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Critical Point Theorems on Finsler Manifolds

László Kozma Alexandru Kristály Csaba Varga [1]

Institute of Mathematics and Informatics, University of Debrecen
H-4010 Debrecen, Pf. 12, Hungary
e-mail: kozma@math.klte.hu

Faculty of Mathematics and Informatics, Babes-Bolyai University
Str. Kogalniceanu nr.1, R-3400 Cluj–Napoca, Romania
e-mail: akristal@math.ubbcluj.ro csvarga@math.ubbcluj.ro

Abstract. In this paper we consider a dominating Finsler metric on a complete Riemannian manifold. First we prove that the energy integral of the Finsler metric satisfies the Palais-Smale condition, and ask for the number of geodesics with endpoints in two given submanifolds. Using Lusternik-Schnirelman theory of critical points we obtain some multiplicity results for the number of Finsler-geodesics between two submanifolds.

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Let M be a finite dimensional manifold and let M_1 respectively M_2 be two submanifolds of M. Many authors studied the problem in the Riemannian case (see [8], [19], [13], [21], [18] and [20]):

> *What is the number of geodesics with endpoints in M_1 and M_2 and which are orthogonal to M_1 and M_2 ?*

The purpose of our study is to examine the existence and the number of Finsler-geodesics joining orthogonally M_1 and M_2 when a Finsler metric is given on a complete Riemannian manifold. The existence of closed geodesics in the case of Finsler space has been studied by F. Mercuri, see [11]. Following its considerations we shall extend some of the Riemannian

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results of K. Grove [8], J.P. Serre [19] and J.T. Schwartz [18] for geodesics of Finsler spaces with endpoints in two given submanifolds. Using the methods of D. Motreanu [13], T. Wang [21] and Cs. Varga – G. Farkas [20] it is possible to extend these results for locally convex cases.

In the first section, following [11] we describe the Riemann-Hilbert manifold $\Lambda_N M$ of absolutely continuous maps from the unit interval $I = [0, 1]$ to M with endpoints in $N \subset M \times M$. The second section is devoted to the study of energy integral of a Finsler metric. We consider only such a Finsler metric which dominates the underlying Riemannian structure of the manifold. We show that the energy integral $\tilde{L}$ is of class C^{2-} on $\Lambda_N M$, and the geodesics of the Finsler metric F joining orthogonally M_1 and M_2 are just the critical points of the energy integral $\tilde{L}: \Lambda_{M_1 \times M_2} M \to \mathbb{R}$. In the third section we prove that the energy functional $\tilde{L}: \Lambda_N M \to \mathbb{R}$ of a Finsler metric satisfies the Palais-Smale condition on a complete manifold (Theorem 3). This generalizes the analogous result of [8] for Finsler metrics. In the last section, applying the results of [19] and [18] we deduce some multiplicity results for geodesics of Finsler spaces joining M_1 and M_2.

1. Preliminaries

Let M be an n-dimensional Riemannian manifold and $I = [0, 1]$ the unit interval. Let $c \in C^\infty(I, M)$ and consider the pull-back diagram:

$$\begin{array}{ccc} c^*TM & \xrightarrow{c^*_\pi} & TM \\ \pi^*_c \downarrow & & \downarrow \pi \\ I & \xrightarrow{c} & M \end{array}$$

where $\pi : TM \to M$ is the canonical projection, and

$$c^*TM = \{(t, y) \in I \times TM |\ c(t) = \pi(y)\}$$

$$c^*_\pi(t, y) = y, \qquad \pi^*_c(t, y) = t.$$

The Riemannian metric and connection on M can then be pulled back to a Riemannian metric and a connection on π^*_c and we will denote them by $\langle , \rangle_c$ and ∇_c, respectively.

Let $\Sigma(\pi^*_c) = \{s : I \to c^*TM |\ \pi^*_c \circ s = \mathrm{id}\}$ be the set of all sections of π^*_c, and we consider the following spaces:

$$\begin{aligned} H^0(c^*TM) &= \{X \in \Sigma(\pi^*_c) |\ \|X(t)\|_c \in L^2(I)\} \\ H^1(c^*TM) &= \{X \in \Sigma(\pi^*_c) |\ \nabla_c X \text{ exists and } \nabla_c X \in H^0(c^*TM)\}. \end{aligned}$$

We have that $H^i(c^*TM),\ i = \overline{0, 1}$ is a Hilbert space with respect to the scalar products:

$$\langle X, Y \rangle_0 = \int_I \langle X(t), Y(t) \rangle_c dt$$

$$\langle X, Y \rangle_1 = \langle X, Y \rangle_0 + \langle \nabla_c X, \nabla_c Y \rangle_0.$$

We will denote by $\| \cdot \|_i$ the relative norms and $\| \cdot \|_\infty$ the sup norm in $C^0(c^*TM)$, where $C^k(c^*TM)$ will have the usual meaning for $k = 0, 1, \ldots, \infty$.

