Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 45 (2004), No. 1, 217-224.

The Problem of Polygons with Hidden Vertices

Joseph M. Ling

University of Calgary, Calgary, Alberta, Canada T2N 1N4

Abstract. G. Ewald proved that it is possible for a polygon(al path) in $\mathbf{R}^3$ to hide all its vertices behind its edges from the sight of a point M not on the polygon. Ewald also stated that it takes at least 8 vertices to do the job and constructed an example with 14 vertices. It was then suggested that the least number of vertices $n_{\min}$ for such a configuration is closer to 14 than to 8. In this paper, we shall prove that $11 \le n_{\min} \le 12$.

Let $P = [P_1 P_2 \dots P_n]$ be a polygon without self intersection in $\mathbf{R}^3$. So, $P_1, \dots, P_n$ are distinct points, and P is the union of the (closed) line segments $[P_1, P_2], [P_2, P_3], \dots, [P_{n-1}, P_n]$ such that for all $i \ne j$, $[P_{j-1}, P_j] \cap (P_{i-1}, P_i) = \emptyset$, where (P_{i-1}, P_i) denotes the relative interior of of $[P_{i-1}, P_i]$. The points $P_1, \dots, P_n$ are called the vertices of the polygon, and the line segments $[P_{i-1}, P_i]$ the edges.

Ewald [1] proved that there is a polygon and a point M (not on the polygon) such that all vertices are hidden from the sight of M, i.e., for all i, there exists j such that the line segments $[M, P_i]$ and $[P_{j-1}, P_j]$ intersect at a point in (P_{j-1}, P_j). More specifically, Ewald [1] gave an example of such a configuration with $n = 14$. Ewald asked for the smallest number of vertices $n_{\min}$ for which such a configuration exists, argued that $8 \le n_{\min} \le 14$,and suggested that it should be closer to 14 than to 8. We shall prove that $11 \le n_{\min} \le 12$. Here, let us first mention an example with 12 vertices:

$$
\begin{aligned}
M &= (0,0,0) \\
P_1 &= \left(19, \frac{17}{2}, 5\right) = 5P_8 + \frac{7}{2}P_9
\end{aligned}
$$

0138-4821/93 \$ 2.50

$$\begin{aligned}
P_2 &= (-19,-8,-5) = 11P_6 + 3P_7 \\
P_3 &= (0,0,1) = P_7 + \frac{19}{3}P_8 \\
P_4 &= (0,1,0) = 2P_1 + 2P_2 \\
P_5 &= (1,0,0) = 4P_{10} + P_{11} \\
P_6 &= (0,1,1) = P_3 + P_4 \\
P_7 &= \left(-\frac{19}{3},-\frac{19}{3},-\frac{16}{3}\right) = \frac{4}{3}P_{11} + \frac{1}{3}P_{12} \\
P_8 &= (1,1,1) = P_5 + P_6 \\
P_9 &= (4,1,0) = P_4 + 4P_5 \\
P_{10} &= (5,2,1) = P_8 + P_9 \\
P_{11} &= (-19,-8,-4) = P_2 + P_3 \\
P_{12} &= (57,13,0) = 13P_4 + 57P_5
\end{aligned}$$

Thus, $n_{\min} \le 12$. In what follows, we shall prove that $n_{\min} \ge 11$.

To facilitate our discussion, we shall also denote the vertices by $Q_1, \ldots, Q_n$, without any implication that the endpoints of the edges $[Q_i, Q_j]$ should have consecutive integral indices. Thus, $Q_1, \ldots, Q_n$ is a permutation of $P_1, \ldots, P_n$, and $[Q_i, Q_j]$ is an edge of the polygon $[P_1P_2 \ldots P_n]$ if and only if there is some k such that $\{Q_i, Q_j\} = \{P_{k-1}, P_k\}$, but it is not necessary that $|i-j| = 1$. However, we note that as $P_1, \ldots, P_n$ are distinct, so are $Q_1, \ldots, Q_n$.

We say that $[Q_i, Q_j]$ blocks Q_k if the line segment $[M, Q_k]$ intersects $[Q_i, Q_j]$ at a point in (Q_i, Q_j), and that Q_i helps block Q_k if for some vertex Q_j, $[Q_i, Q_j]$ is an edge of the polygon which blocks Q_k. We denote the point of intersection of $[M, Q_k]$ and $[Q_i, Q_j]$ by R_k in case the values of i and j are clear from the context. (See Figure 1) Sometimes, we find it convenient to say that a set S of points blocks or does not block a point Q according as whether or not there is some line segment with endpoints in S that blocks Q.

