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Deviation Measures and Normals of Convex Bodies

Dedicated to Professor August Florian on the occasion of his seventy-fifth birthday

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Abstract. With any given convex body we associate three numbers that exhibit, respectively, its deviation from a ball, a centrally symmetric body, and a body of constant width. Several properties of these deviation measures are studied. Then, noting that these special bodies may be defined in terms of their normals, corresponding deviation measures for normals are introduced. Several inequalities are proved that show that convex bodies cannot deviate much from these special types if their corresponding deviations of the normals are small. These inequalities can be interpreted as stability results.

1. Introduction

Let $\mathcal{K}^n$ denote the class of convex bodies (compact convex sets) in n-dimensional euclidean space $\mathbf{R}^n$, and $\mathcal{K}^n_*$ the subclass of $\mathcal{K}^n$ consisting of all centrally symmetric bodies. For any $K \in \mathcal{K}^n$ let $h_K(u)$ denote the support function and $w_K(u) = h_K(u) + h_K(-u)$ the width of K in the direction u. As underlying metric on $\mathcal{K}^n$ we use the distance concept based on the L_2-norm

$$\|\Phi\| = \left(\int_{S^{n-1}} \Phi(u)^2 d\sigma(u)\right)^{1/2},$$

where the real valued function Φ is defined on the unit sphere S^{n-1} in $\mathbf{R}^n$ (centered at the origin o on $\mathbf{R}^n$) and σ refers to the surface area measure on S^{n-1}. For any pair $K, L \in \mathcal{K}^n$ the corresponding L_2-distance $\delta(K, L)$ is then defined by

$$\delta(K, L) = \|h_L(u) - h_K(u)\|.$$

With each $K \in \mathcal{K}^n$ we now associate three 'deviation measures' that indicate, respectively, the deviation of K from being spherical, being centrally symmetric, and having constant width.

The *spherical deviation* of K is defined by

$$\mathcal{S}(K) = \inf\{\delta(K, B^n(p, r)) : p \in \mathbf{R}^n, r \geq 0\},$$

where $B^n(p, r)$ denotes the n-dimensional ball of radius r centered at p.

As a deviation measure for central symmetry, which will be called *eccentricity*, we introduce the expression

$$\mathcal{E}(K) = \inf\{\delta(K, Z) : Z \in \mathcal{K}^n_*\}.$$

Finally we define the *width deviation* of K by

$$\mathcal{W}(K) = \frac{1}{2}\inf\{\|w_K(u) - w\| : w \geq 0\}.$$

In view of the definitions of $\mathcal{S}$ and $\mathcal{E}$ it appears that it might be more appropriate to define $\mathcal{W}(K)$ as the infimum of $\delta(K, X)$ where X ranges over all convex bodies of constant width. This, however, would lead to difficulties regarding the existence of certain convex bodies of constant width. For an alternative possibility to define the width deviation see (2) below.

In Section 3 some properties of these deviation measures will be discussed. In particular, it will be shown that $\mathcal{S}$, $\mathcal{E}$, and $\mathcal{W}$ are closely related with the mean width, the Steiner point and the Steiner ball of K.

A *normal* at a boundary point q of a convex body K is defined as a line that passes through q and is orthogonal to a support plane of K at q. There are various theorems that characterize particular classes of convex bodies in terms of properties of the normals. For example, balls are characterized as convex bodies such that all their normals pass through a common point, or convex bodies of constant width are characterized by the property that each normal is a 'double normal', that is, a normal at two boundary points (see [3, Sec. 2]). In Section 4 we supply new analytic proofs for such theorems and also prove inequalities relating the deviation measures to certain properties of the normals. These inequalities can be interpreted as stability results. (See [5] concerning the general idea of stability for geometric inequalities.) More specifically, we obtain inequalities that provide estimates for the spherical and width deviation of a convex body from a ball or a body of constant width if, respectively, its normals are close to a fixed point, or any two parallel normals are close to each other. A similar estimate is proved for the eccentricity. One of our results has an interesting implication (formulated as a corollary) concerning physical bodies that are nearly in equilibrium on a flat surface in any position.

2. Definitions and notation

In this section we introduce some definitions and describe the notation that will be essential for the following sections. If $K \in \mathcal{K}^n$ and $u \in S^{n-1}$ then $H_K(u)$ denotes the support plane of K of direction u, i. e., $H_K(u) = \{x + h_K(u)u : x \cdot u = 0\}$, where the dot indicates the inner product in $\mathbf{R}^n$. We repeatedly use the fact that the support function satisfies the translation formula

$$h_{(K+p)}(u) = h_K(u) + p \cdot u.$$

One of the advantages of the L_2-distance in comparison to other distance concepts for convex bodies is the possibility to employ the inner product for functions on S^{n-1}. For any two bounded integrable functions Φ and Ψ on S^{n-1} it is defined by

$$\langle \Phi, \Psi \rangle = \int_{S^{n-1}} \Phi(u)\Psi(u) d\sigma(u).$$

As usual, Φ and Ψ are said to be orthogonal if their inner product is zero. We let $|\cdot|$ denote (in addition to the absolute value) the euclidean norm in $\mathbf{R}^n$. If $F(u)$ is a function on S^{n-1} with values in $\mathbf{R}^n$ we simply write $\|F(u)\|$ instead of $\| \, |F(u)| \, \|$.