Proposition 1. [11] *The following inclusions* $H^1(c^*TM) \hookrightarrow C^0(c^*TM) \hookrightarrow H^0(c^*TM)$ *are continuous. More precisely:*

$$\text{(i)} \quad \textit{if } \ \xi \in C^0(c^*TM), \textit{then } \|\xi\|_0 \leq \|\xi\|_\infty$$

and

$$\text{(ii)} \quad \textit{if } \xi \in H^1(c^*TM), \ \textit{then } \|\xi\|_\infty^2 \leq 2\|\xi\|_1^2.$$

Now, we consider the manifold $L_1^2(I, M)$ of absolutely continuous maps from the unit interval $I = [0, 1]$ to M with locally square integrable derivative. The space $L_1^2(I, M)$ has a natural complete Riemannian-Hilbert structure given by

$$\langle X, Y \rangle_c' = \int_I \langle X_c(t), Y_c(t) \rangle_{c(t)} + \langle \nabla_c X_c(t), \nabla_c Y_c(t) \rangle_{c(t)} dt,$$

where X and Y are arbitrary elements of $T_c L_1^2(I, M) = H^1(c^*TM)$, the set of all absolutely continuous vector fields X along c with square integrable covariant derivative $\nabla_c X$.

Let $P : L_1^2(I, M) \to M \times M$ be the projection, defined by $P(c) = (c(0), c(1))$ for all $c \in L_1^2(I, M)$ and let $N \subset M \times M$ be a submanifold of $M \times M$ of codimension k. From the expression of local coordinate we get that P is submersion. Then we have that $P^{-1}(N)$ is a submanifold of $L_1^2(I, M)$ of codimension k. We denote $P^{-1}(N)$ by $\Lambda_N M$.
Let U be an open set containing the zero section in TM, $c \in C^\infty(I, M)$ and $U_c = (c_\pi^*)^{-1}(U)$. The map $\tilde{\phi}_c : H^1(U_c) \to \Lambda_N M$ given by

$$\tilde{\phi}_c(x)(t) = \exp c_\pi^* x(t)$$

is injective.

For $x \in TM$, $j = 1, 2$ define $(\nabla_j \exp)(x) : T_{\pi(x)}M \to T_{\exp x}M$ by

$$(\nabla_1 \exp)(x)y = (d \exp)(x) \circ (d\pi | T^h TM)^{-1} y,$$

$$(\nabla_2 \exp)(x)y = (d \exp)(x) \circ k(x)^{-1} y,$$

where $k(x) : T_x^v TM \to T_{\pi(x)}M$ is the canonical identification.

For any $c \in \Lambda_N M$, $\dot{c}(t) \in H^0(c^*TM)$. Let

$$H^i(\Lambda_N M^*TM) = \bigcup_{c \in \Lambda_N M} H^i(c^*TM).$$

$c \in \Lambda_N M$ gives a section $\partial_c : \Lambda_N M \to H^0(\Lambda_N M^*TM)$. For $x \in TM$, set $\theta(x) = [\nabla_2 \exp(x)]^{-1} \circ [\nabla_1 \exp(x)]$, and for $c \in C^\infty(I, M)$, $X \in H^0(c^*TM)$

$$\tilde{\theta}_c(X)(t) = (c_\pi^*)^{-1} \circ \theta(c_\pi^* X(t)) \partial c(t).$$

Then

$$\partial_c X = \nabla_c X + \tilde{\theta}_c X.$$

2. The energy integral of a Finsler metric

Definition 1. *A Finsler metric on a manifold M is a continuous function $F : TM \to \mathbb{R}_+$ satisfying the following properties:*

(a) *F is C^∞ on $TM \setminus \{0\}$.*

(b) *$F(u) > 0,\ \forall\ u \in TM \setminus \{0\}$.*

(c) *$F(tu) = |t|F(u),\ \forall t \in \mathbb{R},\ u \in TM$.*

(d) *for any $p \in M$ the indicatrix $I_F(p) = \{u \in T_pM \mid F(u) < 1\}$ is strongly convex.*

A manifold M endowed with a Finsler metric is called a Finsler space [1], [3], [12]. *We say that a Finsler metric F dominates a Riemannian metric g of the manifold if for some $H_0 > 0$: $F(u) \geq H_0\|u\|\ \ \forall u \in TM$, where $\|\,.\,\|$ denotes the Riemannian norm.*

Remarks.

