct the pseudo-line $J_2 = \sigma_y(J_1)$. The union $J_1 \cup J_2$ contains the four symmetric copies of the integer point $E$. We can use one of the two pseudo-lines to calculate the index of two copies of $E$ and the other one for the other two copies. We can use $J_1$ and $J_2$ indifferently to calculate the index of the origin O. To calculate an algebraic index we have to fix a normal vector system for each pseudo-line. We choose two coherent normal vector systems $\nu_1, \nu_2$ for $J_1$ and $J_2$ around the origin.
For $r = 1, 2, 3, 4$ the points O and $E^r$, with an appropriate normal vector system for $J$ around $E^r$, satisfy Lemma 3.1.

$$\alpha(E^r) = \frac{\sum_{B\in(J_1\cap A)} i_{J_1,O}(B)}{2} - \sum_{B\in(\overline{OE^r}\cap A)} i_{J_1,O}(B) \qquad \text{if } E^r \in J_1$$

$$\alpha(E^r) = \frac{\sum_{B\in(J_2\cap A)} i_{J_2,O}(B)}{2} - \sum_{B\in(\overline{OE^r}\cap A)} i_{J_2,O}(B) \qquad \text{if } E^r \in J_2$$

By Lemma 3.3 it follows that

$$\sum_{r=1}^{4} \alpha(E^r) = 0$$

To describe the index situation in $P_1^{\#}$, consider the following additional condition on $J_1$: the pseudo-line $J_1$ contains a vertex of $\Gamma$ for any parity of integer points of $P_1$. Let us denote by $\alpha_s$ the algebraic index of $E^s$. As $P_1$ is equipped with a Harnack distribution, then (Theorem 3.4) $\mid \alpha_i \mid$ is the index of all the vertices of type $\delta_i$ contained in $P^i$ and $\mid \alpha_s \mid$ for $s = j, k, l$ is the index of all non isolated vertices of $P_1^s$ (as all non isolated vertices have same index).

Moreover $J_1$ can be also used to calculate the index of a vertex $F$ of $\Gamma$ of type $\delta_r \neq \delta_i$. If we denote by $\alpha(F^s)$ the algebraic index of the symmetric copy of $F$ in $P_1^s$ we have:

$$\alpha(F^s) = \alpha_s \ \text{ if } s \neq i, r$$

$$\alpha(F^s) = \alpha_s \pm 1 \ \text{ if } s = i, r$$

$$\sum_{s=1}^{4} \alpha(F^s) = 0$$

Therefore if $\alpha(F^i) = \alpha_i \pm 1$ then $\alpha(F^r) = \alpha_r \mp 1$. □

## 4. Modification on cycles and rays

We have seen in Section 2 that a fragmentation of $T$ is associated to a dividing T-curve. In this section we describe two operations, called "modifications on cycles and rays", which allow us to pass from a T-curve of type I to other T-curves of type I with controlled topology and with known associated fragmentations. Consider a T-curve $A$ of type I associated to a pair $(\Gamma, \varepsilon)$ and let $\mathcal{F}$ be the fragmentation of T, equipped with a fragmented Harnack distribution, associated to $A$. Let $\mathcal{L}$ be a cycle or a ray of $T$ having biparity $\delta_{i,j}$; let us remark that $\mathcal{L}$ is not necessary a cycle or a ray of the boundary $\mathcal{B}(\mathcal{F})$ of $\mathcal{F}$. Starting from $A$ and from $\mathcal{L}$, we construct a new fragmentation $\mathcal{F}'$ of $T$ whose boundary $\mathcal{B}'$ is the union of the edges $l$ of $\Gamma$ which verify one of the following two conditions:

1. The edge $l$ belongs to $\mathcal{B}(\mathcal{F})$ and it is not an edge of $\mathcal{L}$.
2. The edge $l$ is not an edge of $\mathcal{B}(\mathcal{F})$ and it is an edge of $\mathcal{L}$.

We can choose a fragmented Harnack distribution $\varepsilon'$ for $\mathcal{F}'$ in such a way $\varepsilon$ and $\varepsilon'$ coincide outside the zone $\mathcal{Z}$ of $\mathcal{L}$, then from Theorem 2.14 the fragmentation $\mathcal{F}'$ and the distribution $\varepsilon'$ allow us to construct a dividing T-curve $A'$. We say that $A'$ is obtained from $A$ by a modification on the cycle (resp. the ray) $\mathcal{L}$.