The volume of the unit ball $B^n(o,1)$ will be denoted by κ_n and its surface area measure by σ_n. We write $\bar{w}(K)$ for the *mean width* of K. Hence,

$$\bar{w}(K) = \frac{2}{\sigma_n} \langle h_K(u), 1 \rangle.$$

Using an obvious notational extension of the integral, one defines the *Steiner point* $s(K)$ of K by

$$s(K) = \frac{1}{\kappa_n} \int_{S^{n-1}} h_K(u) u \, d\sigma(u). \tag{1}$$

The ball $B^n(s(K), \bar{w}(K)/2)$ is called the *Steiner ball* of K and denoted by $B(K)$. We sometimes use the obvious fact that the support function of $B^n(p,r)$ is $r + p \cdot u$ and, in particular, that

$$h_{B(K)}(u) = \frac{1}{2}\bar{w}(K) + s(K) \cdot u.$$

For any $K \in \mathcal{K}^n$ let K_* denote the convex body obtained from K by central symmetrization, and K_o the translate of K that has its Steiner point at the origin of $\mathbf{R}^n$. Hence,

$$K_* = \frac{1}{2}(K + (-K))$$

and

$$K_o = K - s(K).$$

3. Properties of the deviation measures

The following theorem shows where the infima that are used in the definition of the deviation measures are attained, and it provides therefore more explicit representations of these measures.

Theorem 1. *Let K be a convex body in $\mathbf{R}^n$.*

(a) $\delta(K, B^n(p,r))$, *considered as a function of p and r, is minimal exactly if $p = s(K)$ and $r = \bar{w}(K)/2$. Hence,*

$$\mathcal{S}(K) = \delta(K, B(K)) = \|h_{K_o}(u) - \bar{w}(K)/2\|.$$

(b) $\delta(K, Z)$, *considered as a function of $Z \in \mathcal{K}_*^n$ is minimal exactly if $Z = K_* + s(K)$. Consequently,*

$$\mathcal{E}(K) = \delta(K_o, K_*) = \frac{1}{2}\|h_{K_o}(u) - h_{K_o}(-u)\|.$$

(c) $\|w_K(u) - w\|$, *considered as a function of w is minimal exactly if $w = \bar{w}(K)$. Hence,*

$$\mathcal{W}(K) = \frac{1}{2}\|w_K(u) - \bar{w}(K)\| = \delta(K_*, B(K_*)) = \mathcal{S}(K_*). \tag{2}$$

Moreover, the deviation measures have the property that

$$\mathcal{S}(K)^2 = \mathcal{E}(K)^2 + \mathcal{W}(K)^2. \tag{3}$$

Before we turn to the proof of this theorem we add several pertinent remarks. The fact that the Steiner ball is the best L_2-approximation of a convex body by balls (and the Steiner point its center) could be used as the definition of these concepts. It certainly would be a better motivated approach than using (1) as a definition. The relation (3) provides a quantitative version of the well-known theorem that a convex body that is both of constant width and centrally symmetric must be a ball. In fact, it allows one to obtain information on the spherical deviation of a convex body if corresponding data on the width deviation and eccentricity are available.

Proof of Theorem 1. One could prove this theorem using the development of functions on the sphere in terms of spherical harmonics. But, as will be shown here, it can also be proved using only elementary analysis.

The following easily proved relations (cf. [6, Sec. 3.2]) will be used repeatedly without explicitly mentioning it. It is assumed that $u = (u_1, \ldots, u_n) \in S^{n-1}$ (and each u_i is viewed as a function of u on S^{n-1}), $p \in S^{n-1}$ and $i, j = 1, \ldots, n$ $(i \neq j)$.

$$\langle u_i, 1\rangle = 0, \quad \langle u_i, u_j\rangle = 0, \quad \|u_i\|^2 = \kappa_n\,, \quad \|p \cdot u\|^2 = \kappa_n |p|^2.$$

(The last relation is an obvious consequence of the two preceding ones.) We also note that the integral of any odd function on S^{n-1} vanishes.

First let us show that for all $K, L \in \mathcal{K}^n$

$$\delta(K,L)^2 = \delta(K_o, L_o)^2 + \kappa_n |s(K) - s(L)|^2. \tag{4}$$

We obviously have $\delta(K,L)^2 = \|h_K(u) - h_L(u)\|^2 = \|[h_{K_o}(u) - h_{L_o}(u)] + [(s(K) - s(L)) \cdot u]\|^2$. Furthermore, since $s(K_o) = s(L_o)$ it follows from (1) that $\langle h_{K_o}(u) - h_{L_o}(u), u_i\rangle = 0$ and this implies that the functions $h_{K_o}(u) - h_{L_o}(u)$ and $(s(K) - s(L)) \cdot u$ are orthogonal.

