Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 45 (2004), No. 1, 267-273.

Lattice Packings with Gap Defects are not Completely Saturated

Dedicated to the memory of Bernulf Weißbach

Greg Kuperberg[1] **Krystyna Kuperberg**[2] **Włodzimierz Kuperberg**

Department of Mathematics, University of California, Davis, CA 95616
e-mail: greg@math.ucdavis.edu

Department of Mathematics, Auburn University, Auburn, AL 36849
e-mail: kuperkm@math.auburn.edu kuperwl@math.auburn.edu

Abstract. We show that a honeycomb circle packing in $\mathbb{R}^2$ with a linear gap defect cannot be completely saturated, no matter how narrow the gap is. The result is motivated by an open problem of G. Fejes Tóth, G. Kuperberg, and W. Kuperberg, which asks whether of a honeycomb circle packing with a linear shift defect is completely saturated. We also show that an fcc sphere packing in $\mathbb{R}^3$ with a planar gap defect is not completely saturated.

MSC 2000: 52C15, 52C17
Keywords: circle packing, saturated packing, completely saturated packing

1. Introduction

A packing of unit spheres in $\mathbb{R}^n$ is *k-saturated* if it is not possible to replace $k-1$ spheres by k and still have a packing; it is *completely saturated* if it is k-saturated for all k [2]. Every completely saturated packing has maximum density, and every periodic packing with maximum density is completely saturated [2]. Thus the densest lattice packing of circles in $\mathbb{R}^2$ (the *honeycomb* circle packing) is completely saturated. This motivates the question: Is the honeycomb circle packing the only one which is completely saturated?

[1] G. Kuperberg's research was supported in part by the National Science Foundation grant #DMS-0072342
[2] K. Kuperberg's research was supported in part by the National Science Foundation grant #DMS-0204081

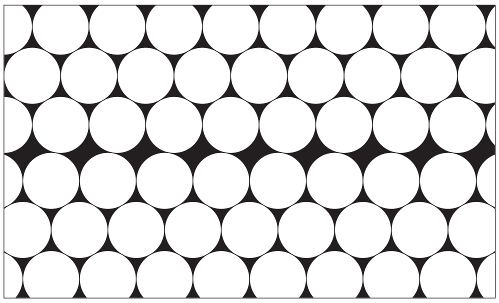

Figure 1. Honeycomb circle packing with a shift defect

G. Fejes Tóth, G. Kuperberg, and W. Kuperberg [1] asked whether a honeycomb circle packing with a linear shift defect, as in Figure 1, is completely saturated. This question appears to be closely related to the (still open) conjecture of L. Fejes Tóth [3] asserting that the same packing is *solid*, meaning that if you obtain another packing by rearranging finitely many circles, it is congruent to the original one.

In this paper we will instead consider packings with a linear or planar gap defect in which the spheres on opposite sides of the gap do not touch.

Definition. *Let $\mathcal{S}$ be a sphere packing in $\mathbb{R}^n$, let $H \subset \mathbb{R}^n$ be a hyperplane, let H^+ be a closed half-space bounded by H, and let $\vec{v}$ be a vector perpendicular to H in the direction of H^+. Let $\widetilde{\mathcal{S}}$ be the packing obtained from $\mathcal{S}$ by moving all spheres with centers in H^+ by the vector $\vec{v}$. (Note that $\widetilde{\mathcal{S}}$ is a packing because the motion does not decrease the distances between sphere centers.) Then $\widetilde{\mathcal{S}}$ is a sphere packing with a hyperplane gap defect and $\|\vec{v}\|$ is the width of the gap.*

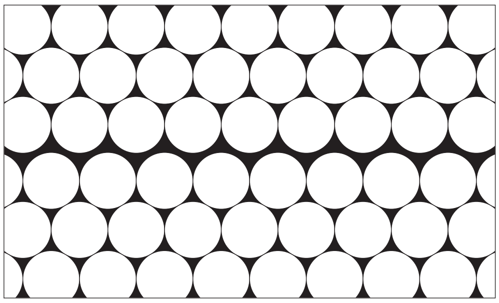

Figure 2. Honeycomb circle packing with a gap defect

Figures 2 and 5 show a honeycomb circle packing with a linear gap defect.