1. If we consider the function $L = F^2$, then L is of class $\mathcal{C}^1$ and dL is locally Lipschitz on TM.
2. The function L is of class $\mathcal{C}^2$ if and only if F is a norm of a Riemannian metric.
3. The condition (d) implies that the second fibre derivative $d_v^2 L$ derives a positive definite quadratic form in the vertical bundle $V\tau_M$. Then $g := d_v^2 L\colon \mathrm{Sec}\, V\tau_M \times \mathrm{Sec}\, V\tau_M \to C^\infty(M)$ makes the vertical bundle $V\tau_M$ a Riemannian vector bundle.
4. It is clear that if the manifold M is compact, then any Finsler metric dominates a Riemannian metric on M. Namely, considering the Loewner ellipsoid of the indicatrix in each tangent space we get a Riemannian metric on the manifold, dominated by the Finsler metric, for
$$H_0 = \inf\{F(u) \mid \|u\| = 1,\ u \in TM\}$$
is positive due to the compactness of M.
5. It is known [2] that if F_1 and F_2 are Finsler metrics on a manifold then $\sqrt{F_1^2 + F_2^2}$ is a Finsler metric as well. This means that for any Riemannian metric g on M the Finsler metric
$$\tilde{F}(u) = \sqrt{F^2(u) + g(u,u)}$$
dominates g with the constant $H_0 = 1$.

Definition 2. *The function $L = F^2$ induces a map $\tilde{L} : \Lambda_N M \to \mathbb{R}$ defined by*
$$\tilde{L}(c) = \int_I L(\dot{c}(t))dt, \quad \forall\ c \in \Lambda_N M$$
and is called the energy integral.

In the following we use the next result:

Lemma 1. [10] *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous, C^∞ on $\mathbb{R}^n \setminus \{0\}$, and positively homogeneous of degree α. Then*
(a) *if $\alpha = 1$, there exists a constant k with*
$$\|f(x) - f(y)\|_{\mathbb{R}^m} \leq k\|x - y\|_{\mathbb{R}^n},$$

(b) *if* $\alpha = 2$, *there exists constants* k_1, k_2 *with*

$$\|f(x) - f(y)\| \leq k_1\|x - y\|^2 + k_2\|x - y\| \cdot \|y\|$$

for all $x, y \in \mathbb{R}^n$.

Theorem 1. *The energy integral* $\tilde{L}$ *is* C^{2-} *on* $\Lambda_N M$, *i.e.* $\tilde{L}$ *is of class* C^1 *and the differential of* $\tilde{L}$ *is locally Lipschitz.*

Proof. Let $c \in C^\infty(I, M)$ be a fixed element and $(\phi_c, H^1(U_c))$ a local coordinate system about c and $\tilde{L}_c = \tilde{L} \circ \phi_c$. Then $\tilde{L}_c$ is the composition of the following maps:

$$H^1(U_c) \xrightarrow{1\times\partial_c} H^1(U_c) \times H^0(c^*TM) \xrightarrow{\tilde{\lambda}_c} L^1(I) \to \mathbb{R},$$

where the last map is the integration and the $\tilde{\lambda}_c$ is induced by the fibre map $\lambda_c : U_c \oplus c^*TM \to I \times \mathbb{R}$, defined by

$$\lambda_c(x, y) = (\pi_c^* x, L((\nabla_2 \exp)(c_\pi^* x) c_\pi^* y)) \quad \text{for } \forall\ (x, y) \in U_c \oplus c^*TM.$$

It is sufficient to show that $\tilde{\lambda}_c$ is of class C^{2-}. We note that the function $\tilde{\lambda}_c$ is well-defined. In fact, for $(X, Y) \in H^1(U_c) \times H^0(c^*TM)$ we have the following inequality:

$$\int_I L(\nabla_2 \exp(c_\pi^* X(t)) c_\pi^* Y(t)) dt \leq k_2 \left(\int_I \|\nabla_2 \exp c_\pi^* X(t)\|^2 \, dt \cdot \int_I \|Y(t)\|^2 dt \right)^{\frac{1}{2}}.$$

Indeed, because of $L(\tilde{0}) = 0$, where $\tilde{0}$ is the zero section of TM and using the main value theorem and the fact that dL is locally Lipschitz we have:

$$L(\nabla_2 \exp(c_\pi^* X(t)) \cdot (c_\pi^* Y(t))) - L(\tilde{0}) \leq k_2 \|\nabla_2 \exp(c_\pi^* X(t)) \cdot (c_\pi^* Y(t))\|.$$

Since $(\nabla_2 \exp)(x)$ is isomorphism, see [9] and using the pull-back, we get:

$$\|\nabla_2 \exp(c_\pi^* X(t)) \cdot (c_\pi^* Y(t))\| \leq \|\nabla_2 \exp(c_\pi^* X(t)\| \cdot \|c_\pi^* Y(t)\|.$$

Combining the above inequalities and integrating we get the inequality:

$$\int_I L(\nabla_2 \exp(c_\pi^* X(t) \cdot (c_\pi^* Y(t)))) \, dt \leq k_2 \int_I \|\nabla_2 \exp(c_\pi^* X(t))\| \cdot \|Y(t)\| \, dt \leq$$

$$k_2 \left(\int_I \|\nabla_2 \exp(c_\pi^* X(t))\|^2 dt \right)^{\frac{1}{2}} \cdot \left(\int_I \|Y(t)\|^2 dt \right)^{\frac{1}{2}},$$

which is bounded since $\|X(t)\|_\infty$ is small and $Y(t) \in H^0(c^*TM) = L^2(c^*TM)$.