Consequently, $\|[h_{K_o}(u) - h_{L_o}(u)] + [(s(K) - s(L)) \cdot u]\|^2 = \|h_{K_o}(u) - h_{L_o}(u)\|^2 + \|(s(K) - s(L)) \cdot u\|^2$ and we obtain (4).

Let us now prove part (b) of the theorem. Using (4) we find

$$\begin{aligned}\delta(K,Z)^2 &= \delta(K_o, Z_o)^2 + \kappa_n |s(K) - s(Z)|^2 \\ &= \|[\tfrac{1}{2}(h_{K_o}(u) + h_{K_o}(-u)) - h_{Z_o}(u)] + \tfrac{1}{2}[h_{K_o}(u) - h_{K_o}(-u)]\|^2 + \kappa_n |s(K) - s(Z)|^2.\end{aligned}$$

Since the first one of the two functions in brackets is even and the other one is odd it follows that they are orthogonal and we obtain

$$\delta(K,Z)^2 = \delta(K_*, Z_o)^2 + \frac{1}{4}\|h_{K_o}(u) - h_{K_o}(-u)\|^2 + \kappa_n |s(K) - s(Z)|^2.$$

This shows that $\delta(K,Z)$ is minimal if and only if $s(K) = s(Z)$ and $K_* = Z_o$. Clearly, this happens exactly if $Z = K_* + s(K)$.

To prove part (c) we note that $\|w_K(u) - w\|^2 = \|[w_K(u) - \bar{w}(K)] + [\bar{w}(K) - w]\|^2$ and that the definition of the mean width shows that the two functions in brackets are orthogonal. Hence,

$$\|w_K(u) - w\|^2 = \|w_K(u) - \bar{w}(K)\|^2 + \sigma_n(\bar{w}(K) - w)^2 \tag{5}$$

and this is apparently minimal if and only if $w = \bar{w}(K)$.

For the proof of part (a) of the theorem we use again (4) and find that

$$\begin{aligned}\delta(K, B^n(p,r))^2 &= \delta(K_o, B^n(o,r))^2 + \kappa_n |s(K) - p|^2 \\ &= \|[\tfrac{1}{2}(h_{K_o}(u) + h_{K_o}(-u)) - r] + \tfrac{1}{2}[h_{K_o}(u) - h_{K_o}(-u)]\|^2 + \kappa_n |s(K) - p|^2.\end{aligned}$$

Since the product of the two functions in brackets is odd they are orthogonal and this implies that

$$\delta(K, B^n(p,r))^2 = \frac{1}{4}\|w_K(u) - 2r\|^2 + \frac{1}{4}\|h_{K_o}(u) - h_{K_o}(-u)\|^2 + \kappa_n |s(K) - p|^2.$$

Thus, using (5) (with $w = 2r$) we obtain

$$\begin{aligned}\delta(K, B^n(p,r))^2 =& \frac{1}{4}\|w_K(u) - \bar{w}(K)\|^2 + \frac{1}{4}\|h_{K_o}(u) - h_{K_o}(-u)\|^2 \\ &+ \frac{1}{4}\sigma_n(\bar{w}(K) - 2r)^2 + \kappa_n |s(K) - p|^2.\end{aligned}$$

This shows that $\delta(K, B^n(p,r))$ is minimal exactly if $r = \bar{w}(K)/2$ and $p = s(K)$, as stated in (a). Letting $B^n(p,r) = B(K)$, one also finds that

$$\mathcal{S}(K)^2 = \delta(K, B(K))^2 = \frac{1}{4}\|w_K(u) - \bar{w}(K)\|^2 + \frac{1}{4}\|h_{K_o}(u) - h_{K_o}(-u)\|^2 = \mathcal{W}(K)^2 + \mathcal{E}(K)^2,$$

which proves the last statement of the theorem.

We note that (4) shows that $\delta(K, L+p)$, considered as a function of p is minimal if and only if p is such that the respective Steiner points of K and $L+p$ coincide. This has been proved previously by Arnold [1]. See also Groemer [6, Proposition 5.1.2] where spherical harmonics are used to prove this result and to establish related properties of the Steiner point, in particular the minimal property of the Steiner ball stated under (a) in the above theorem.

4. Normals and deviation measures

We now consider the relationship between the deviation measures and the normals of convex bodies. If $K \cap H_K(u)$ consists of only one point then u is said to be a *regular direction* of K. The set of all regular directions of K will be denoted by $\mathcal{R}(K)$. It is known that for every convex body almost all directions are regular (see, for example, Schneider [9, sec. 2.2]). If u is a regular direction of K the normal of K at $K \cap H_K(u)$ that is orthogonal to $H_K(u)$ will be denoted by $N_K(u)$. Furthermore, if X and Y are two parallel lines or a point and a line we let $d(X, Y)$ denote the (orthogonal) distance between X and Y. Corresponding to the three deviation measures we now define certain average values associated with the normals. It will be shown (in the lemma below) that all these average values, i. e., the corresponding integrals, exist.