**Lemma 4.1.** *If $\tilde{\varepsilon}|_{P_1^i} = H^{\mu}_{\delta_i}$ then $\tilde{\varepsilon}'|_{P_1^i} = H^{-\mu}_{\delta_j}$.*

*Proof.* The extended distributions of $\varepsilon$ and $\varepsilon'$ coincide on $\mathcal{L}^{\#}$, then $\tilde{\varepsilon}'|_{P_1^i}$ and $H^{\mu}_{\delta_i}$ coincide on the vertices of parity $\delta_i$ and $\delta_j$. Besides $\tilde{\varepsilon}'|_{P_1^i}$ must be a Harnack distribution different from $H^{\mu}_{\delta_i}$, then $\tilde{\varepsilon}'|_{P_1^i}$ and $H^{\mu}_{\delta_i}$ are opposite on the vertices of parity $\delta_k$ and $\delta_l$. This means $\tilde{\varepsilon}'|_{P_1^i} = H^{-\mu}_{\delta_j}$. □

There exists the following relation between the connected components of $\mathcal{Z}^{\#} \setminus A'$ and the connected components of $\mathcal{Z}^{\#} \setminus A$.

**Lemma 4.2.** *Take two T-curves $A$ and $A'$ obtained one from the other by a modification on a cycle (a ray) $\mathcal{L}$ of biparity $\delta_{i,j}$. Then $A$ and $A'$, in the union of the symmetric copies of the zone $\mathcal{Z}$ of $\mathcal{L}$, are obtained one from the other by the symmetry $\sigma_{i,j}$.*

*Proof.* From Lemma 4.1 and from Proposition 2.4, it follows that the part of $A$ contained in $\sigma_{i,s}(\mathcal{Z})$ coincides with the part of $A'$ contained in $\sigma_{j,s}(\mathcal{Z})$. Then the part of $A$ and $A'$ contained in $\mathcal{Z}^{\#}$ are obtained one from the other by the symmetry $\sigma_{i,j}$ too. □

**Corollary 4.3.** *Take two T-curves $A$ and $A'$ obtained one from the other by a modification on a cycle (a ray) $\mathcal{L}$ of biparity $\delta_{i,j}$. If $\mathcal{Z}$ is the zone of $\mathcal{L}$ then a connected component of the complement of the curve $A$ in $\mathcal{Z}^{\#} \cap Q_s$ is also a connected component of the complement of the curve $A'$ in $\mathcal{Z}^{\#} \cap (\sigma_{i,j}(Q_s))$.*

Let $E \in P_1$ be an integer point of type $\delta_i$ and $\alpha_s$ with $s = 1, 2, 3, 4$ be as in Lemma 3.5. Let us denote by $C_r^s$ for $r = 1, \dots, \gamma_s$ the connected components of the complement of $A$ in $\mathcal{Z}^s$ for $s = i, j, k, l$. For each $C_r^s$, we define an integer number $t_r^s$ called the *relative algebraic index* of $C_r^s$ with respect to $(E^s, \alpha_s)$.

Choose an integer point $X_r^s$ in $C_r^s$. Construct a pseudo-line $J$ (symmetric with respect to the origin) through $E^s$ and $X_r^s$ such that $\overline{E^sX_r^s} \subset \mathcal{Z}^s$. Choose a normal vector system $\nu$ for $J$ such that the algebraic index of $E^s$ with respect to $(A, J, \nu)$ is exactly $\alpha_s$ (recall that it could be also $-\alpha_s$). With this choices, we define: $t_r^s = t_{E^s}^{X_r^s}$.

By Lemma 4.2 the symmetric copy $\sigma_{i,j}(C_r^s)$ of $C_r^s$ is a connected component of $\mathcal{Z}^\# \setminus A'$ in the quadrant $Q_{s'} = \sigma_{i,j}(Q_s)$, then

$$\mathcal{Z}^{s'} \setminus A' = \bigcup_{r=1}^{\gamma_s} \sigma_{i,j}(C_r^s)$$

For each $C_r^s$, another integer number $t'^s_r$ is also defined: it is the relative algebraic index of $\sigma_{i,j}(C_r^s)$ with respect to $(E^{s'}, \alpha_{s'})$.