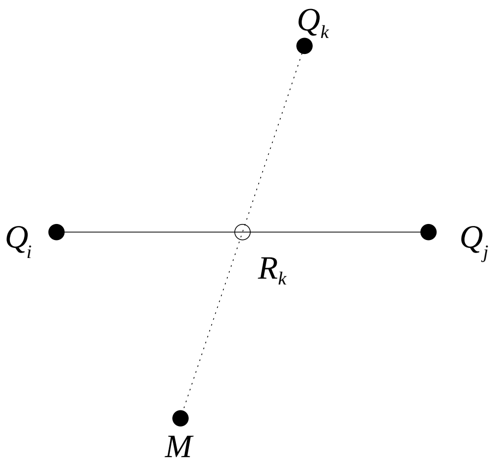

Figure 1

The plane that passes through three noncollinear points A, B, C will be denoted by π_{ABC}. Thus, $\pi_{ABC} = \pi_{ACB} = \pi_{BCA}$, etc. If A happens to be some Q_j, we also write π_{ABC} as π_{jBC}, etc. Thus, π_{ijk} is the plane that passes through Q_i, Q_j, Q_k, and π_{Mij} is the plane that pases through M, Q_i, Q_j. We find it convenient to write $\pi_{Mijk\ldots}$ for the plane, when exists, that passes through $M, Q_i, Q_j, Q_k, \ldots$.

Each plane π determines two closed half-spaces in $\mathbf{R}^3$, denoted by $\pi^{\geq}$ and $\pi^{\leq}$, with common boundary π. We denote the corresponding open half-spaces by $\pi^{>}$ and $\pi^{<}$, respectively. We note that the choice of designations $<$ and $>$ for the pair of half-spaces is arbitrary, and the same applies to $\leq$ and $\geq$.

The following facts will be used frequently. We omit their easy proofs.

1. If (P, Q) intersects π, then $P \in \pi^{>} \Rightarrow Q \in \pi^{<}$ (and $P \in \pi^{<} \Rightarrow Q \in \pi^{>}$). We also describe this scenario by saying that (P, Q) pierces through π.
2. If Q_i helps block Q_j, then Q_j does not help block Q_i.
3. If $M \in \pi$ and $Q \in \pi^{>}$, then Q is not blocked by any line segments with endpoints in $\pi^{\leq}$.
4. If $M \in \pi$ and if Q_k is blocked by $[Q_i, Q_j]$, where $(Q_i, Q_j) \subseteq \pi^{>}$, then $Q_k \in \pi^{>}$.
5. If all the vertices of a polygon are hidden from sight of M, then M must lie in the interior of the convex hull of the polygon. (This was proved in [1].)

Below, we shall assume that $[P_1 \ldots P_n]$ is a polygon whose vertices are hidden from the sight of the point M.

To begin, it is quite easy to see that $n \geq 10$.

Lemma 1. *If π is any plane through M, then there are at least three vertices in $\pi^{>}$ and at least three vertices in $\pi^{<}$.*

Proof. By [1] Theorem 1, not all vertices lie on the same plane. So, there is some vertex $P \in \pi^{>}$, say. But then, since M lies in the interior of the convex hull of the polygon, there must be some vertex Q in $\pi^{<}$. Since P is blocked from M, and since points in $\pi^{\leq}$ are not sufficient to block P, there must be some P' in $\pi^{>}$ which helps block P. Now, as P' helps block P, P cannot help block P'. Thus, there must be yet another vertex P'' in $\pi^{>}$ which helps block P'. Thus, $\pi^{>}$ contains at least three vertices. The same argument applies to $Q \in \pi^{<}$. □

Theorem 1. $n_{\min} \geq 10$.

Proof. Since the polygon $[P_1 \ldots P_n]$ has n vertices hidden behind $n-1$ edges, at least one of the edges must block more than one vertex. Suppose Q_1, Q_2 are two vertices that are blocked by $[Q_3, Q_4]$. Then M, Q_1, Q_2, Q_3, Q_4 all lie on the same plane π. By Lemma 1, there are at least three vertices in $\pi^{>}$ and at least three vertices in $\pi^{<}$. So, the total number of vertices is at least $4 + 3 + 3 = 10$. □

We need another lemma to establish $n_{\min} \geq 11$.