If $K \in \mathcal{K}^n$ and $p \in \mathbf{R}^n$, then the average distance of the normals of K from p is defined by

$$\rho_K(p) = \sqrt{1/\sigma_n}\, \|d(p, N_K(u))\|,$$

It is easily shown (and will follow from our Theorem 3) that K is a ball with p as center exactly if $\rho_K(p) = 0$.

As mentioned before, bodies of constant width are characterized by the property that every normal is a double normal. This means that $N_K(-u) = N_K(u)$. Suggested by this fact we consider for regular directions u and $-u$ the distance between $N_K(u)$ and $N_K(-u)$ and the corresponding mean value

$$\omega_K = \sqrt{1/\sigma_n}\, \|d(N_K(u), N_K(-u))\|.$$

Note that ω_K does not depend on a particular point p.

Finally, to describe in terms of the normals those convex bodies that are centrally symmetric with respect to a given point p consider first the case $p = o$. In this case the central symmetry of K with respect to o implies that for any $u \in \mathcal{R}(K)$ and $-u \in \mathcal{R}(K)$ we have $N_K(-u) = -N_K(u)$. Clearly, the corresponding relation for symmetry with respect to an arbitrary point p can be expressed by $N_{K-p}(-u) = -N_{K-p}(u)$. Motivated by these considerations we define

$$\eta_K(p) = \sqrt{1/\sigma_n}\, \|d(N_{K-p}(-u), -N_{K-p}(u))\|.$$

Similarly as in the case of the deviation measures in Section 2 of particular interest are the respective minima of $\rho_K(p)$ and $\eta_K(p)$ for all possible choices of p. Thus, we define the following 'normal deviation measures' that correspond, respectively, to the spherical deviation, the eccentricity, and the width deviation. (The superscript $\perp$ indicates that these are deviation measures concerning the normals.)

$$\mathcal{S}^{\perp}(K) = \inf\{\rho_K(p) : p \in \mathbf{R}^n\}, \quad \mathcal{E}^{\perp}(K) = \inf\{\eta_K(p) : p \in \mathbf{R}^n\}, \quad \mathcal{W}^{\perp}(K) = \omega_K.$$

In analogy to Theorem 1 we now formulate a theorem that provides explicit evaluations of $\mathcal{S}^{\perp}(K)$ and $\mathcal{E}^{\perp}(K)$. It also exhibits a relationship of the same kind as (3).

Theorem 2. *Let $K \in \mathcal{K}^n$ and $p \in \mathbf{R}^n$. Then, for any $p \in \mathbf{R}^n$ we have*

$$\rho_K(p)^2 = \rho_K(s(K))^2 + \frac{(n-1)}{n}|p - s(K)|^2, \tag{6}$$

$$\eta_K(p)^2 = \eta_K(s(K))^2 + 4\frac{(n-1)}{n}|p - s(K)|^2, \tag{7}$$

and

$$4\rho_K(p)^2 = \eta_K(p)^2 + \omega_K^2. \tag{8}$$

Hence, considered as functions of p, both $\rho_K(p)$ and $\eta_K(p)$ are minimal exactly if $p = s(K)$ and it follows that

$$\mathcal{S}^{\perp}(K) = \rho_K(s(K)), \quad \mathcal{E}^{\perp}(K) = \eta_K(s(K)),$$

and

$$4\mathcal{S}^{\perp}(K)^2 = \mathcal{E}^{\perp}(K)^2 + \mathcal{W}^{\perp}(K)^2.$$

Next we consider the stability of the characterization of balls, symmetric bodies, and convex bodies of constant width in relation to the corresponding properties of their normals. In other words, we estimate $\mathcal{S}(K)$, $\mathcal{E}(K)$, and $\mathcal{W}(K)$ in terms of $\mathcal{S}^{\perp}(K)$, $\mathcal{E}^{\perp}(K)$, and $\mathcal{W}^{\perp}(K)$, respectively.

Theorem 3. *For any $K \in \mathcal{K}^n$ we have*

$$\mathcal{S}(K) \leq \sqrt{\kappa_n/2}\,\mathcal{S}^{\perp}(K). \tag{9}$$

$$\mathcal{E}(K) \leq \sqrt{\sigma_n/12(n+1)}\,\mathcal{E}^{\perp}(K), \tag{10}$$

$$\mathcal{W}(K) \leq \sqrt{\kappa_n/8}\,\mathcal{W}^{\perp}(K), \tag{11}$$

Equality holds in (9) exactly if the support function h_K of K is of the form $Q_0 + Q_1 + Q_2$, where Q_k denotes a spherical harmonic of order k. In (10) and (11) equality holds if and only if the expansion of h_K is, respectively, of the form $Q_1 + Q_3 + \sum_{k=0}^{\infty} Q_{2k}$ and $Q_0 + Q_2 + \sum_{k=0}^{\infty} Q_{2k+1}$.

If there is a point p such that $\rho_K(p) = 0$, $\eta_K(p) = 0$, or $\omega_K = 0$ and therefore, respectively, $\mathcal{S}^{\perp}(K) = 0$, $\mathcal{E}^{\perp}(K) = 0$, or $\mathcal{W}^{\perp}(K) = 0$ then Theorem 3 implies the previously mentioned characterizations of balls, symmetric bodies, or convex bodies of constant width.