**Lemma 4.4.** *Let $A$ be a T-curve associated to a pair $(\Gamma, \varepsilon)$. Let $\mathcal{L}$ be a cycle of $T$ of biparity $\delta_{i,j}$, then the extended distribution $\bar{\varepsilon}$ satisfies one of the following relations:*

a) $\bar{\varepsilon}(E^i) = \bar{\varepsilon}(E^j)$ $\forall$ *integer points* $E \in \mathcal{L}$

b) $\bar{\varepsilon}(E^i) = -\bar{\varepsilon}(E^j)$ $\forall$ *integer points* $E \in \mathcal{L}$

*Similarly for the extended distribution $\bar{\varepsilon}$ restricted to $\mathcal{L}^\# \cap Q_k$ and $\mathcal{L}^\# \cap Q_l$.*

*Proof.* Consider the quadrants $Q_i$ and $Q_j$ (resp. the quadrants $Q_k$ and $Q_l$) and look at the polygons $P_1^i$ and $P_1^j$ (resp. $P_1^k$ and $P_1^l$). The vertices of parity $\delta_i$ and $\delta_j$ (resp. $\delta_k$ and $\delta_l$) are isolated respectively in the first polygon and in the second one. This means that the distributions of signs on $\mathcal{L}^\# \cap P_1^i$ and $\mathcal{L}^\# \cap P_1^j$ are both alternate (resp. the distributions of signs on $\mathcal{L}^\# \cap P_1^k$ and $\mathcal{L}^\# \cap P_1^l$ are both constant), that is $\bar{\varepsilon}(E^i) = \bar{\varepsilon}(E^j)$ for each integer point $E \in \mathcal{L} \cap P_1$ or $\bar{\varepsilon}(E^i) = -\bar{\varepsilon}(E^j)$ for each integer point $E \in \mathcal{L} \cap P_1$ (resp. $\bar{\varepsilon}(E^k) = \bar{\varepsilon}(E^l)$ for each integer point $E \in \mathcal{L} \cap P_1$ or $\bar{\varepsilon}(E^k) = -\bar{\varepsilon}(E^l)$ for each integer point $E \in \mathcal{L} \cap P_1$). As the vertices of $\mathcal{L}$ are all of parities $\delta_i$ and $\delta_j$ and the signs of all the vertices having same parity change in the same way with respect to a symmetry, the lemma is proved. □

**Proposition 4.5.** *If $C_r^s$ is a connected component of $\mathcal{Z}^s \setminus A$ with index $\mid \alpha_s - t_r^s \mid$, then $\sigma_{i,j}(C_r^s)$ is a connected component of $\mathcal{Z}^\# \setminus A'$ with index $\mid \alpha_{s'} + t_r^s \mid$.*

*Proof.* We can choose symmetric orientations for the T-curves $A$ and $A'$ such that the two orientations coincide outside $\mathcal{Z}^\#$ and are opposite inside $\mathcal{Z}^\#$ that is $\sigma_{i,j}(A_{|\mathcal{Z}^s})$ and $A'_{|\sigma_{i,j}(\mathcal{Z}^s)}$ have opposite orientations. If $Q_s$ and $Q_{s'} = \sigma_{i,j}(Q_s)$ are symmetric quadrants with respect to the origin, then we need only a pseudo-line $J$ (constructed as before) to prove the proposition. Otherwise we consider also $J' = \sigma_{xy}(J)$. If a connected component $C_r^s$ in $\mathcal{Z}^s \setminus A$ has index $\mid \alpha_s - t_r^s \mid$ then the sum of the algebraic values of the intersection points between the curve $A$ and the piece $\overline{E^sX_r^s}$ is equal to $t_r^s$.

After the modification the point $\sigma_{i,j}(X_r^s) \in \sigma_{i,j}(C_r^s)$ belongs to the quadrant $Q_{s'}$. By definition, the index of $\sigma_{i,j}(C_r^s)$ with respect to the curve $A'$ is $\mid \alpha_{s'} - t'^s_r \mid$. On the other hand $t'^s_r$ is the sum of the algebraic values of the intersection points between $A'$ and the piece $\overline{E^{s'}X_r^{s'}}$ which is the symmetric copy of the piece $\overline{E^sX_r^s}$. As $A \cap \mathcal{Z}^s = A' \cap \mathcal{Z}^{s'}$ with opposite orientations, the two numbers $t_r^s$ and $t'^s_r$ must be opposite. □

There exists another important property of cycles which can be formulated in terms of the algebraic relative index. The most interesting aspect of this relation, which connects the relative algebraic indices and the Euler characteristics of the connected components of $\mathcal{Z}^{\#} \setminus A$, is that it is really independent from the curve outside the zone of the cycle. Therefore it describes a property of the curve contained in the zone of the cycle and, in particular, it is invariant under translations.

