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Bisectors in Minkowski 3-Space

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Abstract. We discuss the concept of the bisector of a segment in a Minkowski normed n-space. We prove that all bisectors are topological images of a plane of the embedding Euclidean 3-space iff the shadow boundaries of the unit ball K are topological circles. To a conjectured proving strategy for dimensions n, we introduce the concept of general parameter sphere of the unit ball K, corresponding to a direction vector of the n-space and to a positive parameter. We prove that the Haussdorff limit of these "spheres" is the shadow boundary of K of the same direction.

1. Introduction, historical remarks

If K is a 0-symmetric, bounded, convex body in the Euclidean n-space E^n (with a fixed origin O) then it defines a norm whose unit ball is K itself (see [6] or [8]). Such a space is called *Minkowski normed space*. In fact, the norm is a continuous function on the vectors of E^n which is considered (in the geometric terminology as in [6]) as a gauge function. The metric (the so-called Minkowski metric), i.e. the distance of any two points, induced by this norm, is invariant with respect to the translations of the space.

The unit ball is said to be *strictly convex* if its boundary bdK contains no line segment.

In a previous paper [5] of this topic we examined the boundary of the unit ball of the norm and proved two theorems (Theorem 2 and 3 in [5]) similar to the characterization of the Euclidean norm investigated by H. Mann, A. C. Woods and P. M. Gruber in [7], [12], [1], [2] and [3], respectively. We proved that if the unit ball of a Minkowski normed space is strictly convex then every bisector (which is the collection of those points of the

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embedding Euclidean space which have the same distance with respect to the Minkowskian norm to two given points of the space) is a topological hyperplane (meaning that there is a homeomorphism of E^n onto itself sending the bisector to a usual hyperplane) (Theorem 2). Example 3 in [5] has shown that strict convexity does not follow from the fact that all bisectors are topological hyperplanes.

In [5] we recalled the concept of normal subdivision: the Dirichlet-Voronoi cell system of a lattice L yields a *normal subdivision* of the embedding Euclidean space if the boundary of any cell does not contain Euclidean n-ball and we showed that the Dirichlet-Voronoi cell system of an arbitrary lattice L gives a normal subdivision of the embedding Euclidean space if and only if the bisectors are topological hyperplanes. Especially the strict convexity of the unit ball ensures the normality of Dirichlet-Voronoi type K-subdivision of any point lattice (Theorem 4).

The purpose of the present paper is to examine the connections between the shadow boundaries of the unit ball K and the bisectors of the Minkowski space. We strongly believe that the following statement is true: *The bisectors are topological hyperplanes if and only if the corresponding shadow boundaries are $(n-2)$-dimensional topological spheres*, however, we shall prove this conjecture only in the three-dimensional case. (Theorem 2 and Theorem 4) We examine the topological properties of the shadow boundary (Section 2), and define the so-called *general parameter spheres* for $n \geq 3$, as a tool for a prospective proof of our conjecture.

2. Shadow boundary

Shadow boundaries have been considered frequently in convexity theory. I mention only two interesting results in context of Baire categories, see [4] and [11]. In [4] the authors proved that a typical shadow boundary under parallel illumination from a direction vector has infinite $(n-2)$-dimensional Hausdorff measure, while having Hausdorff dimension $(n-2)$. In [11] it is shown that, in the sense of Baire categories, most n-dimensional convex bodies have infinitely long shadow boundaries if the light vector comes along one of $(n-2)$-dimensional subspaces.

Definition 1. *Let K be a compact convex body in n-dimensional euclidean space E^n and let S^{n-1} denote the $(n-1)$-dimensional unit sphere in E^n. For $\mathbf{x} \in S^{n-1}$ the shadow boundary $S(K,\mathbf{x})$ of K in direction $\mathbf{x}$ consists of all points P in bdK such that the line $\{P+\lambda\mathbf{x} : \lambda \in R$ (real numbers)$\}$ supports K, i.e. it meets K but not the interior of K. The shadow boundary $S(K,\mathbf{x})$ is sharp if any above supporting line of K intersects K exactly in the point P. If $S(K,\mathbf{x})$ is not sharp, in general, it may have sharp point for that the above uniqueness holds.*

It is clear that the shadow boundary decomposes the boundary of K into three disjoint sets. These are $S(K,\mathbf{x})$ itself, moreover

$$\begin{aligned} K^+ &:= \{\mathbf{y} \in bdK | \text{ there is } \tau > 0 \text{ such that } \mathbf{y} - \tau \cdot \mathbf{x} \in int(K)\}, \qquad (1)\\ K^- &:= \{\mathbf{y} \in bdK | \text{ there is } \tau > 0 \text{ such that } \mathbf{y} + \tau \cdot \mathbf{x} \in int(K)\}, \end{aligned}$$

respectively. We call the congruent (thus homeomorphic) sets K^+ and K^- the *positive and negative part* of bdK, respectively.

Example 1. In general, the shadow boundary of a central symmetric convex body is not a nice set from topological point of view . There exists a central symmetric convex body K and a direction $\mathbf{x}$ of the space E^3 such that every supporting line of K parallel to $\mathbf{x}$ contains a point of K having no relative neighborhood in $S(K, \mathbf{x})$ homeomorphic to an open segment. This means that $S(K, \mathbf{x})$ is not a 1-dimensional manifold. In fact, consider a unit circle C in E^2 and the diadic rational points of it with respect to the usual parametrization (Fig. 1). More precisely, take the parameter values $t_{i,j} = \frac{j}{2^i} 2\pi$, where $0 \leq i$ is integer and $1 \leq j \leq 2^i$ is odd number. The diadic rational points of the circle are the points $S_{i,j} = (\cos(t_{i,j}), sin(t_{i,j}))$ of the subspace E^2 with respect to an orthonormed basis. (Note that we define the points $S_{0,1}$ and $S_{1,1}$ in the 0^{th} and 1^{th} steps, respectively, and – in the i^{th} step – we consider further 2^{i-1} points of form $S_{i,j}$ of the circle.) Let now $s_{i,j}$ be a segment orthogonal to the subspace E^2 whose midpoint is $S_{i,j}$ and its length is equal to $\frac{1}{2^{i-2}}$ if $i \geq 2$ and is equal to 2 if $i = 0, 1$. The point sets

$$C^* := C \cup (\cup_{i,j}\{s_{i,j}\}) \text{ and } K := convC^*$$

are central symmetric, here *conv* abbreviates convex hull. This body is also closed, see it in Fig. 1. If l is a supporting line of K orthogonal to the plane E^2 then it does not intersect