Theorem 1. *A honeycomb circle packing in $\mathbb{R}^2$ with a linear gap defect is not completely saturated, regardless of the width and direction of the gap. An fcc (face-centered cubic lattice) packing of spheres in $\mathbb{R}^3$ with an arbitrary planar gap defect is also not completely saturated.*

Since the honeycomb circle packing in $\mathbb{R}^2$ [6] and the fcc sphere packing in $\mathbb{R}^3$ [4] are the unique densest lattice sphere packings in 2 and 3 dimensions, we obtain the following corollary.

Corollary 2. *In $\mathbb{R}^2$ and $\mathbb{R}^3$, no lattice sphere packing with a hyperplane gap defect is completely saturated.*

Remark. In contrast to the above corollary, in which the width of the gap can be arbitrarily small, the corresponding problem for circle packings with a shift defect is still open, even with the largest possible size of the defect.

2. Honeycomb packing with gap defects

2.1. Gap along a lattice line

By a *lattice line* we understand here a line determined by two closest points of the lattice.

Let $\mathcal{C}$ be the honeycomb circle packing in $\mathbb{R}^2$ with a circle center at every point of the form $(2i+j, \sqrt{3}j)$, with $i, j \in \mathbb{Z}$. Let $d > 0$ and move all circles with center on or above the x-axis by the vector $\vec{v} = (0, d)$ to obtain the packing $\widetilde{\mathcal{C}}$.

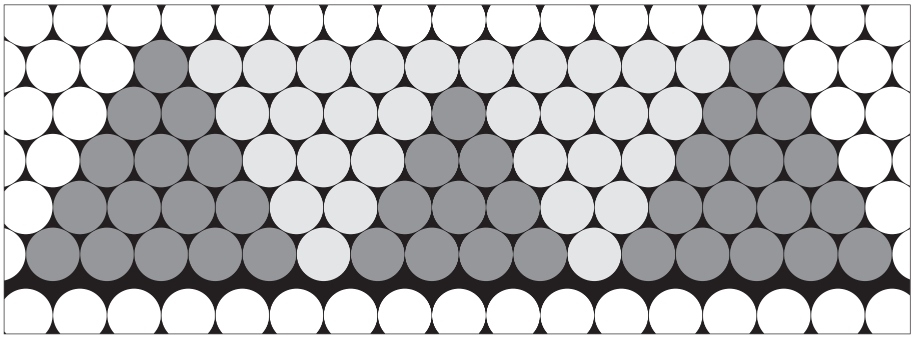

Figure 3. Five triangular blocks

We will show that $\widetilde{\mathcal{C}}$ is not completely saturated. It suffices to rearrange finitely many circles in $\widetilde{\mathcal{C}}$ to create enough empty space for an extra circle. To achieve that, we will widen the gap in a sufficiently large finite region. If we can create a gap of width $d' = d + \delta$, with δ a non-decreasing function of d, then by repeating the operation we can create a gap of any width. In the end, if the width is at least 2, there is room for another circle.

We define a *triangular block of size n* to be a set of $n(n+1)/2$ circles whose centers lie in an equilateral triangle of edge length $2n-2$, *i.e.*, with n circles along each edge. Note that a triangular block in $\widetilde{\mathcal{C}}$ must lie entirely on one side of the gap. If the block points up, we will call it a Δ_n-block; otherwise we will call it a ∇_n-block. The rearrangement to widen the gap is as follows: Take two Δ_n-blocks, two ∇_n-blocks, and one Δ_{n-1}-block that form a trapezoid, as in Figure 3. Move the two outer Δ_n-blocks down by a distance of $d_1 \leq d$, then move the two ∇_n-blocks at 60 degree angles from vertical by d_2, then move the middle Δ_n-block up by d_3, as in Figure 4. If we choose the distances d_1, d_2, and d_3 to maximize d_3, then d_3 is at least weakly monotonic in d, because increasing d relaxes the constraints on the other parameters. Thus the gap is widened by δ monotonic in d, as desired. Since the parameter n is arbitrary, it can be taken large enough to repeat the operation with smaller blocks to reach any desired width.