**Proposition 4.6.** *Let $\mathcal{Z}$ be the zone of a cycle $\mathcal{L}$ of biparity $\delta_{i,j}$. The relative algebraic indices and the Euler characteristic of the connected components $C_r^s$ of $\mathcal{Z}^{\#} \setminus A$ satisfy the following relation:*

$$\sum_{\mathcal{Z}^j \setminus A} \chi(C_r^j) t_r^j + \sum_{\mathcal{Z}^i \setminus A} \chi(C_r^i) t_r^i - \sum_{\mathcal{Z}^k \setminus A} \chi(C_r^k) t_r^k - \sum_{\mathcal{Z}^l \setminus A} \chi(C_r^l) t_r^l = 0$$

*Proof.* We prove the proposition by induction on the number $n$ of cycles and rays which intersect the zone $\mathcal{Z}$.

$n = 0$: In this case $P_1$ coincides with $\mathcal{Z}$ then the polygon $\mathcal{Z}$ is equipped with a Harnack distribution and the index situation in $\mathcal{Z}^{\#}$ is the one described in Table 3.5. Denote by $n_s$ the number of integer points of parity $\delta_s$ contained in the interior part of $\mathcal{Z}$ and by $p_s$ the number of integer points of parity $\delta_s$ contained in $\mathcal{L}$. From Proposition 2.4 it follows that the complement of the curve in $\mathcal{Z}^s$ has $n_s + p_s + 1$ connected components one of which has Euler characteristic equal to $1 - n_s$ and each of all the others $n_s + p_s$ connected components encloses an isolated vertex and has Euler characteristic equal to 1.

The following relation comes immediately from the proof of Lemma 3.5:

$$\begin{aligned}
&\sum_{\mathcal{Z}^j \setminus A} \chi(C_r^j) t_r^j + \sum_{\mathcal{Z}^i \setminus A} \chi(C_r^i) t_r^i - \sum_{\mathcal{Z}^k \setminus A} \chi(C_r^k) t_r^k - \sum_{\mathcal{Z}^l \setminus A} \chi(C_r^l) t_r^l = \\
&= \pm(n_j + p_j) \mp (1 - n_i) \mp n_k \mp n_l = \\
&= \pm(n_j + p_j - 1 + n_i - n_k - n_l)
\end{aligned}$$

The number of vertices on $\mathcal{L}$ is exactly $2p_j$ as a cycle of biparity $\delta_{i,j}$ has an even number of vertices and half of them is of parity $\delta_i$ and the others are of parity $\delta_j$. The statement follows then from Proposition 2.11.

$n \Rightarrow (n + 1)$: Suppose now that there are $n + 1$ cycles and rays intersecting the zone $\mathcal{Z}$ and let $\mathcal{L}'$ be one of these cycles and rays having biparity $\delta_{i,l}$ (if $\mathcal{L}'$ is of biparity $\delta_{i,j}$ or $\delta_{k,l}$ the statement can be proved in similar way).

We want to prove:

$$\sum_{\mathcal{Z}^j \setminus A} \chi(C_r^j) t_r^j + \sum_{\mathcal{Z}^i \setminus A} \chi(C_r^i) t_r^i - \sum_{\mathcal{Z}^k \setminus A} \chi(C_r^k) t_r^k - \sum_{\mathcal{Z}^l \setminus A} \chi(C_r^l) t_r^l = 0 \tag{1}$$

where $t_r^s$ is the relative algebraic index of the component $C_r^s$ of $\mathcal{Z}^s$ with respect to $(E^s, \alpha_s)$.

Let $A'$ be the dividing T-curve obtained from the T-curve $A$ by a modification on $\mathcal{L}'$. The cycle $\mathcal{L}$ belongs to the boundary of the fragmentation of $A'$ and its zone is intersected by $n$ cycles and rays.

Let $\mathcal{Z}'$ be the zone of $\mathcal{L}'$, $\mathcal{Z}'^{\circ}$ be its interior part, $\mathcal{Z}_1 = \left(\mathcal{Z} \setminus \mathcal{Z}'^{\circ}\right)$, $\mathcal{Z}_2 = (\mathcal{Z} \cap \mathcal{Z}')$ and $\bar{E}$ be a point of $(\mathcal{L} \setminus \mathcal{Z}')$ having parity $\delta_i$ and such that $ind_A(\bar{E}^s) =\mid \alpha_s - m_s \mid$ for $s = i, j, k, l$, with $m_s$ such that $m_i + m_j + m_k + m_l = 0$. Observe that for each $C_r^s$ the relative algebraic index with respect to $(\bar{E}^s, \alpha_s - m_s)$ is $b_r^s$ defined by $t_r^s = m_s + b_r^s$.