Lemma 2. *If Q_i, Q_j, Q_k are the only vertices in $\pi^>$ for some plane π through M, then Q_i, Q_j, Q_k will help block one another in a cyclical manner, i.e., either*

$$Q_i \text{ helps block } Q_k,\ Q_k \text{ helps block } Q_j,\ Q_j \text{ helps block } Q_i,$$

or

$$Q_i \text{ helps block } Q_j,\ Q_j \text{ helps block } Q_k,\ Q_k \text{ helps block } Q_i.$$

Proof. One of Q_i and Q_j must help block Q_k. If Q_i helps block Q_k, then Q_k does not help block Q_i, and so Q_j must help block Q_i. Then Q_i does not help block Q_j, and so, Q_k must help block Q_j. We arrive at the first scenario. Similarly, if Q_j helps block Q_k, we arrive at the second scenario. (We may imagine the vertices Q_i, Q_j, Q_k together with their blocking edges forming something like a tripod. See Figure 2.) □

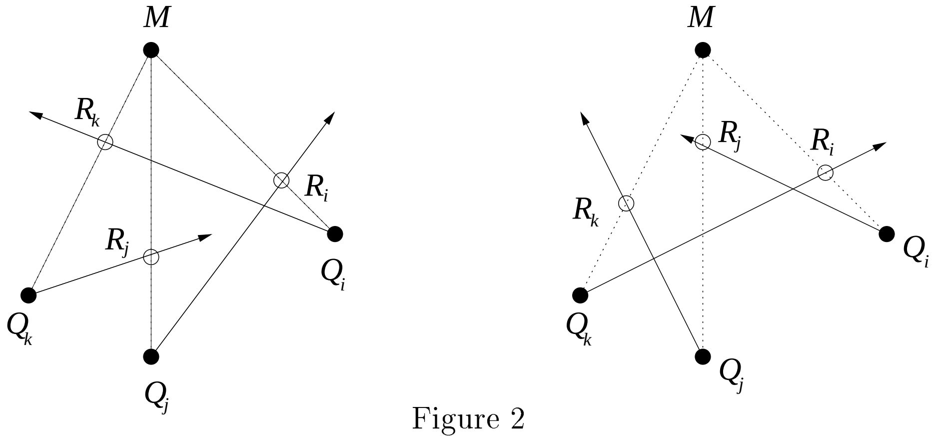

Figure 2

Theorem 2. $n_{\min} \geq 11$.

Proof. As in the proof of Theorem 1, we suppose that Q_1, Q_2 are two vertices that are blocked by $[Q_3, Q_4]$, and we denote the plane π_{M1234} by π. We look at various ways Q_3 and Q_4 may be blocked. The cases are separated according to whether the line segments blocking them lie on or pierce through the plane π. In each case, we prove that it is impossible for the configuration to have just 10 vertices. In view of Lemma 1, this implies that no plane through M can contain more than 4 vertices.

CASE 1. Q_3 and Q_4 are blocked by two edges piercing through the plane $\pi = \pi_{M1234}$, and that the two edges share a vertex. That is, there are points $Q_5, Q_6, Q_7 \notin \pi$ such that $[Q_5, Q_6]$ blocks Q_3 and $[Q_6, Q_7]$ blocks Q_4. (See Figure 3.) Since (Q_5, Q_6) and (Q_6, Q_7) pierce through π, we may assume without loss of generality that $Q_6 \in \pi^<$ and $Q_5, Q_7 \in \pi^>$. In view of Lemma 2, we have vertices $Q_8, Q_{10} \in \pi^<$ and we may assume that

$$Q_6 \text{ helps block } Q_8;\ Q_8 \text{ helps block } Q_{10};\ Q_{10} \text{ helps block } Q_6.$$

Also, there is a vertex $Q_9 \in \pi^>$. Suppose that $Q_1, \dots, Q_{10}$ are all the vertices. Since $[Q_5, Q_6]$ blocks Q_3, $[Q_6, Q_7]$ blocks Q_4, and $[Q_3, Q_4]$ blocks Q_1 and Q_2, π_{567} strictly separates Q_1, Q_2, Q_3, Q_4 from M, i.e., $M \in \pi_{567}^{>}$ and $Q_1, Q_2, Q_3, Q_4 \in \pi_{567}^{<}$. (See Figure 4.) Since