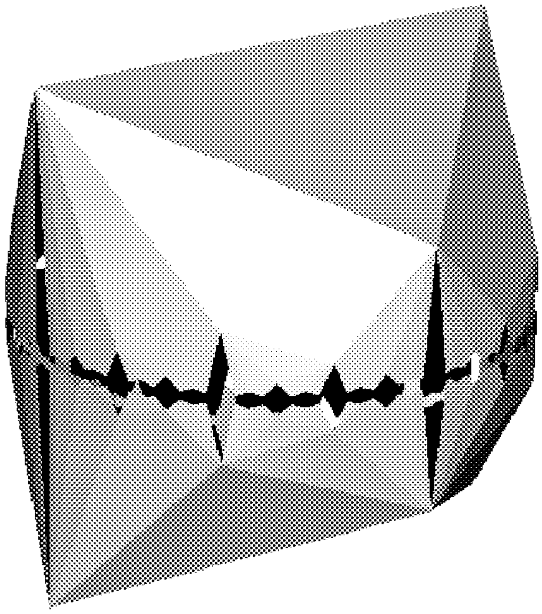

Figure 1. Shadow boundary which is not a topological manifold

the relative interior of the disc in E^2 bounded by the circle C, so it intersects the circle C. If $l \cap C$ is a point of form $S_{i,j}$ then $l \cap K = l \cap C^* = s_{i,j}$, while if $l \cap C$ is another point of C then $l \cap K = l \cap C$. We conclude to $S(K, \mathbf{x}) = C^*$ being not a 1-manifold, as we claimed.

Example 2. It is easy to show that the shadow boundary is connected as a topological space but it is not necessarily arcwise connected. Take a two dimensional cylinder and draw up a "$\sin(\frac{1}{x})$-type" curve on it, as in Fig. 2, whose set of accumulation points contains a segment s lying on a generator of the cylinder. (See Fig. 2 as a local picture.) If we consider

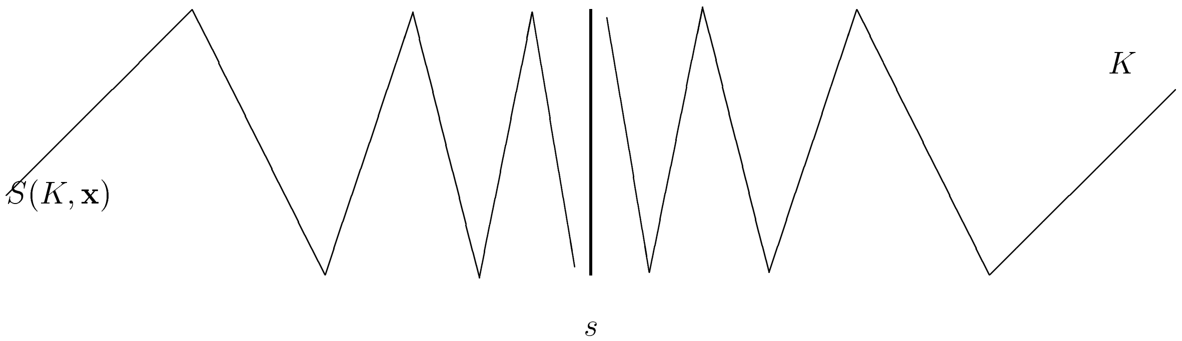

Figure 2. Shadow boundary with more than one arcwise connected components

the union of this curve and this segment then the shadow boundary of the convex hull K of this union (from the direction of the generators) contains two arcwise connected components (the segment s and the remaining curve). (In Fig. 2 K is not central symmetric. Of course, similar central symmetric examples can be constructed, too.)

In order to describe the connection between the bisectors and the shadow boundaries of the unit ball we introduce some parametrized sets on the boundary of K, corresponding to a given direction of the space. These tend to the shadow boundary of K of the same direction if the parameter tends to infinity. As we shall see in the case of a nice unit ball these sets give a parametrization of the closed "positive part" of bdK. In this way we can define the *general parameter spheres* according to this direction.

Definition 2. *Let K be the Minkowski unit ball above and $\mathbf{x}$ a fixed direction of the space E^n. Let*

$$\lambda_0 := \inf\{0 < t \in R \mid tK \cap (tK + \mathbf{x}) \neq \emptyset\} \tag{2}$$

be the smallest value t for which tK and $tK + \mathbf{x}$ intersect. Then a general parameter sphere of bdK corresponding to the direction $\mathbf{x}$ and to any fixed parameter $\lambda \geq \lambda_0$ is the following set:

$$\gamma_\lambda(K,\mathbf{x}) := \frac{1}{\lambda}[bd(\lambda K) \cap bd(\lambda K + \mathbf{x})] \subset bdK. \tag{3}$$

In general, the above set is not a topological sphere of dimension $n-2$, and they are not homeomorphic to each other for different λ's. For example the dimension of $\gamma_{\lambda_0}(K,\mathbf{x})$ may be $0, 1, \ldots, n-1$ while the dimension of $\gamma_\lambda(K,\mathbf{x})$ for $\lambda > \lambda_0$ is at least $n-2$ because it dissects the boundary of K (Fig. 3, 4). We remark that the two parts of $bdK \setminus \gamma_\lambda(K,\mathbf{X})$ for $\lambda > \lambda_0$ are also homeomorphic to each other by the projection from $\frac{1}{2\lambda}\mathbf{x}$ (since $\lambda K \cap \lambda K + \mathbf{x}$ is central symmetric in $\frac{1}{2}\mathbf{x}$ for any $\lambda \geq \lambda_0$).

Lemma 1. *Let $\Pi(\mathbf{x},\mathbf{y})$ be a 2-plane parallel to the vectors $\mathbf{x}$ and $\mathbf{y} \in S(K,\mathbf{x})$, through the origin. Then we have two possibilities for $\Pi(\mathbf{x},\mathbf{y}) \cap \gamma_\lambda(K,\mathbf{x})$:*