Not to create confusion, let us use the following notions: ${C'}_r^s$ denotes a connected component of $(\mathcal{Z}^s \setminus A')$, ${t'}_r^s$ will be its relative algebraic index with respect to $(E^s, \alpha_s)$. $C_r^s$ is a connected component of $\mathcal{Z}^s \setminus A$ and its relative algebraic index with respect to $(E^s, \alpha_s)$ is denoted by $t_r^s$.

Consider a point of $\mathcal{Z}_1^s$. It lies outside the zone $\mathcal{Z}'$ of the cycle (the ray) $\mathcal{L}'$, then its index is the same with respect to the two T-curves. Then the indices of $\bar{E}$ and of its symmetric copies are the same for the two T-curves.

On the other hand, consider a point belonging to a connected component $C_r^s$ of $\mathcal{Z}_2^s$: its index with respect to the T-curve $A$ is $\mid \alpha_s - t_r^s \mid = \mid \alpha_s - m_s - b_r^s \mid$ while its index with respect to the T-curve $A'$ is $\mid \alpha_{s'} - m_{s'} + b_r^s \mid$ (Proposition 4.5), where $s$ and $s'$ are related in this way: $s = i$, $s' = l$; $s = j$, $s' = k$; $s = k$, $s' = j$; $s = l$, $s' = i$ (let us observe that if $\mathcal{L}'$ is of biparity $\delta_{i,j}$ or $\delta_{k,l}$, the relations between $s$ and $s'$ are different. Moreover all we are going to explain must be modified, for these cases, keeping attention to relate in the exact way $s$ and $s'$).

We can apply the inductive hypothesis to the T-curve $A'$ using the relative algebraic index with respect to $(E^s, \alpha_s)$:

$$\sum_{\mathcal{Z}^j \setminus A'} \chi({C'}_r^j){t'}_r^j + \sum_{\mathcal{Z}^i \setminus A'} \chi({C'}_r^i){t'}_r^i - \sum_{\mathcal{Z}^k \setminus A'} \chi({C'}_r^k){t'}_r^k - \sum_{\mathcal{Z}^l \setminus A'} \chi({C'}_r^l){t'}_r^l = 0 \tag{2}$$

We can split equalities (1) and (2) as the sum over $\mathcal{Z}_1 \setminus A$ and over $\mathcal{Z}_2 \setminus A$. Recalling that $t_r^s = m_s + b_r^s$, equality (1) is equivalent to the following one:

$$\begin{aligned} \sum_{s \in \{i,j\}} \sum_{\mathcal{Z}_2^s \setminus A} \chi(C_r^s)(m_s + b_r^s) - \sum_{s \in \{k,l\}} \sum_{\mathcal{Z}_2^s \setminus A} \chi(C_r^s)(m_s + b_r^s) = \\ = \sum_{\mathcal{Z}_2^j \setminus A} \chi(C_r^k)(m_j + b_r^k) + \sum_{\mathcal{Z}_2^i \setminus A} \chi(C_r^l)(m_i + b_r^l) + \\ - \sum_{\mathcal{Z}_2^k \setminus A} \chi(C_r^j)(m_k + b_r^j) - \sum_{\mathcal{Z}_2^l \setminus A} \chi(C_r^i)(m_l + b_r^i) \end{aligned} \tag{3}$$

On the other hand using the relation $\sum_{s=1}^4 m_s = 0$ equation (3) is equivalent to the following one:

$$\sum_{\mathcal{Z}_2^j \setminus A} \chi(C_r^j) - \sum_{\mathcal{Z}_2^i \setminus A} \chi(C_r^i) - \sum_{\mathcal{Z}_2^k \setminus A} \chi(C_r^k) + \sum_{\mathcal{Z}_2^l \setminus A} \chi(C_r^l) = 0 \tag{4}$$

Let us prove this equality: denote by $p_s$ the number of intersection points between $A$ and the boundary $\mathcal{B}(\mathcal{Z}_2^s)$ of the polygon $\mathcal{Z}_2^s$. For $s = i, j, k, l$ the following equality holds:

$$\sum_{\mathcal{Z}_2^s \setminus A} \chi(C_r^s) = 1 + \frac{p_s}{2}$$

On the other hand, consider $p_s$ as the sum $\beta_s + \beta'_s$ where $\beta_s$ and $\beta'_s$ are respectively the number of points in $A \cap \mathcal{B}(\mathcal{Z}_2^s) \cap \mathcal{L}$ and in $A \cap \mathcal{B}(\mathcal{Z}_2^s) \cap \mathcal{L}'$. From Lemma 4.4:

$$\beta_i = \beta_j \qquad \beta_k = \beta_l \qquad \beta'_i = \beta'_l \qquad \beta'_j = \beta'_k$$