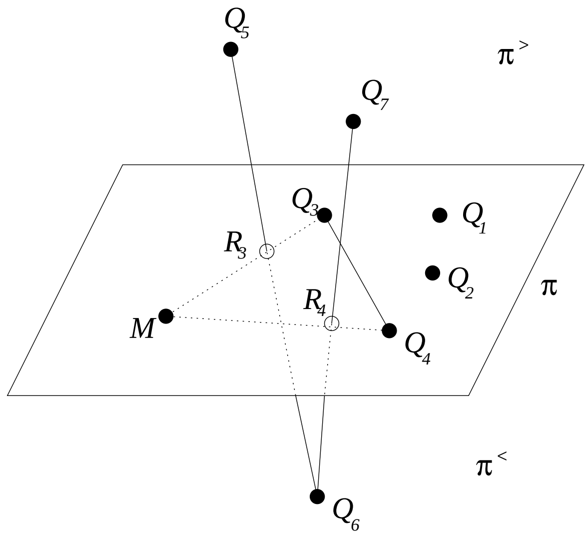

Figure 3

Q_6 helps block Q_8 and the edges in the polygon that connect to Q_6 are $[Q_5, Q_6]$ and $[Q_6, Q_7]$, Q_8 must be blocked by one of $[Q_5, Q_6]$ and $[Q_6, Q_7]$. In either case, the plane π_{567} strictly separates Q_8 from M. So, the vertices $Q_1, \ldots, Q_8$ are all in $\pi_{567}^{\leq}$ whereas $M \in \pi_{567}^{\geq}$. So, if $\pi_{//}$ denotes the plane through M parallel to π_{567}, then one of the half-spaces $\pi_{//}^{>}$ and $\pi_{//}^{<}$ contains no more than two vertices Q_9 and Q_{10}. This contradicts Lemma 1. This completes CASE 1.

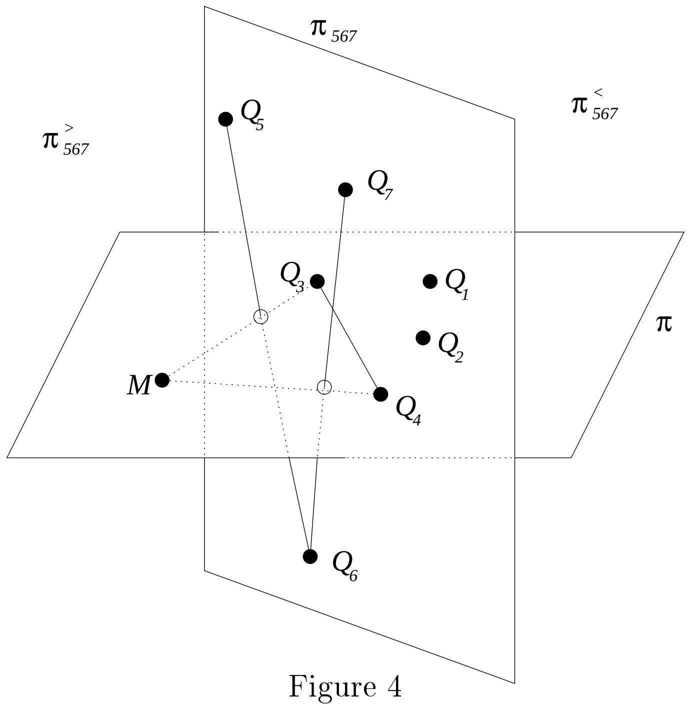

Figure 4

CASE 2. Q_3 and Q_4 are blocked by two edges piercing through the plane $\pi = \pi_{M1234}$, and that they do not share a common vertex. That is, there are (distinct) points Q_5, Q_6, Q_7, Q_8 not on π such that $[Q_5, Q_6]$ blocks Q_3 and $[Q_7, Q_8]$ blocks Q_4. (See Figure 5.)