For any $t \in I$, consider the restriction of λ_c to the fibre $\lambda_t : (U_c)_t \oplus (c^*TM)_t \to \mathbb{R}$. If we denote by x, y the variable in the first and second factor, respectively, we have:

1. λ_t and $\dfrac{\partial \lambda_t}{\partial x}$ are positively homogeneous of degree 2 in y, and
2. $\dfrac{\partial \lambda_t}{\partial y}$ is positively homogeneous of degree 1 in y.

We show that $\tilde{\lambda}_c$ is differentiable and $(d\tilde{\lambda}_c)(X,Y)(t) = (d_f\lambda_c)(X(t),Y(t))$, where d_f denotes the fibre derivative. Let $(X_1,Y_1) \in H^1(U_c) \times H^0(c^*TM)$ be small enough. Then for some $\theta \in [0,1]$, we have

$$\|\tilde{\lambda}_c((X+X_1),(Y+Y_1)) - \tilde{\lambda}_c(X,Y) - (d_f\lambda_c)(X(t),Y(t))(X_1(t),Y_1(t))\|_{L^1} =$$

$$= \int_I \|[(d_f\lambda_c)[X(t)+\theta X_1(t), Y(t)+\theta Y_1(t)) - (d_f\lambda_c)(X(t),Y(t))]](X_1(t),Y_1(t))\|dt \leq$$

$$\leq \int_I \left\| \frac{\partial \lambda_t}{\partial x}(X(t)+\theta X_1(t), Y(t)+\theta Y_1(t)) - \frac{\partial \lambda_t}{\partial x}(X(t),Y(t)) \right\| \cdot \|X_1(t)\|\, dt +$$

$$+ \int_I \left\| \frac{\partial \lambda_t}{\partial y}(X(t)+\theta X_1(t), Y(t)+\theta Y_1(y)) - \frac{\partial \lambda_t}{\partial y}(X(t),Y(t)) \right\| \cdot \|Y_1(t)\| dt \leq$$

$$\leq \|X_1\|_\infty \int_I \left\| \frac{\partial \lambda_t}{\partial x}(X(t)+\theta X_1(t), Y(t)+\theta Y_1(t)) - \frac{\partial \lambda_t}{\partial x}(X(t),Y(t)) \right\| dt +$$

$$+ \|Y_1\|_0 \left(\int_I \left\| \frac{\partial \lambda_t}{\partial y}(X(t)+\theta X_1(t), Y(t)+\theta Y_1(t)) - \frac{\partial \lambda_t}{\partial y}(X(t),Y(t)) \right\|^2 dt \right)^{\frac{1}{2}} \leq$$

$$\leq \|X_1\|_\infty \left(\int_I k_1 \|X_1(t)\|^2 dt + \int_I k_2 \|Y_1(t)\| \Big[\|Y_1(t)\| + \|Y(t)\| \Big] dt \right) +$$

$$+ \|Y_1\|_0 \left\{ \int_I k_3 \|X_1(t)\|^2 dt + \int_I k_4 \|Y_1(t)\|^2 dt \right\}^{\frac{1}{2}}$$

by the lemma.

Since (X_1,Y_1) is small, then $\tilde{\lambda}_c$ is differentiable. A similar calculation shows that $\tilde{\lambda}_c$ is C^{2-}. □

Now we generalize the notion that a curve orthogonally joins two submanifolds of a Finsler manifold. Here we use the machinery of Abate and Patrizio's book [1].

Definition 3. *We say that a curve $c\colon I \to M$ orthogonally joins two submanifolds M_1 and M_2 if $\langle U^H | T^H \rangle_{\dot{c}(0)} = 0$ and $\langle V^H | T^H \rangle_{\dot{c}(1)} = 0$ hold for all $U \in T_{c(0)}M_1$, and $V \in T_{c(1)}M_2$ respectively, where $T = \dot{c}$.*

Remark. This orthogonality property was given by H. Rund ([17], page 26), which is, of course, not a symmetrical relationship, in general. The symmetry property of orthogonality is, however, not required in this investigation.