With these relations it is easy to verify equality (4). □

## 5. Proof of Rokhlin's formula for T-curves

We are able now to prove the following theorem:

**Theorem 5.1.** (Rokhlin's Formula for T-curves) *For every primitive dividing T-curve A of degree m*

$$\int_{\hat{T}\setminus A} ind_A^2(x)\ d\chi(x) = \frac{m^2}{4}$$

Observe that this theorem is not a corollary of Rokhlin's formula. In fact if we construct a T-curve $A$ using a convex triangulation we are sure that there exists a real algebraic projective curve $C$ such that the pair $(\mathbb{R}\mathrm{P}^2, \mathbb{R}\mathrm{C})$ is homeomorphic to the pair $(\hat{T}, A)$. For these T-curves, that is for algebraic T-curves, we just know that Rokhlin's formula is verified. On the other hand we will prove the formula also for primitive T-curves constructed using a non convex triangulation.

Let us explain the structure of the proof and then we will prove the necessary steps. Consider a dividing T-curve $A$ of degree $m$ associated to the pair $(\Gamma, \varepsilon)$ and the T-curve $M$ obtained from $A$ by a modification on all the cycles and rays of the boundary of the fragmentation associated to $A$. Let $A_0 = A, A_1 \dots, A_{k-1}, A_k = M$ be the dividing T-curves such that for $i = 1, \dots, k$ the T-curves $A_{i-1}$ and $A_i$ differ by a modification on a cycle or on a ray.

We first prove that the T-curve M satisfies Rokhlin's formula (Lemma 5.2 below) and, second, that, for $i = 1, \dots, k$, the T-curve $A_{i-1}$ satisfies Rokhlin's formula if and only if the T-curve $A_i$ satisfies the formula (Theorem 5.3 below). Then we obtain that $A$ satisfies Rokhlin's formula.

Let us prove the first step:

**Lemma 5.2.** *The T-curve M satisfies Rokhlin's formula.*

*Proof.* The T-curve $M$ is associated to a pair $(\Gamma, H)$ where $H$ is a Harnack distribution. In [15] Itenberg proved that in this situation the real scheme of the T-curve $M$ is $< 1 < \frac{(h-1)(h-2)}{2} > \amalg \frac{3h^2-3h}{2} >$ if $m = 2h$ or $< \mathcal{J} \amalg < \frac{(m-1)(m-2)}{2} >>$ if $m = 2h+1$. Consider a complex orientation of $M$: for even degrees the index of each exterior oval is 1 and the index of each interior oval is 0, while for odd degrees we have $\frac{h(h+1)}{2}$ ovals with index $\frac{3}{2}$ and $\frac{(m-1)(m-2)-h(h+1)}{2}$ ovals with index $\frac{1}{2}$. It is easy to verify that Rokhlin's formula is satisfied for such schemes. □

Let $\mathcal{Z}$ be the zone of a cycle or a ray of $T$. The value $\mathcal{R}ok(A, \mathcal{Z}) = \int_{(\mathcal{Z}\#\setminus A)} ind_A^2(x)\ d\chi(x)$ is called *the contribution of $\mathcal{Z}$ in the Rokhlin's formula for the T-curve A.*

Let us prove now the last step to complete the proof of Rokhlin's formula.

**Theorem 5.3.** *Let $A$ and $A'$ be two T-curves of type* I *and degree $m$ obtained one from the other by a modification on a cycle or a ray. Then*

$$\int_{(\hat{T}\setminus A)} ind_A^2(x)\ d\chi(x) = \int_{(\hat{T}\setminus A')} ind_{A'}^2(x)\ d\chi(x)$$

*Proof.* Let $\mathcal{Z}$ be the zone of the cycle or the ray, then it is possible to split the two integrals as the sum between the integral on $\mathcal{Z}^{\#}$ and on $\hat{T}\setminus\mathcal{Z}^{\#}$. As $A$ and $A'$ coincide outside $\mathcal{Z}^{\#}$ the statement of the theorem is verified if and only if the contributions of $\mathcal{Z}$ in Rokhlin's formula for the T-curves $A$ and $A'$ are equal.

We will study separately the case of dividing T-curves whose fragmentations differ by a modification on a cycle and the case of two curves of type I differing by a modification on a ray.