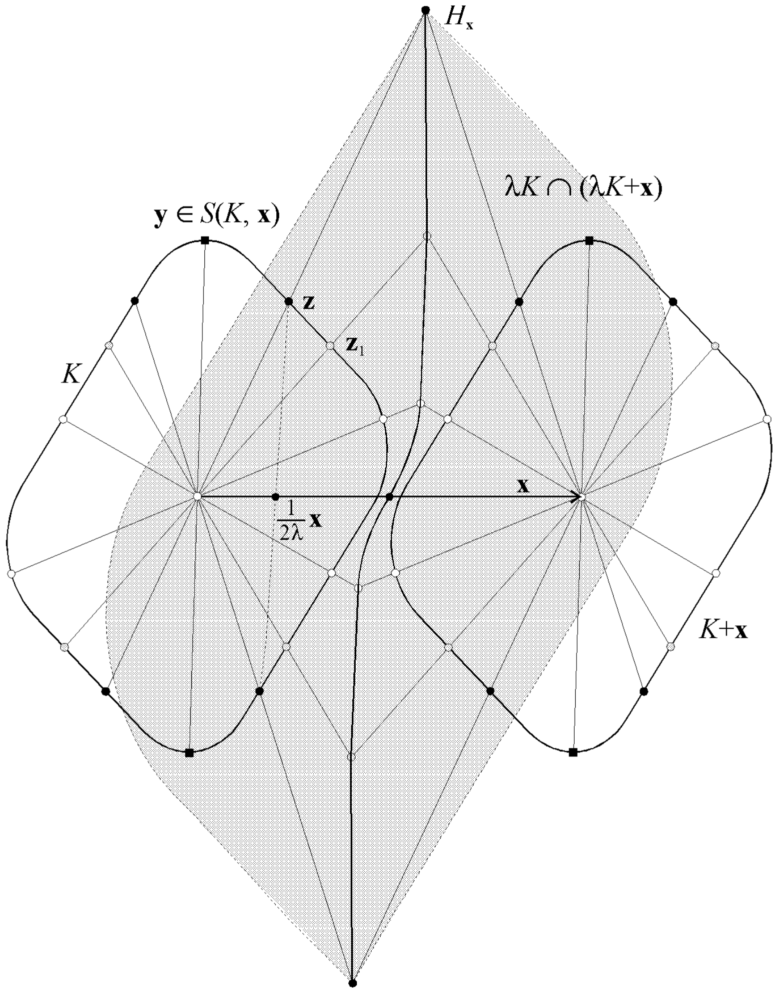

Figure 3. General parameter spheres to a sharp point $\mathbf{y} \in S(K, \mathbf{x})$

1. *If the shadow boundary $S(K, \mathbf{x})$ is sharp for the point $\mathbf{y} \in S(K, \mathbf{x})$ then $\Pi(\mathbf{x}, \mathbf{y}) \cap \gamma_\lambda(K, \mathbf{x})$ contains two opposite points with respect to $\frac{1}{2\lambda}\mathbf{x}$ (Fig. 3)*
2. *There is a uniquely defined parameter value $\lambda(\mathbf{y})$ that for every $\lambda > \lambda(\mathbf{y})$ the intersection $\Pi(\mathbf{x}, \mathbf{y}) \cap \gamma_\lambda(K, \mathbf{x})$ is the union of a pair of segments parallel to $\mathbf{x}$, opposite with respect to $\frac{1}{2\lambda}\mathbf{x}$. (Fig. 4)*

In the second case the segments of the parameter spheres $\gamma_\lambda(K, \mathbf{x})$ belong to the shadow boundary $S(K, \mathbf{x})$.

Proof. Let $\lambda > \lambda_0$ be an arbitrary real number and consider the generalized parameter sphere $\gamma_\lambda(K, \mathbf{x})$. Then $\gamma_\lambda(K, \mathbf{x}) = \frac{1}{\lambda} S(\lambda K \cap (\lambda K + \mathbf{x}), \mathbf{x})$. In fact, $\mathbf{y} \in \gamma_\lambda(K, \mathbf{x})$ iff $\lambda \mathbf{y} \in bd(\lambda K) \cap bd(\lambda K + \mathbf{x}) \subset bd(\lambda K \cap (\lambda K + \mathbf{x}))$. Let the line $l(\tau)$ be of the form $\lambda \mathbf{y} + \tau \mathbf{x}$ where τ runs through real numbers.

There is no $\tau_0 \neq 0$ for which e.g. $\tau_0 < 0$ holds and $\lambda \mathbf{y} + \tau_0 \mathbf{x} \in int(\lambda K \cap (\lambda K + \mathbf{x}))$. Indirectly, $\lambda \mathbf{y} + \tau_0 \mathbf{x} \in int(\lambda K)$ and $\lambda \mathbf{y} + \tau_0 \mathbf{x} \in int(\lambda K + \mathbf{x})) = int(\lambda K) + \mathbf{x}$ hold. The second relation implies $\lambda \mathbf{y} + (\tau_0 - 1)\mathbf{x} \in int(\lambda K)$, while $\lambda \mathbf{y} \in bd(\lambda K)$ and $\lambda \mathbf{y} \in bd(\lambda K + \mathbf{x})$ involve

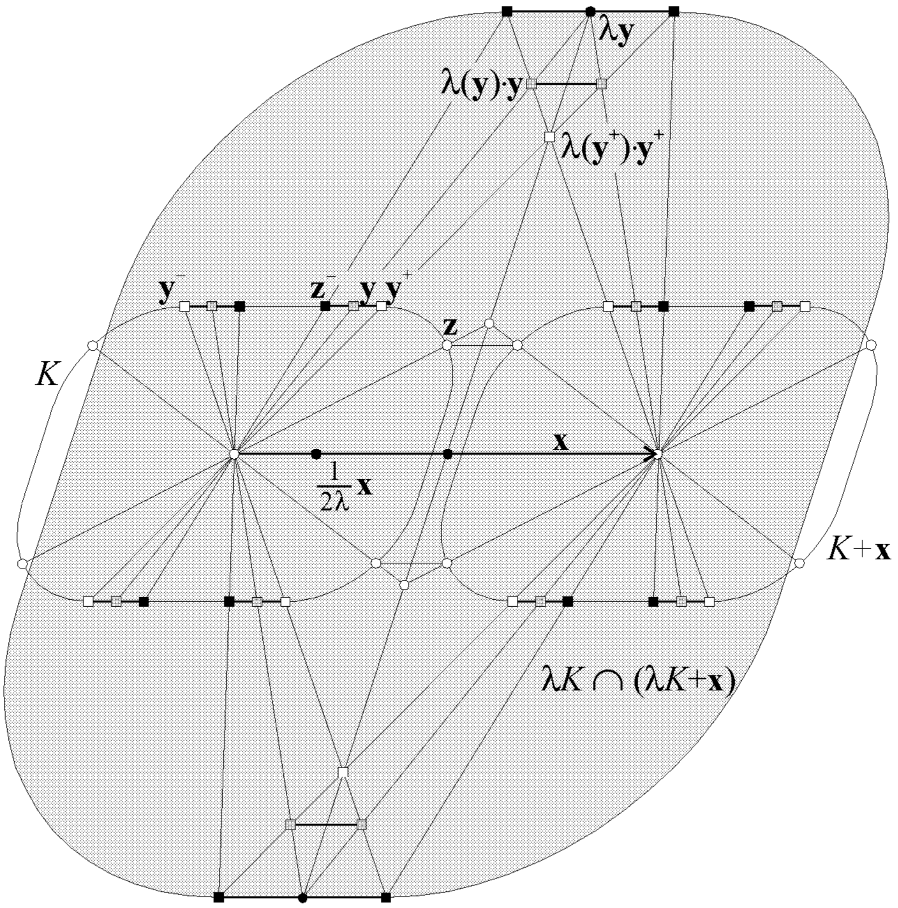

Figure 4. General parameter spheres to points $\mathbf{y} \in [\mathbf{y}^-, \mathbf{y}^+] \subset S(K, \mathbf{x})$ which are not sharp