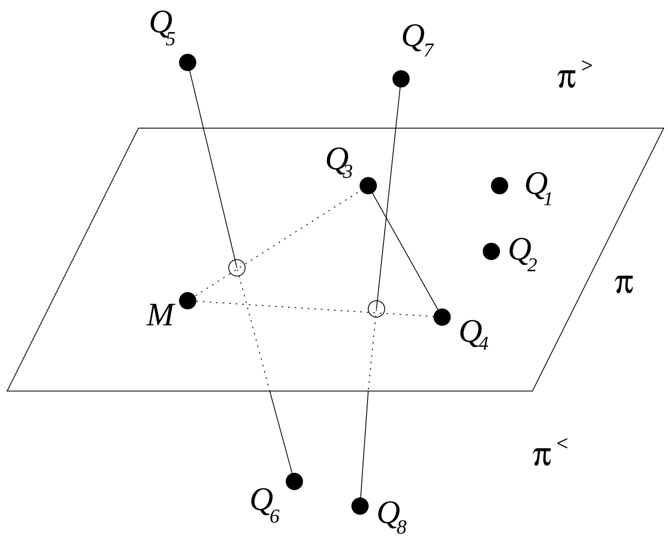

Figure 5

We may assume that $Q_5, Q_7 \in \pi^>$ and $Q_6, Q_8 \in \pi^<$. In view of Lemma 1, we have Q_9 in $\pi^>$, Q_{10} in $\pi^<$, and we assume that

$$Q_5 \text{ helps block } Q_7; Q_7 \text{ helps block } Q_9;\ Q_9 \text{ helps block } Q_5.$$

Now, suppose that these are all the vertices. We shall derive a contradiction.

Consider the plane π_{M478}. Since $Q_7 \notin \pi = \pi_{M34}$, we have $Q_3 \notin \pi_{M478}$. Suppose $Q_3 \in \pi^<_{M478}$. Then $(Q_3, Q_4) \subseteq \pi^<_{M478}$. It follows that $Q_1, Q_2 \in \pi^<_{M478}$. Also, $R_3 = (M, Q_3) \cap [Q_5, Q_6] \in \pi^<_{M478}$. This implies that at least one of Q_5, Q_6 is in $\pi^<_{M478}$. But then, $Q_1, Q_2, Q_3, Q_4, Q_7, Q_8$, and at least one of Q_5, Q_6 are in $\pi^{\leq}_{M478}$. It follows by Lemma 1 that $Q_9, Q_{10} \in \pi^>_{M478}$, and exactly one of Q_5 and Q_6 must be in each of $\pi^>_{M478}$ and $\pi^<_{M478}$. As a summary,

$$\begin{aligned} Q_1, Q_2, Q_3, \text{ and one of } Q_5, Q_6 &\in \pi^<_{M478}; \\ Q_9, Q_{10}, \text{ and one of } Q_5, Q_6 &\in \pi^>_{M478}. \end{aligned}$$

By a similar argument,

$$\begin{aligned} Q_1, Q_2, Q_4, \text{ and one of } Q_7, Q_8 &\in \pi^<_{M356}; \\ Q_9, Q_{10}, \text{ and one of } Q_7, Q_8 &\in \pi^>_{M356}. \end{aligned}$$

Subcase 2(a). $Q_5 \in \pi^<_{M478}$ (and so $Q_6 \in \pi^>_{M478}$).
As Q_5 helps block Q_7, Q_7 must be blocked by $[Q_5, Q_i]$ for some $Q_i \in \pi^>_{M478}$. So, $i = 6, 9$, or 10. Since Q_7 helps block Q_9, Q_9 does not help block Q_7, and so, $i \neq 9$. Also, since $Q_5, Q_6 \in \pi_{M356}$ but $Q_7 \notin \pi_{M356}$, $[Q_5, Q_6]$ does not block Q_7. So, $i \neq 6$, and we conclude that $i = 10$, and so

$$[Q_5, Q_{10}] \text{ blocks } Q_7.$$

This in turn implies that

$$Q_7 \in \pi^>_{M356}, \text{ and so } Q_8 \in \pi^<_{M356}.$$

Now, Q_9, Q_{10}, Q_7 are the only vertices in $\pi^{>}_{M356}$, and as Q_{10} helps block Q_7, it follows from Lemma 2 that

$$Q_7 \text{ helps block } Q_9, \text{ and } Q_9 \text{ helps block } Q_{10}.$$

Then, as the only vertices in $\pi^{>}_{M478}$ are Q_9, Q_{10}, Q_6, and as Q_9 helps block Q_{10}, it follows from Lemma 2 again that

$$Q_{10} \text{ helps block } Q_6, \text{ and } Q_6 \text{ helps block } Q_9.$$

Next, as the only vertices in $\pi^{<}$ are Q_6, Q_8, Q_{10}, and as Q_{10} helps block Q_6, it follows once more from Lemma 2 that

$$Q_6 \text{ helps block } Q_8, \text{ and } Q_8 \text{ helps block } Q_{10}.$$

So, Q_6 helps block the vertices Q_3, Q_8, and Q_9. As there are only two edges connecting Q_6, one of them is going to block two of Q_3, Q_8, Q_9.