Inequality (9) can be used to obtain a result concerning bodies that are nearly in equilibrium in any position on a horizontal plane. To describe this result, K will now be assumed to be a three-dimensional physical convex body. It is known (see [4] or [8]) that K must be a ball if it rests in equilibrium in any position on a horizontal plane. (This and a more general problem is mentioned in the well-known 'Scottish Book,' see Mauldin [7, Problem 19].) We wish to estimate the deviation of K from a ball if for any position on a horizontal plane the corresponding moment, say $M_K(u)$, of K is small. Here $M_K(u)$ is defined as follows: If u is a regular direction of K then $M_K(u)$ is the total mass of K multiplied by the distance of the normal $N_K(u)$ from the center of mass of K. In this connection K is not assumed to have uniform mass distribution, it may be endowed with any mass distribution, that is, an integrable density function. In terms of physics, if u is considered the vertical direction and K is assumed to be placed on a horizontal plane, then $M_K(u)$ is the moment that has to be applied to keep K in equilibrium. Applying (9) and Theorem 1 (part (a) with $n = 3$) and observing that $\mathcal{S}^{\perp}(K) \leq \rho_K(p)$, where p is the center of mass of K, we obtain immediately the following result.

Corollary. *Let K be a convex body in $\mathbf{R}^3$ having a given mass distribution with total mass $m(K) > 0$. If for some $\mu \geq 0$ and every regular direction u of K its corresponding moment $M_K(u)$ has the property that $M_K(u) \leq \mu$, then there is a ball B such that*

$$\delta_2(K, B) \leq \frac{\sqrt{2\pi/3}}{m(K)}\, \mu.$$

As a suitable ball one may choose the Steiner ball of K. Letting $\mu = 0$, one obtains the result stated before, that K must be a ball if it rests in equilibrium in any position on a horizontal plane. We also mention that this corollary can even be applied to the case when K is not convex since one may apply it to the convex hull $\tilde{K}$ of K and put the density function zero at all points of $\tilde{K} \setminus K$. Then, the above inequality holds if on the left hand side K is replaced by $\tilde{K}$ (and in the assumptions the regular directions of K are replaced by the regular directions of $\tilde{K}$). Clearly, one could also state an n-dimensional version of this corollary but this would not have the physical interpretation mentioned before.

To prove Theorems 2 and 3 we first show a lemma relating for any $K \in \mathcal{K}^n$ the distances $d(o, N_K(u))$, $d(N_K(u), N_K(-u))$, and $d(N_K(-u), -N_K(u))$ with the gradient of the support function of K and an associated series of spherical harmonics. A real valued function Φ on S^{n-1} will be said to be *smooth* if it is twice continuously differentiable. The (spherical) gradient of Φ will be denoted by $\nabla_o\Phi$. Thus, if Φ is a (differentiable) function on S^{n-1} and if ∇ denotes the ordinary gradient operator for functions on open subsets of $\mathbf{R}^n$ then $\nabla_o\Phi(u)$ is defined as the ordinary gradient of the constant radial extension of Φ evaluated at the point u. A more appropriate notation might be $(\nabla_o\Phi)(u)$ but this would lead to an unsightly accumulation of parentheses. Note, however, that this notational convention implies that for all $u \in S^{n-1}$ we have $\nabla_o\Phi(u) = (\nabla\Phi(x/|x|))_{x=u}$ and $\nabla_o\Phi(-u) = (\nabla\Phi(x/|x|))_{x=-u}$. It is customary, to assume that the support function h_K is extended from S^{n-1} to $\mathbf{R}^n$ by stipulating that it be positively homogeneous. It is of importance to notice that for the evaluation of $\nabla_o h_K$ the constant radial extension of h_K

has to be used. Thus, if h_K is defined in the conventional way then for any $u \in S^{n-1}$ we have $\nabla_o h_K(u) = (\nabla h_K(x/|x|))_{x=u}$ and therefore

$$\begin{aligned}\nabla h_K(u) &= \big(\nabla(|x| h_K(x/|x|))\big)_{x=u} = \left(\frac{1}{|x|} h_K(x)x + |x| \nabla h_K(x/|x|)\right)_{x=u} \\ &= h_K(u)u + \nabla_o h_K(u).\end{aligned} \tag{12}$$