$\lambda\mathbf{y} - \mathbf{x} \in bd(\lambda K)$. This means that the points $\lambda\mathbf{y}$, $\lambda\mathbf{y} - \mathbf{x}$, $\lambda\mathbf{y} + \tau_0\mathbf{x}$, $\lambda\mathbf{y} + (\tau_0 - 1)\mathbf{x}$ are on the line l, ordered as

$$\lambda\mathbf{y} - \mathbf{x}, \lambda\mathbf{y} + (\tau_0 - 1)\mathbf{x}, \lambda\mathbf{y} + \tau_0\mathbf{x}, \lambda\mathbf{y}$$

by the convexity of K. This would imply $\tau_0 = 0$, a contradiction.

Since the shadow boundary of the convex bodies $K_\lambda = \frac{1}{\lambda}(\lambda K \cap (\lambda K + \mathbf{x}))$ to $\mathbf{x}$ are on the boundary of K, it can contain a segment parallel to $\mathbf{x}$ if and only if this segment belongs to the shadow boundary of K, too. An interesting phenomenon that – though $\Pi(\mathbf{x}, \mathbf{y}) \cap S(K, \mathbf{x})$ is a pair of opposite segments (by central symmetry in 0) – for a starting λ (which gives the positive end of $\Pi(\mathbf{x}, \mathbf{y}) \cap S(K, \mathbf{x})$), $\Pi(\mathbf{x}, \mathbf{y}) \cap \gamma_\lambda(K, \mathbf{x})$ is a pair of points. So we are done. □

Fig. 3, 4 show that the set $\{\lambda(\mathbf{y}) | \mathbf{y} \in S(K, \mathbf{x})\}$ is not bounded from up, in general. An important consequence of Lemma 1 is the following

Corollary. *The general parameter spheres for $\lambda > \lambda_0$ provide a natural parametrization of the surface $K^+ \setminus \gamma_{\lambda_0}(K, \mathbf{x})$. In this parametrization any point of $K^+ \setminus \gamma_{\lambda_0}(K, \mathbf{x})$ is determined by a point of a Euclidean unit sphere of dimension $n - 2$, orthogonal to $\mathbf{x}$ in 0, and by a parameter $\lambda > \lambda_0$.*

(Of course, it is possible that the above surface $K^+ \setminus \gamma_{\lambda_0}(K, \mathbf{x})$ is empty, as in the case of a cube (=K) when four of its edges are parallel to $\mathbf{x}$. However, in significant cases it is a useful parametrization. For example, if K is strictly convex, then it has only one singular point $\gamma_{\lambda_0}(K, \mathbf{x})$ on the positive half.)

To prove this corollary, we observe the fact (see Fig. 3, 4) that the common points of two distinct parameter spheres belong to the shadow boundary of K, hence the generalized parameter spheres give a one-fold covering of $K^+ \setminus \gamma_{\lambda_0}(K, \mathbf{x})$, see formula (1).

We recall the concept of *Hausdorff distance* ρ_H of two point sets S_1 and S_2, expressed by the Euclidean distance ρ_E:

$$\rho_H(S_1, S_2) = \max\{\sup_{s_1 \in S_1} \{\rho_E(s_1, S_2)\}, \sup_{s_2 \in S_2} \{\rho_E(s_2, S_1)\}\}. \tag{4}$$

(Here e.g. $\rho_E(s_1, S_2) = \inf_{s_2 \in S_2}\{\rho_E(s_1, s_2)\}$.)

The main result of this section is the following:

Theorem 1. *The shadow boundary $S(K, \mathbf{x})$ is the limit of the general parameter spheres $\gamma_\lambda(K, \mathbf{x})$, with respect to the Hausdorff metric, when λ tends to infinity.*

Proof. According to the previous lemma we have two cases (Fig.3,4). In the first one the 2-plane $\Pi(\mathbf{x}, \mathbf{y})$, with $\mathbf{y} \in S(K, \mathbf{x})$, intersects both $S(K, \mathbf{x})$ and $\gamma_\lambda(K, \mathbf{x})$ in two point pairs, respectively (Fig.3); while in the second case the intersection $\Pi(\mathbf{x}, \mathbf{y}) \cap S(K, \mathbf{x})$ is a 0-opposite pair of segments, and the intersection $\Pi(\mathbf{x}, \mathbf{y}) \cap \gamma_\lambda(K, \mathbf{x})$, if $\lambda > \lambda(\mathbf{y}) \geq \lambda_0$, is an opposite pair of segments with respect to $\frac{1}{2\lambda}\mathbf{x}$ (Fig.4). We will mention the necessary intersections as a point or a segment, shortly. Introduce now the following notations. Let S' be the set of sharp points of $S(K, \mathbf{x})$ and S'' be the set of the remaining points of $S(K, \mathbf{x})$, decomposed to (disjoint) segments parallel to $\mathbf{x}$. We say that the points $\mathbf{y} \in S(K, \mathbf{x})$ and $\mathbf{z} \in \gamma_\lambda(K, \mathbf{x})$ correspond to each other, if $\mathbf{y}, \mathbf{z} \in \Pi(\mathbf{x}, \mathbf{y})$ and the line of direction $\mathbf{x}$ through the origin does not separate them in $\Pi(\mathbf{x}, \mathbf{y})$. If $\mathbf{y} \in S'$ then there exists one corresponding point $\mathbf{z} \in \gamma_\lambda(K, \mathbf{x})$ (see Lemma 1). Denote this simply by $\mathbf{z}$.