*The case of a cycle.* Suppose that $A$ and $A'$ differ by a modification on a cycle $\mathcal{L}$ of biparity $\delta_{i,j}$ and let $E$ be an integer point of $\mathcal{L}\cap P_1$ of type $\delta_i$, $p_s = card\left((A\cap\mathcal{L}^{\#})\cap Q_s\right)$ and $\alpha_r$ for $r=i,j,k,l$ be as in Lemma 3.5; observe that $\sum_{(\mathcal{Z}^s\setminus A)}\chi(C_r^s) = 1+\frac{p_s}{2}$ and recall that from Lemma 4.4 one has that $p_j = p_i$ and $p_k = p_l$, then the contribution of $\mathcal{Z}$ in the Rokhlin's formula for the T-curve $A$ can be written as follows:

$$\begin{aligned}
\mathcal{R}ok(A,\mathcal{Z}) = &\sum_{(\mathcal{Z}^{\#}\setminus A)} \chi(C_r^s) ind^2_{C_r^s}(X_r^s) = \\
= &\sum_{s\in\{j,i,k,l\}}\sum_{(\mathcal{Z}^s\setminus A)}\chi(C_r^s){\alpha_s}^2 + \sum_{(\mathcal{Z}^{\#}\setminus A)}\left(-2\alpha_s\chi(C_r^s)t_r^s + \chi(C_r^s)(t_r^s)^2\right) = \\
= &\left({\alpha_j}^2+{\alpha_i}^2\right)\left(1+\frac{p_j}{2}\right) + \left({\alpha_k}^2+{\alpha_l}^2\right)\left(1+\frac{p_k}{2}\right) + \sum_{(\mathcal{Z}^{\#}\setminus A)}\chi(C_r^s)\left(-2\alpha_s t_r^s + (t_r^s)^2\right)
\end{aligned}$$

Consider now the T-curve $A'$. From Proposition 4.5 it follows that if $C_r^s$ is a connected component of $\mathcal{Z}^s\setminus A$ with index $\mid \alpha_s - t_r^s\mid$ then $\sigma_{i,j}(C_r^s)$ is a connected component of $\mathcal{Z}^{s'}\setminus A'$ with index $\mid \alpha_{s'} + t_r^s\mid$, where $s$ and $s'$ are related in this way: $s=i$, $s'=j$; $s=j$, $s'=i$; $s=k$, $s'=l$; $s=l$, $s'=k$. Denote by $p'_s = card\left((A'\cap\mathcal{L}^{\#})\cap Q_s\right)$. As the distributions on $\mathcal{L}$ coincide for the two T-curves, we have $card\left((A'\cap\mathcal{L}^{\#})\cap Q_s\right) = card\left((A\cap\mathcal{L}^{\#})\cap Q_s\right)$ that is ${p'}_s = p_s$ for $s\in\{i,j,k,l\}$. Using these facts the contribution of $\mathcal{Z}$ in Rokhlin's formula for the T-curve $A'$ can be written as follows:

$$\begin{aligned}
\mathcal{R}ok(A',\mathcal{Z}) = &\sum_{(\mathcal{Z}^{\#}\setminus A')}\chi(\sigma_{i,j}(C_r^s)) ind^2_{\sigma_{i,j}(C_r^s)}(X_r^{s'}) = \\
= &\left({\alpha_j}^2+{\alpha_i}^2\right)\left(1+\frac{p_j}{2}\right) + \left({\alpha_k}^2+{\alpha_l}^2\right)\left(1+\frac{p_k}{2}\right) + \sum_{(\mathcal{Z}^{\#}\setminus A)}\chi(C_r^s)\left(2\alpha_{s'}t_r^s + (t_r^s)^2\right)
\end{aligned}$$

We can now compare the contribution of $\mathcal{Z}$ in Rokhlin's formula for the T-curve $A$ with the one for the T-curve $A'$:

$$\begin{aligned}&\mathcal{R}ok(A',\mathcal{Z})-\mathcal{R}ok(A,\mathcal{Z})=\\&=\sum_{(\mathcal{Z}^{\#}\setminus A)}(2\alpha_{s'}\chi(C_r^s)t_r^s)+\sum_{(\mathcal{Z}^{\#}\setminus A)}(2\alpha_{s}\chi(C_r^s)t_r^s)\end{aligned}$$