If $\sum_{k=0}^{\infty} Q_k$ is the expansion of Φ as a series of spherical harmonics, where Q_k is of order k, we indicate this by writing

$$\Phi \sim \sum_{k=0}^{\infty} Q_k.$$

Lemma. *Let K be a convex body in $\mathbf{R}^n$.*

(i) *If $u \in \mathcal{R}(K)$ then $\nabla_o h_K(u)$ exists and*

$$d(o, N_K(u)) = |\nabla_o h_K(u)|. \tag{13}$$

If both $u \in \mathcal{R}(K)$ and $-u \in \mathcal{R}(K)$ then $\nabla_o(h_K + h_{-K})(u)$ and $\nabla_o(h_{-K} - h_K)(u)$ exist and

$$d(N_K(-u), -N_K(u)) = |\nabla_o(h_K - h_{-K})(u)|, \tag{14}$$

$$d(N_K(u), N_K(-u)) = |\nabla_o(h_K + h_{-K})(u)|. \tag{15}$$

Thus, (13), (14), *and* (15) *hold for almost all $u \in S^{n-1}$.*

(ii) *$d(o, N_K(u))$, $d(N_K(-u), -N_K(u))$, and $d(N_K(u), N_K(-u))$ are bounded integrable functions on S^{n-1}. Consequently, $\rho_K(o)$, $\eta_K(o)$, and ω_K exist.*

(iii) *If*

$$h_K \sim \sum_{k=0}^{\infty} Q_k, \tag{16}$$

then

$$\sigma_n \rho_K(o)^2 \geq \sum_{k=1}^{\infty} k(n+k-2)\|Q_k\|^2, \tag{17}$$

$$\sigma_n \eta_K(o)^2 \geq 4 \sum_{\substack{k \geq 1 \\ k \text{ odd}}} k(n+k-2)\|Q_k\|^2, \tag{18}$$

and

$$\sigma_n \omega_K^2 \geq 4 \sum_{\substack{k \geq 2 \\ k \text{ even}}} k(n+k-2)\|Q_k\|^2. \tag{19}$$

Moreover, each of the relations (17), (18), (19) *holds with equality if the respective functions h_K, $h_K - h_{-K}$, $h_K + h_{-K}$ are smooth.*

Proof. It is known (see [2, Sec. 16]) that for any $u \in \mathcal{R}(K)$ the gradient $\nabla h_K(u)$ exists and equals the support point $K \cap H_K(u)$. Furthermore, (12) shows that $\nabla_o h_K(u)$ is the vector from the intersection point of the line of direction u with $H(u)$ to the support point $H(u) \cap K$. Hence, obvious geometric considerations together with the fact that $\nabla_o h_K(-u) = -\nabla_o h_{-K}(u)$ show that

$$d(o, N_K(u)) = |\nabla_o h_K(u)|,$$

$$d(N_K(-u), -N_K(u)) = |\nabla_o h_K(-u) + \nabla_o h_K(u)| = |\nabla_o (h_K - h_{-K})(u)|,$$

and

$$d(N_K(u), N_K(-u)) = |\nabla_o h_K(u) - \nabla_o h_K(-u)| = |\nabla_o (h_K + h_{-K})(u)|.$$

These relations show the validity of (13), (14), and (15).

Turning to the proof of part (ii) we use the known fact (see for example Schneider [9, Sec. 3.3]) that for any $K \in \mathcal{K}^n$ there exists a sequence $\{K^j\}$ of strictly convex bodies that converges in the Hausdorff metric to K and is such that for every j the support function h_{K^j} is smooth. Routine convergence arguments show that for any $u \in \mathcal{R}(K)$ we have $\lim_{j\to\infty} K^j \cap H_{K^j}(u) = K \cap H_K(u)$ and therefore

$$\lim_{j\to\infty} d(o, N_{K^j}(u)) = d(o, N_K(u)), \tag{20}$$

$$\lim_{j\to\infty} d(N_{K^j}(-u), -N_{K^j}(u)) = d(N_K(-u), -N_K(u)), \tag{21}$$

$$\lim_{j\to\infty} d(N_{K^j}(u), N_{K^j}(-u)) = d(N_K(u), N_K(-u)). \tag{22}$$

Moreover, since the functions $d(o, N_{K^j}(u))$, $d(N_{K^j}(-u), -N_{K^j}(u))$, and $d(N_{K^j}(u) + N_{K^j}(-u))$ are obviously uniformly bounded and integrable the same is true for their respective limits $d(o, N_K(u))$, $d(N_K(-u), -N_K(u))$, and $d(N_K(u), N_K(-u))$.

Finally, for the proof of part (iii) we assume that (16) holds and consider first the case when h_K is smooth. Then it follows from (13) and known facts about spherical harmonics (see [6, Sec. 3.2]) that

$$\sigma_K \rho_K(o)^2 = \|d(o, N_K(u))\|^2 = \|\nabla_o h_K(u)\|^2 = \sum_{k=1}^{\infty} k(n+k-2)\|Q_k\|^2. \tag{23}$$

Also, (16) implies that

$$h_{-K}(u) = h_K(-u) \sim \sum_{k=0}^{\infty} (-1)^k Q_k(u)$$

and therefore

$$(h_K - h_{-K})(u) = h_K(u) - h_{-K}(u) \sim 2 \sum_{\substack{k\geq 1 \\ k \text{ odd}}} Q_k(u), \tag{24}$$

and

$$(h_K + h_{-K})(u) = h_K(u) + h_K(-u) \sim 2 \sum_{\substack{k\geq 0 \\ k \text{ even}}} Q_k(u). \tag{25}$$

Similarly as before, but using (14) and (15), and assuming that, respectively, $h_K - h_{-K}$ or $h_K + h_{-K}$ is smooth we find

$$\sigma_n \eta_K(o)^2 = 4 \sum_{\substack{k \geq 1 \\ k \text{ odd}}} k(n+k-2)\|Q_k\|^2 \tag{26}$$

and

$$\sigma_n \omega_K^2 = 4 \sum_{\substack{k \geq 2 \\ k \text{ even}}} k(n+k-2)\|Q_k\|^2. \tag{27}$$

This proves the assertion expressed in the last sentence of the lemma.