If $\mathbf{y} \in S''$ then either it has only one corresponding point in $\gamma_\lambda(K, \mathbf{x})$ (see Lemma 1, $\lambda_0 < \lambda \leq \lambda(\mathbf{y})$) or the corresponding points form a segment belonging to S'' (Lemma 1, $\lambda > \lambda(\mathbf{y})$). We focus on the negative end of the segment of S'', containing $\mathbf{y}$ denoted by $\mathbf{y}^-$, and the negative end of the corresponding segment of $\gamma_\lambda(K, \mathbf{x})$ denoted by $\mathbf{z}^-$. Let S''' be the set of those points $\mathbf{z}$ of $\gamma_\lambda(K, \mathbf{x})$ which correspond to a point of S', and S'''' be the collection of the remaining points of $\gamma_\lambda(K, \mathbf{x})$. Now the claimed convergence follows from the inequalities below:

$$\rho_H(S(K, \mathbf{x}), \gamma_\lambda(K, \mathbf{x})) = \max\{\sup_{\mathbf{y} \in S(K,\mathbf{x})} \{\rho_E(\mathbf{y}, \gamma_\lambda(K, \mathbf{x}))\}, \sup_{\mathbf{z} \in \gamma_\lambda(K,\mathbf{x})} \{\rho_E(S(K, \mathbf{x}), \mathbf{z})\}\} =$$

$$= \max\{\sup_{\mathbf{y} \in S'}\{\rho_E(\mathbf{y}, \gamma_\lambda(K, \mathbf{x}))\}, \sup_{\mathbf{y} \in S''}\{\rho_E(\mathbf{y}, \gamma_\lambda(K, \mathbf{x}))\},$$

$$\sup_{\mathbf{z} \in S'''}\{\rho_E(S(K, \mathbf{x}), \mathbf{z})\}, \sup_{\mathbf{z} \in S''''}\{\rho_E(S(K, \mathbf{x}), \mathbf{z})\}\} \leq$$

$$\leq \max\{\sup_{\mathbf{y}\in S'}\{\rho_E(\mathbf{y},\mathbf{z})\}, \sup_{\mathbf{y}^-\in S''}\{\rho_E(\mathbf{y}^-,\gamma_\lambda(K,\mathbf{x}))\}, \sup_{\mathbf{z}^-\in S'''}\{\rho_E(\mathbf{y}^-,\mathbf{z}^-)\}, \sup_{\mathbf{z}\in S''''}\{\rho_E(S(K,\mathbf{x}),\mathbf{z})\}\} \leq$$

$$\leq \max\{\sup_{\mathbf{y}\in S'}\{\rho_E(\mathbf{y},\mathbf{z})\}\}, \sup_{\mathbf{y}^-\in S''}\{\rho_E(\mathbf{y}^-,\mathbf{z}^-\}, \sup_{\mathbf{z}\in S''''\setminus S(K,\mathbf{x})}\{\rho_E(S(K,\mathbf{x}),\mathbf{z})\}\} \leq$$

$$\leq \max\{\sup_{\mathbf{y}\in S'}\{\rho_E(\mathbf{y},\mathbf{z})\}, \sup_{\mathbf{y}^-\in S''}\{\rho_E(\mathbf{y}^-,\mathbf{z}^-)\}, \sup_{\mathbf{z}\in S''''\setminus S(K,\mathbf{x})}\{\rho_E(\mathbf{y}^-,\mathbf{z})\}\}$$

since each of these three Euclidean distances tend to zero, if λ tends to infinity, since K and its two dimensional intersections are convex and compact, respectively. □

3. Bisectors, general parameter spheres and shadow boundaries in three-space

In this paragraph we prove strong connections among the bisector $H_{\mathbf{x}}$ of the points 0 and $\mathbf{x}$, the general parameter spheres $\gamma_\lambda(K,\mathbf{x})$ and the shadow boundary $S(K,\mathbf{x})$.

According to Definition 2 we define the *bisector* $H_{\mathbf{x}}$ as

$$\begin{aligned} H_{\mathbf{x}} &= \cup_\lambda\{bd(\lambda K)\cap bd(\lambda K+\mathbf{x}) \mid \lambda_0\leq\lambda\}, \text{ i.e.} \\ H_{\mathbf{x}} &= \cup_\lambda\{\lambda\gamma_\lambda(K,\mathbf{x}) \mid \lambda_0\leq\lambda\}. \end{aligned} \tag{5}$$

The closed negative halfspace $H_{\mathbf{x}}^-$, containing O, is

$$H_{\mathbf{x}}^- = \cup_{\lambda,\lambda'}\{\lambda'\gamma_\lambda(K,\mathbf{x})|\lambda_0\leq\lambda \ , \ \lambda'\leq\lambda\}. \tag{6}$$

Its complementary open positive halfspace

$$H_{\mathbf{x}}^+ = E^3\setminus H_{\mathbf{x}}^- \tag{7}$$

contains $\mathbf{x}$, of course.

Definition 3. *A point set $H\subset E^3$ is said to be a topological plane iff there is a homeomorphism of E^3 onto itself, sending H onto a usual 2-plane.*

Theorem 2. *Assume that the bisector $H_{\mathbf{x}}$ is a topological plane of E^3. Then the general parameter spheres $\gamma_\lambda(K,\mathbf{x})$ for $\lambda>\lambda_0$ and the shadow boundary $S(K,\mathbf{x})$ are topological 1-manifolds (topological circles). For $\lambda=\lambda_0$ the parameter sphere can form a point, a segment or a convex disk of dimension 2, respectively.*

Before the proof we recall a nice theorem of two-dimensional topology, characterizing the topological circles on a two-sphere. (See for example [13].)

Definition 4. *A point a is called arcwise accessible from a point set B if $b\in B$ implies the existence of an arc T with end points a and b such that $T\setminus a\subset B$. If A is a point set whose every point is arcwise accessible from some point set B, then we call A arcwise accessible from B.*

The following theorem was discovered by A. Schoenflies [9] and refined to this form by P. M. Swingle [10].

Theorem 3. (Schoenflies, Swingle) *A necessary and sufficient condition that a subset M of S^2 should be an S^1 is that it be a common boundary of two disjoint domains D_1 and D_2, from which M is arcwise accessible.*

Remarks. 1. We shall take bdK in the role of the sphere S^2 and take either the shadow boundary $S(K,\mathbf{x})$ or a general parameter sphere $\gamma_\lambda(K,\mathbf{x})$, $\lambda > \lambda_0$, in the role M, respectively, in Theorem 3. The two domains D_1 resp. D_2 will be either the positive resp. negative parts of $bdK \setminus S(K,\mathbf{x})$, homeomorphic to each other under the projection from the origin O; or the positive and negative part of $bdK \setminus \gamma_\lambda(K,\mathbf{x})$, respectively, that are also homeomorphic to each other under the projection from the point $\frac{1}{2\lambda}\mathbf{x}$, leaving $\gamma_\lambda(K,\mathbf{x})$ invariant (see before Lemma 1). *Thus, arcwise accessibility is enough to guarantee from one domain, only, in the above cases.*

2. On the other hand, *a point in the common boundary of two complementary domains on bdK is not necessarily accessible by arc from any of both domains.* Namely, take Example 2 (Fig. 2). Any point P in the relative interior of a segment s of $S(K,\mathbf{x})$ lies also in a $\gamma_\lambda(K,\mathbf{x})$ for a λ large enough (Theorem 1). All of the arcs on bdK with end P intersect $S(K,\mathbf{x})$ (or $\gamma_\lambda(K,\mathbf{x})$) in a point set such that P is its accumulation point, meaning that this arc lies neither in the two considered (open) domains.