We can prove now the following

Theorem 2. *Let M_1 and M_2 be submanifolds of M. Then $c \in \Lambda_{M_1 \times M_2}M$ is a critical point for $\tilde{L}\colon \Lambda_{M_1 \times M_2}M \to \mathbb{R}$ iff c is a Finsler-geodesic on M joining orthogonally M_1 and M_2.*

Proof. We consider a curve $c : [0,1] \to M$ with unit speed $F(\dot c) = 1$. The first variational formula gives (see [1], page 36)

$$d\tilde L(c)(U) = \langle U^H | T^H \rangle_{\dot c}|_0^1 - \int_0^1 \langle U^H | \nabla_{T^H} T^H \rangle_{\dot c}\, dt.$$

If $c \in \Lambda_{M_1 \times M_2} M$ is a critical point for $\tilde L \colon \Lambda_{M_1\times M_2} M \to \mathbb{R}$ and we consider regular fixed variations, then we get

$$\int_0^1 \langle U^H | \nabla_{T^H} T^H \rangle_{\dot c}\, dt = 0.$$

Since $U \in T_c(\Lambda_{M_1\times M_2} M)$ is arbitrary, it follows $\nabla_{T^H} T^H = 0$, which means c is a geodesic curve. Then for an arbitrary (not fixed) variation U we obtain

$$\langle U^H | T^H \rangle_{\dot c(0)} = \langle U^H | T^H \rangle_{\dot c(1)}$$

which implies that both sides vanish.

The converse statement simply follows from the first variation formula. □

3. The Palais-Smale condition

In this section we consider a dominating Finsler metric on a Riemannian manifold and prove that the functional $\tilde L$ satisfies the Palais-Smale condition on $\Lambda_N M$. This generalizes a result of K. Grove [8] for the energy integral of a Finsler metric. For its proof we need the following

Lemma 2. *Let $S \subset L_1^2(I, M)$ be a subset of $L_1^2(I, M)$ on which $\tilde L$ is bounded. Then S is an equicontinuous family of curves on M with uniformly bounded length.*

Proof. If we denote by $d_M(p, q)$ the distance function on M, i.e. the infimum of the lengths of all piecewise differentiable curves joining p to q, then we have:

$$d_M^2(c_k(t_0), c_k(t_1)) \le \left(\int_{t_0}^{t_1} \|\dot c_k(t)\| dt\right)^2 \le (t_1 - t_0) \int_I \|\dot c_k(t)\|^2 dt.$$

Because $c_k \in L_1^2(I, M)$, we have $\int_I \|\dot c_k(t)\|^2 dt < \infty$, and using the fact that F is a dominating Finsler norm, then there exists a real number $H_0 > 0$ such that

$$\int_I \|\dot c_k(t)\|^2 dt \le H_0 \int_I F^2(\dot c_k(t)) dt = H_0 \int_I L(\dot c_k(t)) dt.$$

Then we have $d_M^2(c_k(t_0), c_k(t_1)) \le (t_1 - t_0) H_0 S_0$, where $\tilde L(c_k) \le S_0,\ k \in \mathbb{N}$. It follows that S is an equicontinuous family of curves of M. □

Proposition 2. *Let $N \subset M \times M$ be a closed submanifold of $M \times M$ with compact $P_1(N) \subset M$ or $P_2(N) \subset M$, and suppose that M is complete. Then any sequence $\{c_n\}$ in $\Lambda_N M$ on which $\tilde L$ is bounded, has a subsequence converging uniformly to a continuous path $h \in C_N^0(M)$ in M.*

Proof. Without loss of generality we can assume that $P_1(N) \subset M$ is compact. From Lemma 2 we have that $\{c_n\}_{n\in\mathbb{N}}$ is an equicontinuous family of curves on M of bounded length, i.e. there exists a closed and bounded set $K \subset M$ such that $c_n(I) \subset K$ for all $n \in \mathbb{N}$ since $c_n(0) \in P_1(N)$, $\forall\ n \in \mathbb{N}$. Since M is a complete manifold, from the Hopf-Rinow theorem, see [1], we get that the set K is compact and hence we can apply Arzela-Ascoli's theorem to obtain the statement of the proposition. □

The main result of this section is the following.

Theorem 3. *Let F be a dominating Finsler metric on a complete Riemannian manifold M, and $N \subset M \times M$ be a closed submanifold of $M \times M$ such that $P_1(N) \subset M$ or $P_2(N) \subset M$ is compact. Then $\tilde{L} : \Lambda_N M \to \mathbb{R}_+$ satisfies the Palais-Smale condition, i.e. any sequence $c_n \in \Lambda_N M$ with $|\tilde{L}(c_n)| < const.$ and $\|d\tilde{L}(c_n)\| \to 0$ as $n \to \infty$ contains a convergent subsequence.*

Proof. Let $\{c_n\}_{n\in\mathbb{N}}$ be a sequence in $\Lambda_N M$, on which $\tilde{L}$ is bounded i.e. $\tilde{L}(c_n) \leq k$, $k \in \mathbb{R}_+$, $\forall\ n \in \mathbb{N}$ and for which $\|(\operatorname{grad}\tilde{L})(c_n)\| \to 0$, where $\operatorname{grad}\tilde{L}$ is a C^{1-}-vector field on $\Lambda_N M$ induced by $\tilde{L}$ due to the Riesz representation theorem, i.e.

$$\langle \operatorname{grad}\tilde{L}(c), \eta\rangle_1 = (d\tilde{L})(c)\eta, \quad \text{for } c \in \Lambda_N M,\ \eta \in T_c(\Lambda_N M).$$

We notice $\operatorname{grad}\tilde{L}$ by $d\tilde{L}$. We want to show that $\{c_n\}$ has a convergent subsequence. Now by Proposition 2 we can assume that c_n converges uniformly to a continuous map $h \in C^0_N(M)$. Let $c \in C^\infty(I, M)$ be uniformly close to $h \in C^0_N(M)$ ($C^\infty(I, M)$ is dense in $\Lambda_N M$).