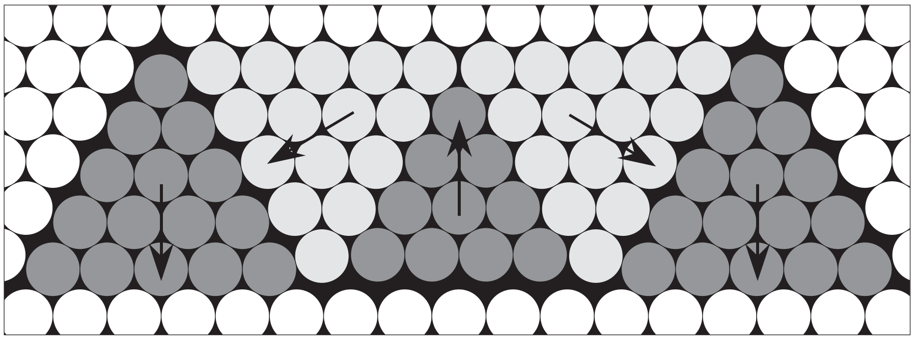

Figure 4. The rearrangement in three moves

2.2. Gap along an arbitrary line

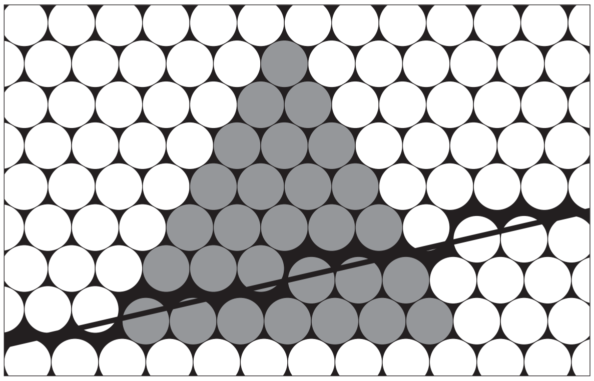

Figure 5. A triangular block split by the gap

Let $\mathcal{C}$ be the above honeycomb circle packing and let ℓ be any line, and now let $\widetilde{\mathcal{C}}$ be the packing $\mathcal{C}$ with a gap along ℓ. Our plan in this case is to create another (finite) gap parallel to a lattice line of $\mathcal{C}$ and reduce to the case of Section 2.1. The gap can be made any length without sacrificing width. For each $n \geq 3$, we can choose a triangular block T in $\mathcal{C}$ of length n so that ℓ intersects the triangle formed by its circle centers, moreover so that ℓ does not meet the vertices. Then ℓ divides T into two sub-blocks T_1 and T_2, one of which, say T_1, has only one corner of T. In $\widetilde{\mathcal{C}}$, these two sub-blocks are separated by the gap, as in Figure 5. In $\widetilde{\mathcal{C}}$, then, T_1 can be moved towards T_2 to introduce gaps on its other two sides. The length n is arbitrary and the width of the new gaps does not depend on n. This completes the reduction.

3. The fcc packing with a gap

The fcc sphere packing in $\mathbb{R}^3$ decomposes into honeycomb layers, and, as noted by Kepler [5], it also decomposes into square-lattice layers. We prove below that the fcc-lattice sphere packing with a gap defect is never completely saturated, no matter how narrow the gap is and no matter along which plane the gap is formed. As before, we first establish special cases.

3.1. Gap parallel to a square layer

Let $\mathcal{B}$ be the fcc packing in $\mathbb{R}^3$ whose sphere centers are at $(2i+k, 2j+k, \sqrt{2}k)$ with $i, j, k \in \mathbb{Z}$. Let $\widetilde{\mathcal{B}}$ be the packing $\mathcal{B}$ with sphere centers in the upper half-space moved by $\vec{v} = (0, 0, d)$ for some $d > 0$. In this model the square-lattice layers are parallel to coordinate planes of $\mathbb{R}^3$, so $\widetilde{\mathcal{B}}$ has a square-layer gap.