Now we prove a technical lemma.

Lemma 2. *Assume that the shadow boundary $S(K,\mathbf{x})$ contains a segment s parallel to $\mathbf{x}$ having the property that it is a subset of accumulation points of $S(K,\mathbf{x})\setminus s$. Then the bisector $H_\mathbf{x}$ can not be a topological plane.*

Proof. Let $\mathbf{y}$ be a relative inner point of the segment s of accumulation points of $S(K,\mathbf{x})$. There exists such a λ (large enough) and also an ε (small enough) for which the segment with negative end $\mathbf{y}$ and positive end $\mathbf{y}^+$ of s lies in $\gamma_{\lambda'}(K,\mathbf{x})$ where $\lambda-\varepsilon < \lambda' < \lambda+\varepsilon$ and the accumulation points of the sets $\gamma_{\lambda'}(K,\mathbf{x}) \setminus s$ contain also the segment $[\mathbf{y},\mathbf{y}^+]$ (by Theorem 1). This means, there is a domain – namely the union of segments

$$\cup_{\lambda'}\{\lambda'[\mathbf{y},\mathbf{y}^+] | \lambda-\varepsilon < \lambda' < \lambda+\varepsilon\}$$

– in the bisector $H_\mathbf{x}$ which lies in the set of accumulation points of the complementary set with respect to $H_\mathbf{x}$. Drawing in this domain a little circle we get a closed curve which relative interior points are also boundary points of its complementary sets. Thus the Jordan Curve Theorem (as a special case of the Schoenflies-Swingle theorem) does not hold on $H_\mathbf{x}$, consequently $H_\mathbf{x}$ could not be a topological plane. □

Proof of Theorem 2. Firstly, we deal with general parameter spheres.

The statement on $\gamma_{\lambda_0}(K,\mathbf{x})$ follows from the convexity and central symmetry of the compact body K (and $K+\mathbf{x}$ as well).

For $\lambda > \lambda_0$ we prove that $\lambda(\gamma_\lambda(K,\mathbf{x})) \subset H_\mathbf{x}$ is arcwise accessible from the negative sets

$$H_1' = \cup_{\lambda'}\{\lambda'(\gamma_{\lambda'}(K,\mathbf{x})) | \lambda_0 \le \lambda' < \lambda\} \subset H_\mathbf{x} \subset H_\mathbf{x}^-, \tag{8}$$

If $\mathbf{v}$ is a point of $\lambda(\gamma_\lambda(K,\mathbf{x}))$ then there is an arc, parametrized by λ' in the intersection $H_\mathbf{x} \cap \Pi(\mathbf{x},\mathbf{v})$ which connects the point $\mathbf{v}$ with the point $\frac{1}{2}\mathbf{x}$, with the property that its points, different from $\mathbf{v}$, lie in H_1'. Since also $\lambda\gamma_\lambda(K,\mathbf{x})$ is the common boundary of H_1' and its complementary set in $H_\mathbf{x}$, by the Schoenflies-Swingle theorem, we get that $\lambda\gamma_\lambda(K,\mathbf{x}))$ is a topological circle, i.e. by the projection from 0, $\gamma_\lambda(K,\mathbf{x}))$ is a topological circle, too, which is arcwise accessible also from the open disk component of $Int(K^+ \setminus \gamma_\lambda(K,\mathbf{x})$ by Theorem 3 (see Fig. 3, 4 for illustration). □

Now let's turn to the case of the shadow boundary: We assume that $H_\mathbf{x}$ by (5) is a topological plane. We check that the conditions of Schoenflies-Swingle theorem hold for $S(K,\mathbf{x})$, too. It is enough to prove that $S(K,\mathbf{x})$ is arcwise accessible from K^+. Let $\mathbf{y}$ an arbitrary point of $S(K,\mathbf{x})$.

If $S(K,\mathbf{x})$ is sharp at this point (Fig. 3) then, by Lemma 1, the set

$$\cup_\lambda\{\Pi(\mathbf{x},\mathbf{y}) \cap \gamma_\lambda(K,\mathbf{x}) | \lambda \geq \lambda_0\} \cup \mathbf{y}$$

is a good arc which connects the interior of K^+ and $\mathbf{y}$. (Since K^+ is arcwise connected $\mathbf{y}$ is accessible from points K^+ by arcs.)

If $\mathbf{y}$ is not a sharp point of $S(K,\mathbf{x})$ (Fig. 4) then (by Lemma 1) we have the segment s of $S(K,\mathbf{x})$ through $\mathbf{y}$ as a union of the monotone increasing sequence of segments $\Pi(\mathbf{x},\mathbf{y}) \cap \gamma_\lambda(K,\mathbf{x})$, parallel to $\mathbf{x}$ where $\lambda > \lambda(\mathbf{y})$, and the negative end $\mathbf{y}^-$ of s (Fig. 4).

Observe that all of this segments are arcwise accessible from K^+, so is their union, too. To prove this, let s' denote one of the segments $\Pi(\mathbf{x},\mathbf{y}) \cap \gamma_\lambda(K,\mathbf{x})$ for fixed $\lambda > \lambda(y)$. Observe that the points of K^+ by (1) belong to one of the following three sets:

$$H_1 = \cup_{\lambda'}\{\gamma_{\lambda'}(K,\mathbf{x}) \mid \lambda > \lambda' \geq \lambda_0\} \cap K^+,\ \gamma_\lambda(K,\mathbf{x}) \cap K^+ \text{ and } K^+ \setminus (\gamma_\lambda(K,\mathbf{x}) \cup H_1). \quad (9)$$

From the points of the first set (by the first part of this proof) there are arcs connecting a point $\mathbf{y}'$ of the considered segment with the required property. We can connect the points of the second set with a point of H_1 by such an arc whose points belong to K^+, and this latter point can be connected again with a required arc, showing that from these points there also exist arcs to $\mathbf{y}'$. Finally, a point $\mathbf{v}$ of the third set (by Lemma 1) lies in a plane $\Pi(\mathbf{x},\mathbf{v})$ intersecting $S(K,\mathbf{x})$ in a sharp point. The arc from $\mathbf{v}$ to a point of H_1 in the intersection $\Pi(\mathbf{x},\mathbf{v}) \cap bdK$ can be extended to a required arc which ends at $\mathbf{y}'$.