We can assume that all c_n belong to a coordinate neighborhood $\phi_c(H^1(U_c))$. Set $X_n = \phi_c^{-1}(c_n)$, $\forall\ n \in \mathbb{N}$.

We show that the function $\tilde{L}$ is locally coercive, i.e. there exist $\alpha > 0$ and $c_1, c_2 \in \mathbb{R}$ such that

$$\begin{gathered}(d\tilde{L}_c(X_n) - d\tilde{L}_c(X_m))(X_n - X_m) \geq \\ \geq \alpha\|X_n - X_m\|_1^2 - c_1\|X_n - X_m\|_\infty^2 - c_2\|X_n - X_m\|_\infty.\end{gathered}$$

We write

$$\begin{aligned}&(d\tilde{L}_c(X_n) - d\tilde{L}_c(X_m))(X_n - X_m) = d\tilde{L}_c(X_n)(X_n - X_m) - d\tilde{L}_c(X_m)(X_n - X_m) = \\ &\int_0^1 (d\tilde{\lambda}_c)(X_n(t), \partial_c X_n(t))(X_n(t) - X_m(t), \nabla_c(X_n - X_m)(t) + d\tilde{\theta}_c(X_n)(X_n - X_m))dt \\ &- \int_0^1 d\tilde{\lambda}_c(X_m(t), \partial_c X_m(t))(X_n(t) - X_m(t), \nabla_c(X_n - X_m)(t) + d\tilde{\theta}_c(X_m)(X_n - X_m))dt.\end{aligned}$$

Remembering that $\partial_c X = \nabla_c X + \tilde{\theta}_c X$, from the relation above we obtain

$$\begin{gathered}(d\tilde{L}_c(X_n) - d\tilde{L}_c(X_m))(X_n - X_m) = \\ = \int_0^1 d\tilde{\lambda}_c(X_n(t), \partial_c X_n(t))((X_n - X_m)(t), \partial_c(X_n - X_m) - \tilde{\theta}_c(X_n - X_m) + d\tilde{\theta}_c(X_n)(X_n - X_m))dt\end{gathered}$$

$$-\int_0^1 d\tilde{\lambda}_c(X_m(t),\partial_c X_m(t))((X_n-X_m)(t),\partial_c(X_n-X_m)-\tilde{\theta}_c(X_n-X_m)+d\tilde{\theta}_c(X_m)(X_n-X_m))dt=$$

$$\int_0^1 (d\tilde{\lambda}_c)(X_n(t),\partial_c X_n(t))(0,\partial_c X_n-\partial_c X_m)dt-\int_0^1 (d\tilde{\lambda}_c)(X_m(t),\partial_c X_m(t))(0,\partial_c X_n-\partial_c X_m)dt$$

$$+\int_0^1 (d\tilde{\lambda}_c)(X_n(t),\partial_c X_n(t))((X_n-X_m)(t),\tilde{\theta}_c(X_m)-\tilde{\theta}_c(X_n)+d\tilde{\theta}_c(X_n)(X_n-X_m))dt-$$

$$-\int_0^1 (d\tilde{\lambda}_c)(X_m(t),\partial_c X_m(t))((X_n-X_m)(t),\tilde{\theta}_c(X_m)-\theta_c(X_n)+d\tilde{\theta}_c(X_m)(X_n-X_m))dt$$

We introduce the following notations

$$M_1=\int_0^1 (d\tilde{\lambda}_c)(X_n(t),\partial_c X_n(t))(0,\partial_c X_n-\partial_c X_m)dt$$

$$M_2=\int_0^1 (d\tilde{\lambda}_c)(X_m(t),\partial_c X_m(t))(0,\partial_c X_n-\partial_c X_m)dt$$

$$M_3=\int_0^1 (d\tilde{\lambda}_c)(X_n(t),\partial_c X_n(t))((X_n-X_m)(t),\tilde{\theta}_c(X_m)-\tilde{\theta}_c(X_n)+d\tilde{\theta}_c(X_n)(X_n-X_m))dt$$

$$M_4=\int_0^1 (d\tilde{\lambda}_c)(X_m(t),\partial_c X_m(t))((X_n-X_m)(t),\tilde{\theta}_c(X_m)-\tilde{\theta}_c(X_n)+d\tilde{\theta}_c(X_m)(X_n-X_m))dt,$$

then we get

$$(d\tilde{L}_c(X_n)-d\tilde{L}_c(X_m))(X_n-X_m)=\sum_{i=1}^{4}(-1)^{i+1}M_i.$$

In the following we estimate M_3, respectively M_4. Since the functions $\frac{\partial\lambda_t}{\partial x}(X_n,\partial_c X_n)(\cdot)$ and $\frac{\partial\lambda_t}{\partial y}(X_n,\partial_c X_n)(\cdot)$ are continuous on the interval [0,1] they attain their minimum and maximum, therefore we get

$$|M_3|\le k_1\int_0^1 \|X_n(t)-X_m(t)\|dt+k_2\int_0^1 \|\tilde{\theta}_c(X_m)-\tilde{\theta}_c(X_n)+d\tilde{\theta}_c(X_n)(X_n-X_m)\|dt.$$