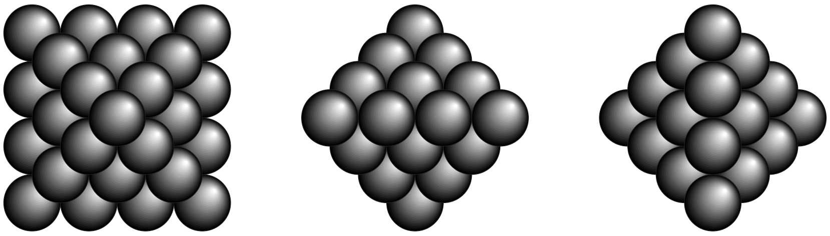

Figure 6. A square pyramid block and two simplex blocks (top view)

To show that $\widetilde{\mathcal{B}}$ is not completely saturated, we will delete finitely many spheres, then rearrange finitely many others to create a void large enough to accommodate more spheres than the ones deleted. We define a *square pyramid block of size* n to be a collection of $n(n+1)(2n+1)/6$ spheres in $\widetilde{\mathcal{B}}$ whose sphere centers lie in a square pyramid of edge length $2n-2$, *i.e.*, with n spheres along each of its edges. We define a *simplex block of size* n to be a collection of $n(n+1)(n+2)/6$ spheres in $\widetilde{\mathcal{B}}$ whose spheres centers lie in a regular simplex of edge length $2n-2$. Both kinds of blocks have n spheres along each edge; see Figure 6 for examples.

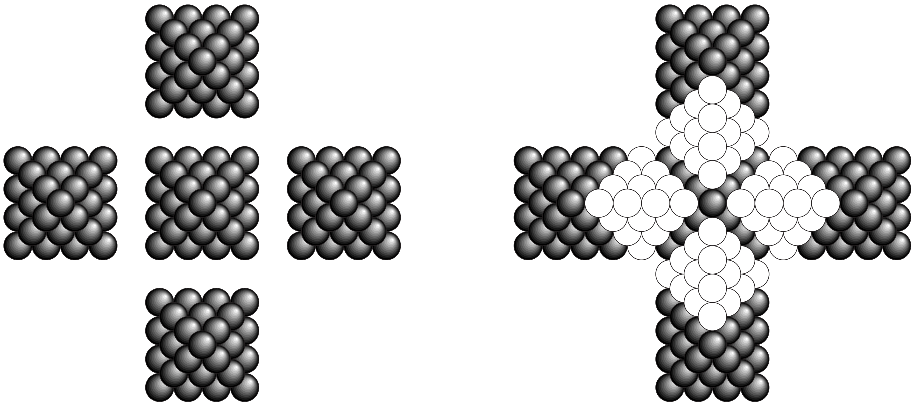

Figure 7. Nine blocks forming a cross gable roof

By analogy with Section 2.1, we will widen the gap by moving 5 square pyramid blocks and 4 simplex blocks forming a cross gable roof, as in Figure 7. The bases of the square pyramid blocks lie on the gap. We move the four outer square pyramid blocks down, then move the simplex blocks down and out, then move the middle square pyramid up after removing its top sphere. As before, we choose the motions in order to maximize the size of the last motion; in this arrangement the last motion is weakly monotonic in d. Thus the gap widens by δ with δ weakly monotonic in d. As before, δ does not depend on n.

As in Section 2.1, n can be large enough to iterate the procedure many times. Let k be the number of iterations needed to widen the gap to 2; k only depends on d. One sphere is deleted in each iteration, for a total of k spheres removed. Since the void at the end grows with n, we can choose n large enough to accommodate these k spheres and then some.

3.2. Gap parallel to a honeycomb layer

Our strategy in this case is to expose a square-layer gap in a large finite region by moving a large block that abuts the honeycomb gap. This reduces the problem to the case considered in Section 3.1. The shape of the block is a cuboctahedron sawed in half to expose a regular hexagon face, as in Figure 8. As the figure indicates, the three square faces lie in square-lattice directions. Moving the block creates gaps along these faces.