It remains to examine of the negative end point $\mathbf{y}^-$ of s (see Fig. 5). Since $\mathbf{y}^-$ is a boundary point of the segment s whose other points belong to the boundary of K^+, then it is a boundary point of K^+. Consider now a sequence $(\mathbf{z}_i)$ of points of K^+ that tends to $\mathbf{y}^-$. First we introduce a parametrization of $S(K,\mathbf{x}) \cup K^+$. Let (φ,ψ) denote the coordinates of any point $\mathbf{z} \in bdK$. Here φ is the angle of the planes $\Pi(\mathbf{x},\mathbf{z})$ and $\Pi(\mathbf{x},\mathbf{y}^-)$, $-\pi < \varphi \leq \pi$, with respect to a fixed orientation, and ψ the angle of the vectors $\mathbf{x}$ and $\mathbf{z}$, $0 < \psi < \pi$. Then we have $(\mathbf{z}_i) = ((\alpha_i,\beta_i)^T)$ and $\mathbf{y}^- = (0,\beta)^T$, T means transposed. We can assume, without loss of generality, that the sequence (α_i) is monotone decreasing. Now we connect the points $\mathbf{z}_i$ and $\mathbf{z}_{i+1}$ by an arc γ_i lying in K^+. We define ψ_i^* for later arcs, near enough $S(K,\mathbf{x})$, by

$$\psi_i^* := \inf\{\psi | \text{ there exists } \alpha_i \geq \varphi \geq \alpha_{i+1} \text{ for which } (\varphi,\psi)^T \in S(K,\mathbf{x})\} - \frac{1}{2^i}.$$

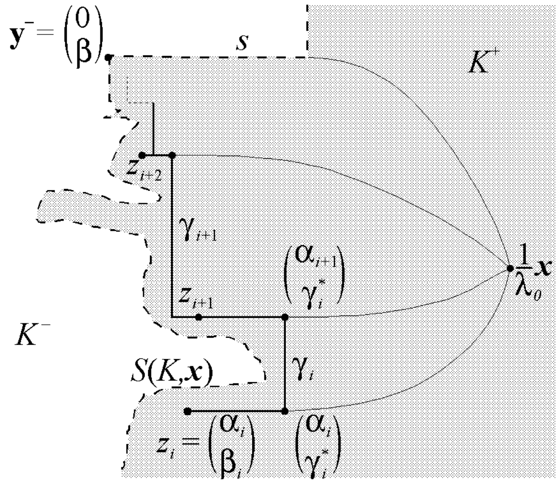

Figure 5. The negative end is accessible by arc

From now on the notation $x \in [a, b]$ ($x \in (a, b)$) means that either $a \leq x \leq b$ ($a < x < b$) or $a \geq x \geq b$ ($a > x > b$) hold. Then the arc γ_i connecting $\mathbf{z}_i$ and $\mathbf{z}_{i+1}$ is the following:

$$\gamma_i := \{(\alpha_i, \psi)^T \text{ with parameter } \psi \in [\beta_i, \psi_i^*]\} \cup$$

$$\cup \{(\varphi, \psi_i^*)^T \text{ with } \varphi \in (\alpha_i, \alpha_{i+1})\} \cup$$
$$\cup \{(\alpha_{i+1}, \psi)^T \text{ with } \psi \in [\beta_{i+1}, \psi_i^*]\}.$$

Of course, the simple union of these arcs is considered only one curve for which one of its accumulation points is $\mathbf{y}^- = (0, \beta)^T$. However, the following set

$$\gamma := cl(\cup_i \gamma_i \setminus \cup_i(\gamma_i \cap \gamma_{i+1}))$$

(in which we do not take multiple points) is an appropriate arc if and only if

$$\gamma \setminus \cup_i \gamma_i = \{\mathbf{y}^-\}.$$

Since the set of accumulation points of γ is a subset of $\gamma \cup s$, thus the indirect assumption implies a subsegment s' of s with non-zero length. This is also a subset of accumulation points of $S(K, \mathbf{x}) \setminus s$ and applying Lemma 2 we get that the bisector would not be a topological plane.

Thus the conditions of the Schoenflies-Swingle theorem are fulfilled so $S(K, \mathbf{x})$ is a topological circle as we claimed. □

Lemma 3. *Assume that the shadow boundary of K in the direction $\mathbf{x}$ is a topological circle. Then the general parameter spheres are also topological circles for $\lambda > \lambda_0$.*

The proof is an easy consequence of Theorem 1 and of the arguments before it. Now we are ready to prove the main result of this section:

Theorem 4. *Let K be a central symmetric compact convex body in E^3. All of the bisectors $H_{\mathbf{x}}$ of the corresponding Minkowski normed space are topological planes if and only if all of the shadow boundaries $S(K,\mathbf{x})$ are topological circles (1-spheres).*

Proof. The necessity is a consequence of Theorem 2.

We prove that if the shadow boundary is a topological circle then the corresponding bisector $H_{\mathbf{x}}$ by (5) is a topological plane. By the assumption and Lemma 3, $\gamma_\lambda(K,\mathbf{x})$ is a topological circle for any fixed $\lambda > \lambda_0$, and $\gamma_{\lambda_0}(K,\mathbf{x})$ is a topological closed ball of dimension 0,1 or 2, respectively. Consider now $S(K,\mathbf{x})$.

First we note that, for a fixed λ, on $\gamma_\lambda(K,\mathbf{x})$ there are only finitely many segments parallel to $\mathbf{x}$. In the contrary case there would be infinitely many corresponding segments on $S(K,\mathbf{x})$, too, but $S(K,\mathbf{x})$ is compact and homeomorphic to a circle, this would easily lead to a contradiction with Theorem 3. Then the set of lengthes of these segments of $S(K,\mathbf{x})$ has a positive lower bound. Thus there are only finitely many parameter values λ_i with the property that $\gamma_{\lambda_i}(K,\mathbf{x})$ ($\lambda_i > \lambda_0$) contains such a positive end of a segment s_i of the shadow boundary parallel to $\mathbf{x}$, which is not lying on a $\gamma_{\lambda'}(K,\mathbf{x})$ for $\lambda' < \lambda_i$.