Sub-subcase 2(a)(i): $[Q_6, Q_i]$ blocks both Q_8 and Q_9 for some i.
Consider the plane π_{Mi689}. Certainly, $i \neq 6, 8, 9$. As $Q_6 \in \pi^{<}$ and $Q_9 \in \pi^{>}$, $i \neq 1, 2, 3, 4, 10$. If $i = 5$, then $[Q_5, Q_6]$ blocks Q_3, Q_8, Q_9, and so π_{M56} contains at least five vertices, a contradiction. If $i = 7$, then $[Q_6, Q_7]$ blocks Q_4, Q_8, Q_9 and we have five vertices on the same plane π_{M67}, a contradiction again.

Sub-subcase 2(a)(ii): $[Q_6, Q_i]$ blocks Q_3 and one of Q_8 and Q_9, for some i.
Since $[Q_5, Q_6]$ blocks Q_3, we have $Q_i \in \pi_{M356}$, and so one of Q_8 or Q_9 must be on π_{M356}. But this contradicts the basic setup of Subcase 2(a), in which $Q_8 \in \pi^{<}_{M356}$ and $Q_9 \in \pi^{>}_{M356}$.

This completes Subcase 2(a).

Subcase 2(b). $Q_5 \in \pi^{>}_{M478}$ (and so $Q_6 \in \pi^{<}_{M478}$).
Since Q_5, Q_9, Q_{10} are the only vertices in $\pi^{>}_{M478}$, and as Q_9 helps block Q_5, it follows from Lemma 2 that Q_5 helps block Q_{10}, and Q_{10} helps block Q_9. But then Q_5 helps block Q_3, Q_7, Q_{10}. As Q_5 is connected to at most two vertices, there must be some i such that $[Q_5, Q_i]$ blocks two of the vertices Q_3, Q_7, Q_{10}.

Sub-subcase 2(b)(i): $[Q_5, Q_i]$ blocks Q_7 and Q_{10}.
Certainly, $i \neq 5, 7, 10$. Since $Q_5 \in \pi^{>}$ and $Q_{10} \in \pi^{<}$, it is easy to see that $i \neq 1, 2, 3, 4, 9$. Since $Q_5 \in \pi^{>}_{M478}$, $[Q_5, Q_8]$ does not block Q_7. So, $i \neq 8$. Finally, if $[Q_5, Q_6]$ blocks Q_7, Q_{10}, then π_{M356} contains five vertices, a contradiction. Thus, $i \neq 6$.

Sub-subcase 2(b)(ii): $[Q_5, Q_i]$ blocks Q_3 and one of Q_7, Q_{10}.
One of Q_7, Q_{10} will then be in π_{M356}. This contradicts the basic setup of Subcase 2(b) in which $Q_7 \notin \pi_{M356}$ and $Q_{10} \in \pi^{>}_{M356}$.

This completes Subcase 2(b), and hence CASE 2.

CASE 3. Q_3 and Q_4 are blocked by two edges at least one of which lies on the plane $\pi = \pi_{M1234}$.
The plane π must contain a fifth vertex, a contradiction.

This completes CASE 3, and hence the proof of Theorem 2. □

Remarks. 1. We conjecture that $n_{\min} = 11$.

2. Our proof relies on the fact that the polygon does not close up onto itself: there are $n-1$ edges connecting n vertices. In the example given in Ewald [1], the line segment joining P_1 and P_{14} does not intersect any of the edges $[P_{i-1}, P_i]$, $2 \le i \le 14$. Thus, we get a closed polygon with all vertices hidden from the sight of M. For such a configuration, a pigeon-hole argument does not apply to allow us to conclude that some edge blocks more than one vertex. The natural starting point is then with one edge blocking one vertex. The total number of vertices is at least $3+3+3=9$. The above argument may be repeated to conclude that the number of vertices is at least 10. On the other hand, in our example with 12 vertices, the polygon can also be closed up without introducing a self-intersection Thus, for closed polygons, we may say $10 \le n_{\min} \le 12$.

Reference

[1] Ewald, G.: *Polygons with Hidden Vertices.* Beiträge Algebra Geom. (Contributions to Algebra and Geometry) **42**(2) (2001), 439–442. Zbl 0996.52006

Received May 15, 2003