Using the fact that the function $\tilde{\theta}_c$ is differentiable, $\|X_n-X_m\|_\infty$ is sufficiently small and the inequality $\|\xi\|_0\le\|\xi\|_\infty$ from Proposition 1 we obtain:

$$|M_3|\le k_1\|X_n-X_m\|_0+k_2\varepsilon_1\|X_n-X_m\|_\infty\le k\|X_n-X_m\|_\infty.$$

In the same way we have

$$|M_4|\le k^*\|X_n-X_m\|_\infty.$$

In the next we estimate the expression

$$M_5=\int_0^1\left(\frac{\partial\lambda_t}{\partial y}(X_n(t),\partial_c X_m(t)\right)(0,\partial_c X_n-\partial_c X_m)dt-M_2.$$

In this case we have

$$M_5 = \int_0^1 \left[\frac{\partial\lambda_t}{\partial y}(X_n(t), \partial_c X_m(t)) - \frac{\partial\lambda_t}{\partial y}(X_m(t), \partial_c X_m(t)\right] (\partial_c X_n - \partial_c X_m)dt.$$

Since $\|\partial_c X_n - \partial_c X_m\|_0$ is bounded, see [11, p. 240], and $\frac{\partial\lambda_t}{\partial y}$ is positively homogeneous, we get that

$$|M_5| \leq k_3 \int_0^1 \|X_n(t) - X_m(t)\|dt \leq k_3\|X_n - X_m\|_0 \leq k_3\|X_n - X_m\|_\infty.$$

Using the main value theorem and condition (d) from Definition 1, we estimate the following expression:

$$\begin{aligned}
M_6 = \int_0^1 & (d\tilde{\lambda}_c)(X_n(t), \partial_c X_n(t))(0, \partial_c X_n(t) - \partial_c X_m(t)) - \\
& -(d\tilde{\lambda}_c)(X_n(t), \partial_c X_m(t))(0, \partial_c X_n(t) - \partial_c X_m(t))dt = \\
& = \int_0^1 \left(\frac{\partial\lambda_t}{\partial y}\right)(X_n(t), \partial_c X_n(t))(\partial_c X_n(t) - \partial_c X_m(t)) - \\
& - \left(\frac{\partial\lambda_t}{\partial y}\right)(X_n(t), \partial_c X_m(t))(\partial_c X_n(t) - \partial_c X_m(t))dt =
\end{aligned}$$

$$= \int_0^1 \frac{\partial^2\lambda_t}{\partial y^2}(X_n(t), s\partial_c X_n(t) + (1-s)\partial_c X_m(t))(\partial_c X_n(t) - \partial X_m(t))(\partial_c X_n(t) - \partial_c X_m(t))dt \geq$$
$$\geq \alpha\|\partial_c X_n - \partial_c X_m\|_0^2,$$

where α is a positive constant. Because we have the inequality

$$\begin{aligned}
\|X_n - X_m\|_1^2 &= \|X_n - X_m\|_0^2 + \|\nabla_c X_n - \nabla_c X_m\|_0^2 \leq \\
&\leq \|X_n - X_m\|_0^2 + 2\|\tilde{\theta}_c(X_n) - \tilde{\theta}_c(X_m)\|_0^2 + 2\|\partial_c X_n - \partial_c X_m\|_0^2
\end{aligned}$$

and $\tilde{\theta}_c$ is differentiable and $d\tilde{\theta}(sX_n + (1-s)X_m)$ is linear and continuous, we get

$$\begin{aligned}
\|\tilde{\theta}_c(X_n) - \tilde{\theta}_c(X_m)\|_0 &\leq \|d\tilde{\theta}_c(sX_n + (1-s)X_m)(X_n - X_m)\|_0 \leq \\
&\leq k_4\|X_n - X_m\|_0 \leq k_4\|X_n - X_m\|_\infty.
\end{aligned}$$

Therefore we have the inequality

$$\alpha\|X_n - X_m\|_1^2 \leq \alpha(1 + 2k_4^2)\|X_n - X_m\|_\infty^2 + 2\alpha\|\partial_c X_n - \partial_c X_m\|_0^2,$$

where $\alpha > 0$ is the constant from the estimation for M_6.