If $\mathbf{y_i^+}$ is a positive end of s_i then $\lambda_i\mathbf{y_i^+}$ is an apex of a corner domain belonging to the intersection of $H_{\mathbf{x}}$ and a plane through the origin and s_i. Partition now $H_{\mathbf{x}}$ into non-overlapping rings by the consecutive topological circles $\lambda_i\gamma_{\lambda_i}(K,\mathbf{x})$ $i \geq 1$. A ring between the circles $\lambda_i\gamma_{\lambda_i}(K,\mathbf{x})$ and $\lambda_{i+1}\gamma_{\lambda_{i+1}}(K,\mathbf{x})$ can be partitioned by straight-line boundaries of the corresponding corners to finitely many non-overlapping domains $D_{i,j}$ where $D_{i,j} \cap D_{i,j+1}$ (for every j with respect to a cyclic order, is a segment connecting a point of $\lambda_i\gamma_{\lambda_i}(K,\mathbf{x})$ to a point of $\lambda_{i+1}\gamma_{\lambda_{i+1}}(K,\mathbf{x})$. These closed domains (each homeomorphic to a closed disc for $i \geq 1$) join only finitely many others, thus we can define a sequence of homeomorphisms $\Phi_{i,j}$ on $D_{i,j}$ by induction in the following way.

First, we partition the unit disc B (with center O) into non-overlapping pieces having the same combinatorial structure as the subdivision of $H_{\mathbf{x}} = \lambda_0\gamma_{\lambda_0}(K,\mathbf{x}) \cup_{i,j} D_{i,j}$. We have three cases: $\lambda_0\gamma_{\lambda_0}(K,\mathbf{x})$ is a closed disc, a closed segment or a point.

In the first case we consider the concentric circles C_{λ_i} with respective radii $r_{\lambda_i} = 1 - \frac{\lambda_0}{2\lambda_i}$ for $i \geq 1$ and define the image of $\lambda_0\gamma_{\lambda_0}(K,\mathbf{x})$ as the disk with origin O and radius $\frac{1}{2}$.

In the second case we consider concentric ellipses which converges to a O-symmetric segment of length 1, and the third case the ring structure giving by concentric circles, too, with corresponding radii $r_{\lambda_i} = 1 - \frac{\lambda_0}{\lambda_i}$ for $i \geq 1$.

We map now the shadow boundary $S(K,\mathbf{x})$ onto the boundary of B. A corner domain of $H_{\mathbf{x}}$ corresponds to a segment s of $S(K,\mathbf{x})$ thus also to a closed arc σ of the unit circle. On the other hand the apex a_σ of this corner corresponds to a λ_i. If $i > 0$ let a'_σ a point of $C_{\lambda_i} \cap conv\{O,\sigma\}$. For $i = 0$, in the first case, we may choose a'_σ in the same way; in the second case we have only two possibilities for a_σ (the ends of $\lambda_0\gamma_{\lambda_0}(K,\mathbf{x})$); thus let a'_σ be one of the ends of the corresponding segment C_{λ_0}. (In this case we choose the corresponding arc γ_0 intersecting the line of C_{λ_0}.) Finally in the latter case there is no such apex. Now we subdivide the rings by the sectors $conv\{a'_\sigma,\sigma\}$. Obviously, the domains $Q_{i,j}$ in this process can be corresponded to the domains $D_{i,j}$ in a unique way. This means that we partition B to closed domains $Q_{i,j}$ with the property: $\cap D_{i,j}$ is homeomorphic to $\cap Q_{i,j}$ for indices i,j.

Second, by induction (with respect to the lexicographic order of the pairs (i,j)) it is not

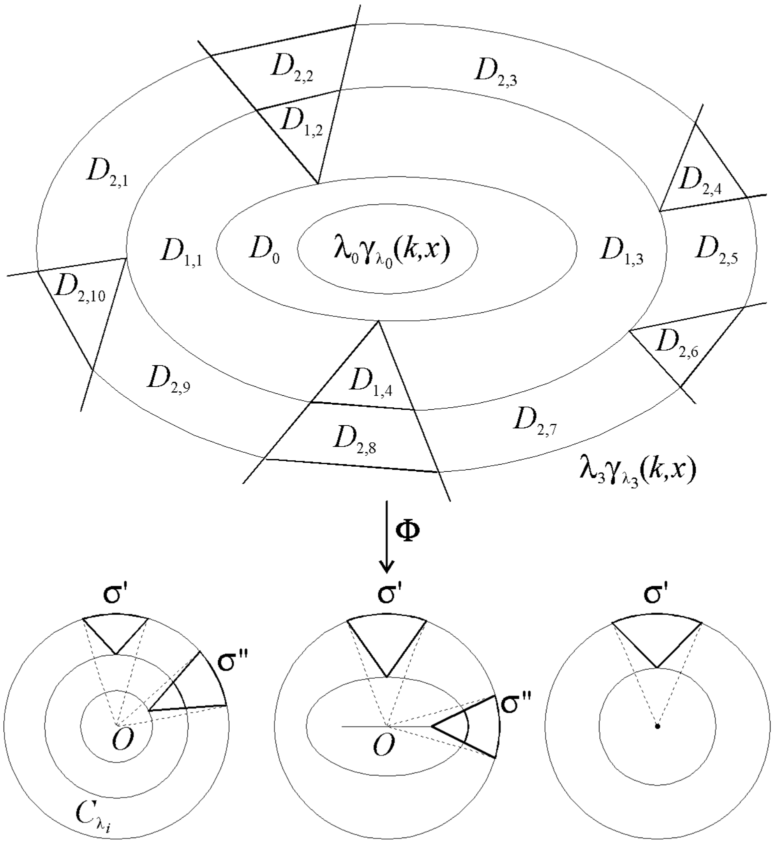

Figure 6. The homeomorphism Φ

too hard to give a family $\{\Phi_{i,j} : D_{i,j} \longrightarrow Q_{i,j}\}$ of homeomorphisms compatible to each other, requiring that if $D_{i,j} \cap D_{k,l} \neq \emptyset$ then $\Phi_{i,j}(\mathbf{v}) = \Phi_{k,l}(\mathbf{v})$ for each point $\mathbf{v}$ of $D_{i,j} \cap D_{k,l}$. (Denote by $\Phi_{0,0}$ the first homeomorphism sending $\lambda_0\gamma_{\lambda_0}(K, \mathbf{x})$ onto the corresponding (not-indicated) subset of B.)

Now the mapping $\Phi : H_{\mathbf{x}} \longrightarrow intB$ (see Fig.6), sending a point $\mathbf{v} \in D_{i,j}$ to the point $\Phi_{i,j}(\mathbf{v})$, is evidently a homeomorphism of $H_{\mathbf{x}}$ onto the interior of the disc B as we stated. □

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