Finally, to prove the remaining part of (iii) we use again the sequence $\{K^j\}$ and note that (20), (21), (22) together with the 'bounded convergence theorem' allows one to deduce that

$$\lim_{j \to \infty} \rho_{K^j}(o) = \rho_K(o), \qquad \lim_{j \to \infty} \eta_{K^j}(o) = \eta_K(o), \qquad \lim_{j \to \infty} \omega_{K^j} = \omega_K. \tag{28}$$

Furthermore, if $h_{K^j} \sim \sum_{k=0}^{\infty} Q_k^j$, then (for all $u \in S^{n-1}$)

$$\lim_{j \to \infty} Q_k^j(u) = Q_k(u). \tag{29}$$

This relation is a consequence of the fact that for fixed k each Q_k^j and Q_k are linear combinations of the members of a finite orthonormal set of spherical harmonics $P_{k,1}, \ldots, P_{k,m}$ whose coefficients are r espectively of the form $\langle h_{K^j}(u), P_{k,i}(u)\rangle$ and $\langle h_K(u), P_{k,i}(u)\rangle$, and that uniformly $\lim_{j \to \infty} h_{K^j} = h_K$. This also shows that for each k the functions Q_k^j are uniformly bounded and it follows from (29) that

$$\lim_{j \to \infty} \|Q_k^j\| = \|Q_k\|. \tag{30}$$

From (23), (26), and (27) (applied to each K^j) one obtains that for any positive integer m

$$\sigma_n \rho_{K^j}(o)^2 \geq \sum_{k=0}^{m} k(n+k-2)\|Q_k^j\|^2, \qquad \sigma_n \eta_{K^j}(o)^2 \geq 4 \sum_{\substack{1 \leq k \leq m \\ k \text{ odd}}} k(n+k-2)\|Q_k^j\|^2,$$

$$\sigma_n \omega_{K^j}^2 \geq 4 \sum_{\substack{2 \leq k \leq m \\ k \text{ even}}} k(n+k-2)\|Q_k^j\|^2.$$

Using (28) and (30) and letting first $j \to \infty$ and then $m \to \infty$ we obtain the desired inequalities (17), (18), and (19).

For the following proofs of Theorems 2 and 3 we assume that (16) holds and repeatedly use the fact that $Q_0 + Q_1$ is the support function of the Steiner ball $B(K)$ (see [6, Sec. 5.1]) and therefore $Q_0 = \bar{w}(K)/2$ and $Q_1 = s(K) \cdot u$. If $K \in \mathcal{K}^n$ is given, the sequence $\{K^j\}$ and the spherical harmonics Q_k^j are defined as in the proof of the Lemma.

Proof of Theorem 2. It suffices to consider only the case $p = o$ since the general result can the be obtained by replacing, if necessary, K by $K - p$. From (23), (26), and (27), applied to K^j (with equality) and Q_k replaced by Q_k^j one sees that (8) holds for each K^j. Letting $j \to \infty$ and observing (28) one obtains (8) for all $K \in \mathcal{K}^n$.

To prove (6) we first note that

$$h_{K-s(K)}(u) = h_K(u) - s(K) \cdot u \sim \sum_{\substack{k \geq 0 \\ k \neq 1}} Q_k(u).$$

Hence, if h_K is smooth it follows from (23) that

$$\sigma_n(\rho_K(o)^2 - \rho_K(s(K))^2) = \sigma_n(\rho_K(o)^2 - \rho_{K-s(K)}(o)^2) = (n-1)\|Q_1\|^2.$$

Since

$$\|Q_1\|^2 = \|s(K) \cdot u\|^2 = \kappa_n |s(K)|^2$$

we obtain (6) for smooth h_K. The general case is again settled by the use of the sequence $\{K^j\}$ and (28). (Note that it can be assumed that for all j we have $s(K^j) = s(K)$, otherwise replace K_j by $K_j - s(K_j) + s(K)$.) Finally, (7) is an obvious consequence of (6) and (8).