Using the estimations above we get

$$(d\tilde{L}_c(X_n) - d\tilde{L}_c(X_m))(X_n - X_m) \geq$$
$$\alpha\|X_n - X_m\|_1^2 - c_1\|X_n - X_m\|_\infty^2 - c_2\|X_n - X_m\|_\infty,$$

where $c_1, c_2 \in \mathbb{R}$ are constants.

Because $\|X_n - X_m\|_\infty \to 0$, $d\tilde{L}_c(X_n) \to 0$ and $d\tilde{L}_c(X_m) \to 0$ if $m, n \to \infty$, from the above relation we obtain $\|X_n - X_m\|_1 \to 0$. Using the fact that $\Lambda_N M$ is a complete Riemann-Hilbert manifold we get that the sequence $\{X_n\}$ contains a convergent subsequence. □

Corollary 1. *Let F be a dominating Finsler metric on a complete Riemannian manifold M, V and V' be two closed submanifolds of M. Then the following holds:*

(a) $\tilde{L} : \Lambda_{V \times V'} M \to \mathbb{R}_+$ *satisfies the Palais-Smale condition if V or V' is compact.*

(b) $\tilde{L} : \Lambda_{\{p\} \times V} M \to \mathbb{R}_+$ *satisfies the Palais-Smale condition.*

(c) $\tilde{L} : \Lambda_{\{p\} \times \{q\}} M \to \mathbb{R}_+$ *satisfies the Palais-Smale condition $\forall\, p, q \in M$.*

4. Multiplicity results

In this section we generalize some results of K. Grove [8], J.P. Serre [19] and J.T. Schwartz [18] for Finsler manifolds. The next result is a generalization of Theorem 2.6. in [8] for Finsler metrics.

Theorem 4. *Let M be a smooth, complete, finite dimensional Riemannian manifold with a dominating Finsler metric F and let M_1 and M_2 be closed submanifolds of M with say M_1 compact. Then in any homotopy class of curves from M_1 to M_2 there exists a Finsler-geodesic joining orthogonally M_1 and M_2 with length smaller than that of any other curve in this class. Furthermore, there are at least $\operatorname{cat} \Lambda_{M_1 \times M_2} M$ geodesics joining orthogonally M_1 and M_2.*

Proof. Since $\Lambda_{M_1 \times M_2} M$ is a complete Hilbert-Riemann manifold and the energy functional satisfies the Palais-Smale condition it follows that the energy integral attains its infimum on any component of $\Lambda_{M_1 \times M_2} M$ and its lower bound. Since any critical point c for $\tilde{L}$ is a geodesic curve of the Finsler metric F, which joins M_1 and M_2 orthogonally, see Theorem 2, we obtain the first part of our theorem.

We note that an infimum of the energy functional is an infimum of the length by using the proof of Lemma 2 and the fact that a change of parameter does not affect the homotopy class of the curve. Using [15, Theorem 7.2] we get easily the second assertion of our theorem. □

Theorem 5. *Let M be a smooth, compact, connected, finite dimensional Finsler manifold. We suppose that M is simply connected and let M_1, M_2 be two closed submanifolds of M such that $M_1 \cap M_2 = \emptyset$, M_1 is contractible. Then there are infinitely many Finsler-geodesics joining orthogonally M_1 and M_2.*

Proof. Since M is compact, the Finsler metric dominates some Riemannian metric on M (see Remarks following Definition 1), and therefore M is a complete Riemannian manifold. Using the inequality $\operatorname{cat} \Lambda_{M_1 \times M_2} M \geq 1 + \operatorname{cuplong} \Lambda_{M_1 \times M_2} M$ and the fact that $\operatorname{cuplong} \Lambda_{M_1 \times M_2} M = \infty$, see [19], from Theorem 4 the statement follows. □

Theorem 6. *Let M be a smooth, complete, non-contractible, finite dimensional Riemannian manifold endowed with a dominating Finsler metric F and let M_1 and M_2 be two closed and contractible submanifolds of M such that M_1 or M_2 is compact. Then there exist infinitely many Finsler-geodesics joining orthogonally M_1 and M_2.*

Proof. Since $M_1 \times M_2$ is a submanifold of $M \times M$, the inclusion $\Lambda_{M_1 \times M_2} M \hookrightarrow C^0_{M_1 \times M_2}(M) = \{\sigma \in C^0([0,1], M) : \sigma(0) \in M_1, \sigma(1) \in M_2\}$ is a homotopy equivalence, see [8, Theorem 1.3]. Since M_1 and M_2 are contractible subsets of M, the sets $C^0_{M_1 \times M_2}(M)$ and $M_1 \times M_2 \times \Omega(M)$

are homotopically equivalent, see [7, Proposition 3.2]. Since M is non-contractible, from [7, Corollary 1.2] we have $\operatorname{cat}\Omega(M) = \infty$. Therefore, $\operatorname{cat}\Lambda_{M_1\times M_2}M = \infty$ and we can apply again Theorem 4 to obtain the desired relation. □

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