As the algebraic values of the indices satisfy the relation $\alpha_j+\alpha_i+\alpha_k+\alpha_l=0$, we obtain:

$$\begin{aligned}&\mathcal{R}ok(A',\mathcal{Z})-\mathcal{R}ok(A,\mathcal{Z})=\\&=\;2(\alpha_j+\alpha_i)\left(\sum_{\mathcal{Z}^j\setminus A}\chi(C_r^j)t_r^j+\sum_{\mathcal{Z}^i\setminus A}\chi(C_r^i)t_r^i\right)+\\&+\;2(\alpha_j+\alpha_i)\left(-\sum_{\mathcal{Z}^k\setminus A}\chi(C_r^k)t_r^k-\sum_{\mathcal{Z}^l\setminus A}\chi(C_r^l)t_r^l\right)\end{aligned}\tag{5}$$

Proposition 4.6 implies that the right hand side of equality (5) is zero.
Therefore if $A$ and $A'$ differ by a modification on a cycle, one has:

$$\int_{(\hat{T}\setminus A)} ind_A^2(x)\;d\chi(x)=\int_{(\hat{T}\setminus A')} ind_{A'}^2(x)\;d\chi(x)$$

*The case of a ray.* Suppose now that the fragmentations of the two T-curves $A$ and $A'$ differ by a modification on a ray $\mathcal{R}$ of biparity $\delta_{i,j}$. The endpoints of $\mathcal{R}$ are different vertices of T; we can suppose (unless we can change coordinate system) that they belong to the $y$-axis in the case they both belong to the same edge of T, or, otherwise, that one endpoint is on the $x$-axis and one on the $y$-axis.

In the first situation (resp. the second) the path $\overline{\mathcal{R}}=\mathcal{R}\cup\sigma_y(\mathcal{R})$ (resp. $\overline{\mathcal{R}}=\mathcal{R}^{\#}$) is a closed path of integer segments having same type.

Let $E$ be an integer point of $\mathcal{R}$ of type $\delta_i$, and $\alpha_i$, for $i=1,\dots,4$, be the algebraic index of the symmetric copy of $E$ in the $i$-th quadrant (as described in Lemma 3.5). Denote by $\overline{\mathcal{Z}}$ the part of $T^{\#}\setminus\overline{\mathcal{R}}$ which is homeomorphic to a disk; let $\vec{z}$ be a vector with even coordinates such that the translation $tr(\overline{\mathcal{R}})$, in $\mathbb{R}^2$, of $\overline{\mathcal{R}}$ by the vector $\vec{z}$, contains no integer points with negative coordinates. If $tr(\overline{\mathcal{R}})$ is not contained in $T$, consider the smallest $m'$ such that the triangle $T'$ of vertices $(0,0),(0,m'),(m',0)$ contains $tr(\overline{\mathcal{R}})$. We can regard $\overline{\mathcal{R}}$ (resp $\overline{\mathcal{Z}}$) as a translation of the cycle $tr(\overline{\mathcal{R}})$ of biparity $\delta_{i,j}$ (resp. of the zone $\mathcal{Z}$ of $tr(\overline{\mathcal{R}})$) of the triangle $T'$.

If $A$ is associated to a pair $(\Gamma,\varepsilon)$, let $(\Gamma^{\#},\varepsilon^{\#})$ be the extended triangulation and distribution of $T^{\#}$. Equip $\mathcal{Z}$ with the triangulation $tr(\Gamma^{\#}|_{\overline{\mathcal{Z}}})$ and assign to a vertex $v$ of this triangulation, the sign of the vertex which corresponds to $v$ via translation. Construct the PL-curve $K$ associated to $\mathcal{Z}$. Consider the point $tr(E)$, its symmetric copies and assign to the point in the $i$-th quadrant the algebraic value $\alpha_i$. In this situation the cycle $tr(\overline{\mathcal{R}})$ satisfies Propositions 2.11 and 4.6.

In the first case we have that $\mathcal{Z}^{\#}$ gives exactly two copies of $\overline{\mathcal{Z}}^{\#}$, then:

$$2\int_{\overline{\mathcal{Z}}^{\#}\setminus A} ind_A^2(x)\;d\chi(x)=\int_{\mathcal{Z}^{\#}\setminus K} ind_K^2(x)\;d\chi(x)$$

In the second case $\overline{\mathcal{Z}}$ coincides with $\overline{\mathcal{Z}}^{\#}$ and $\mathcal{Z}^{\#}$ gives four copies of $\overline{\mathcal{Z}}$, therefore:

$$4\int_{\overline{\mathcal{Z}}^{\#}\setminus A} ind_A^2(x)\ d\chi(x) = \int_{\mathcal{Z}^{\#}\setminus K} ind_K^2(x)\ d\chi(x)$$

then the statement follows from the proof given in the case of a cycle and the proof of Rokhlin's formula is now complete. □

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