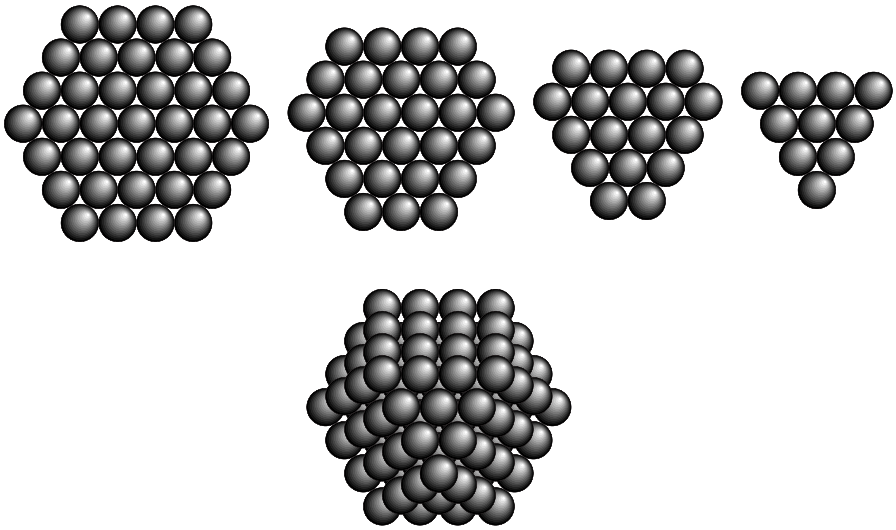

Figure 8. A half-cuboctahedron block, exploded into layers and assembled

3.3. Gap parallel to an arbitrary plane

Assume now that the plane of the gap does not belong to any of the four parallel classes of the honeycomb layers. In a manner similar to that of Section 2.2, we construct a large tetrahedron block which is cut by the gap into two sub-blocks T_1 and T_2. If the plane is not parallel to a square-lattice layer, then we can arrange that T_1 contains one vertex of the tetrahedron and T_2 contains three vertices. The sub-block T_1 can then be moved towards the opposite face, which creates a honeycomb-layer gap and reduces the problem to that of Section 3.1.

4. Open problems and conjectures

1. Is there a sphere packing in $\mathbb{R}^n$ $(n > 1)$ that has a gap defect and still is completely saturated?
2. Let $\mathcal{S}$ be the densest lattice packing of spheres in $\mathbb{R}^4$, which is composed of fcc layers. Let H be a hyperplane in an fcc direction and let $\widehat{\mathcal{S}}$ be a sphere packing obtained from $\mathcal{S}$ by introducing a gap defect along H. We conjecture that $\widehat{\mathcal{S}}$ is not completely saturated.
3. Let $\mathcal{T}$ be a tiling of $\mathbb{R}^2$ with regular hexagons, and let $\widehat{\mathcal{T}}$ be a packing obtained from $\mathcal{T}$ by introducing a gap. If the gap is sufficiently narrow, is $\widehat{\mathcal{T}}$ completely saturated?

References

[1] Fejes Tóth, G.; Kuperberg, G.; Kuperberg, W.: *Highly saturated packings and reduced coverings*. Monatsh. Math. **125**(2) (1998), 127–145. Zbl 0901.52020

[2] Fejes Tóth, G.; Kuperberg, W.: *Packing and covering with convex sets*. Chapter 3.3 in: Handbook of Convex Geometry (P. M. Gruber and J.M. Wills, Eds.), Elsevier 1993, 799–860. Zbl 0789.52018

[3] Fejes Tóth, L.: *Solid circle-packings and circle-coverings*. Studia Sci. Math. Hungar. **3** (1968), 401–409. Zbl 0167.20201

[4] Gauss, C.F.: *Untersuchungen über die Eigenschaften der positiven ternären quadratischen Formen von Ludwig August Seber*. Göttingische gelehrte Anzeigen, Juli 9 [J. Reine Angew. Math. **20** (1840), 312–320 = Werke, Vol. 2 (Königliche Gesellschaft der Wissenschaften, Göttingen 1876), 188–196].

[5] Kepler, J.: *On the six-cornered snowflake*. Oxford Clarendon Press, Oxford 1966 (foreword by L.L. White) (English translation of Kepler's Latin essay printed in 1601).

[6] Lagrange, J. L.: *Recherches d'arithmetique*. Nouv. Mem. Acad. Roy. Sc. Belle Letteres, Berlin 1773, 265–312 = Oeuvres III, 693–758.

Received May 10, 2003; revised version September 2, 2003