Proof of Theorem 3. Observing that the inequalities of Theorem 3 are invariant under translations we may assume that $s(K) = o$. This implies that $K = K_o$ and

$$Q_1 = 0.$$

Since $Q_0 = \bar{w}(K)/2$ we obtain from (16), (24), and (25), combined with Parseval's equality, that

$$\mathcal{S}(K)^2 = \|h_{K_o}(u) - \bar{w}(K)/2\|^2 = \sum_{k=2}^{\infty} \|Q_k\|^2,$$

$$\mathcal{E}(K)^2 = \frac{1}{4}\|h_{K_o}(u) - h_{K_o}(-u)\|^2 = \sum_{\substack{k \geq 3 \\ k \text{ odd}}} \|Q_k\|^2,$$

$$\mathcal{W}(K)^2 = \frac{1}{4}\|h_K(u) + h_K(-u) - \bar{w}(K)\|^2 = \sum_{\substack{k \geq 2 \\ k \text{ even}}} \|Q_k\|^2.$$

Hence, in conjunction with part (iii) of the above lemma, it follows that

$$\sigma_n \mathcal{S}^{\perp}(K) = \sigma_n \rho_K(o) \geq \sum_{k=1}^{\infty} k(n+k-2)\|Q_k\|^2 \geq 2n \sum_{k=2}^{\infty} \|Q_k\|^2 = 2n\mathcal{S}(K)^2, \tag{31}$$

$$\sigma_n \mathcal{E}^{\perp}(K) = \sigma_n \eta_K(o)^2 \geq 4 \sum_{\substack{k \geq 1 \\ k \text{ odd}}} k(n+k-2)\|Q_k\|^2 \geq 12(n+1) \sum_{\substack{k \geq 3 \\ k \text{ odd}}} \|Q_k\|^2 = 12(n+1)\mathcal{E}(K)^2, \tag{32}$$

$$\sigma_n \mathcal{W}^{\perp}(K) = \sigma_n \omega_K^2 \geq 4 \sum_{\substack{k \geq 2 \\ k \text{ even}}} k(n+k-2)\|Q_k\|^2 \geq 8n \sum_{\substack{k \geq 2 \\ k \text{ even}}} \|Q_k\|^2 = 8n\mathcal{W}(K)^2. \quad (33)$$

These inequalities contain the inequalities of Theorem 3. If in (9) equality holds then equality must hold between the third and fourth term in (31). This implies that $Q_k = 0$ if $k \geq 3$. Hence, $h_K \sim Q_0 + Q_2$ and therefore $h_K = Q_0 + Q_2$. Conversely, if $h_K = Q_0 + Q_2$ then h_K is smooth and (according to (23)) equality must also hold between the second and third term of (31) and therefore in (9). Hence, under the assumption that $s(K) = o$ equality holds in (9) if and only if $h_K = Q_0 + Q_2$. In the general case this shows that $h_{K-s(K)} = Q_0 + Q_2$ and consequently $h_K = Q_0 + Q_1 + Q_2$ as stated in Theorem 3. Similarly, equality in (10) implies that $h_K - h_{-K} \sim Q_3$ and therefore $h_K - h_{-K} = Q_3$. This is also sufficient for equality in (32) and therefore in (10). Since (24) shows that this relation is satisfied exactly if in (16) all odd terms except Q_3 vanish we find that $h_K \sim Q_3 + \sum_{k=0}^{\infty} Q_{2k}$. Thus, adding the term Q_1 to remove the condition $s(K) = o$ we see that the conditions for equality in (10) are as stated in Theorem 3. Analogous arguments, applied to (33), and correspondingly to $h_K + h_{-K}$ establish the assertions concerning equality in (11).

References

[1] Arnold, R.: *Zur L^2-Bestapproximation eines konvexen Körpers durch einen bewegten konvexen Körper.* Monatsh. Math. **108** (1989), 277–293. Zbl 0712.52005

[2] Bonnesen T.; Fenchel, W.: *Theorie der konvexen Körper.* Ergebn. d. Math. Bd. 3, Springer Verlag, Berlin 1934. (Engl. transl.: Theory of Convex Bodies, BCS Assoc. Moscow, Idaho, 1987.) Zbl 0008.07708

[3] Chakerian, G. D.; Groemer, H.: *Convex bodies of constant width.* In: Convexity and Its Applications, pp. 49–96. Birkhäuser Verlag, Basel-Boston-Stuttgart 1983. Zbl 0518.52002

[4] Groemer, H.: *Eine kennzeichnende Eigenschaft der Kugel.* Enseign. Math. **7** (1961), 275–276. Zbl 0111.34804

[5] Groemer, H.: *Stability of Geometric Inequalities.* Handbook of Convex Geometry (P. M. Gruber and J. M. Wills Editors), Section 1.4., North Holland Publ., Amsterdam etc. 1993. Zbl 0789.52001

[6] Groemer, H.: *Geometric Applications of Fourier Series and Spherical Harmonics.* Cambridge University Press, Cambridge-New York 1996. Zbl 0877.52002

[7] Mauldin, R. D. (Editor): *The Scottish Book.* Birkhäuser, Boston-Basel-Stuttgart 1981. Zbl 0485.01013

[8] Montejano, L.: *On a problem of Ulam concerning the characterization of the sphere.* Studies in Appl. Math. **53** (1974), 243–248. Zbl 0418.52008

[9] Schneider, R.: *Convex Bodies: the Brunn-Minkowski Theory.* Cambridge University Press, Cambridge-New York 1993. Zbl 0798.52001

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