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Conformally Flat Contact Metric Manifolds with $Q\xi = \varrho\xi$

Florence Gouli-Andreou **Niki Tsolakidou**

Aristotle University of Thessaloniki, Department of Mathematics
Thessaloniki-540 06, Greece
e-mail: fgouli@mailhost.ccf.auth.gr

Abstract. We study conformally flat contact metric manifolds M^{2n+1} $(n > 1)$ for which the characteristic vector field is an eigenvector of the Ricci tensor. We prove that those manifolds are of constant sectional curvature.

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1. Introduction

It is well-known that the curvature of a three-dimensional Riemannian manifold is completely determined by its Ricci tensor. This motivates the study of the properties of this tensor. Let M^{2n+1} be a $(2n+1)$-dimensional contact metric manifold and (φ, ξ, η, g) its contact metric structure. We denote by ∇, R and Q the Levi-Civita connection, the Riemannian curvature and the Ricci operator on M^{2n+1} respectively. If the Ricci operator Q commutes with φ then the characteristic vector field is an eigenvector field of the Ricci tensor, i.e. $Q\xi = (Tr\ell)\,\xi$, $(\ell := R(\cdot, \xi)\,\xi)$, but the converse does not need to be true. We come across the relation $Q\xi = (Tr\ell)\,\xi$ in the study of several problems regarding contact metric manifolds. Many examples of 3-dimensional contact metric manifolds, on which the characteristic vector field is an eigenvector of the Ricci operator, are known such as the 3-dimensional flat torus, the 3-dimensional contact metric manifolds on which the Ricci operator commutes with φ which are not Sasakian [3], [4], etc. This fact led S.Tanno [11] to the study of conformally flat K-contact manifolds M^{2n+1} $(n > 1)$. He proved that those manifolds are of constant curvature $+1$. G.Calvaruso, D.Perrone and L.Vanhecke [5] studied 3-dimensional conformally flat contact

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metric manifolds with $Q\xi = (Tr\ell)\,\xi$. They proved that those manifolds are of constant curvature. R.Sharma [10] studied conformally flat contact metric manifolds of dimension > 3 which satisfy the conditions: *i)* $Q\xi = (Tr\ell)\,\xi$ and *ii)* $K(\xi, X) = K(\xi, \varphi X)$ for every tangent vector field X orthogonal to ξ. He proved that those manifolds are of constant curvature. A.Ghosh and R.Sharma [6] proved that every conformally flat contact strongly pseudo-convex integrable CR-metric manifold of dimension > 3 satisfying $Q\xi = (Tr\ell)\,\xi$ is of constant curvature. We note down that every 3-dimensional contact metric manifold is strongly pseudo-convex integrable CR-manifold [12]. Therefore the respective problem for the dimension 3 has already been studied in [5]. A.Ghosh, Th.Koufogiorgos and R.Sharma [7] proved that every conformally flat contact strongly pseudo-convex integrable CR-metric manifold of dimension > 3 is of constant curvature. In the same paper they proved that every conformally flat contact metric manifold with $Q\xi = (Tr\ell)\,\xi$ and $K(\xi, X) + K(\xi, \varphi X)$ independent of X is of constant curvature.

We should note down that the condition $Q\xi = (Tr\ell)\,\xi$ is invariant under a D-homothetic deformation [8] and it is equivalent to the condition that the characteristic vector field ξ is an eigenvector of the Laplacian $\Delta = g^{ij}\nabla_i\nabla_j$. We note also that it is shown in [2] that there exist three-dimensional conformally flat contact metric spaces which are not real space forms.

The main result of this paper is the following:

Let M^{2n+1} $(n > 1)$ be a conformally flat contact metric manifold with the characteristic vector field an eigenvector of the Ricci operator Q at every point. Then M^{2n+1} is of constant curvature.

This result generalizes S.Tanno's [11] result for the K-contact manifolds and extends the result of G.Calvaruso, D.Perrone and L.Vanhecke [5].

2. Preliminaries

A contact manifold is a C^∞-manifold M^{2n+1} together with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. Since $d\eta$ is of rank $2n$, there exists a unique vector field ξ on M^{2n+1} satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for all X. The vector field ξ is called the characteristic vector field or Reeb vector field of the contact structure η. Every contact manifold has an underlying almost contact structure (η, φ, ξ) where φ is a global tensor field of type $(1,1)$ such that

$$\eta(\xi) = 1,\ \varphi\xi = 0,\ \eta \circ \varphi = 0,\ \varphi^2 = -I + \eta \otimes \xi. \tag{2.1}$$

A Riemannian metric g can be defined (not uniquely) such that

$$\eta(X) = g(\xi, X)\,,\ \Phi(X, Y) = d\eta(X, Y) = g(X, \varphi Y)\,. \tag{2.2}$$

The metric g is said to be associated to the contact structure η. We note that g and φ are not unique for a given contact form η, but g and φ are canonically related to each other.

We refer to $(M^{2n+1}, \eta, \xi, \varphi, g)$ as a contact metric structure.

In what follows, we shall denote by ∇ the Levi-Civita connection of M^{2n+1}, R the corresponding Riemannian curvature tensor, Q the Ricci operator and r the scalar curvature.

In the theory of contact metric manifolds the tensor fields $\ell := R(\cdot, \xi)\,\xi$ and $h := \frac{1}{2}(\pounds_\xi\varphi)$, where $\pounds$ is the Lie derivation, play a fundamental role. h is a symmetric operator which satisfies the following relations:

$$h\varphi = -\varphi h, \quad h\xi = 0, \quad Trh = Trh\varphi = 0. \tag{2.3}$$

On a contact metric manifold we have the following further important relations involving h,

$$\nabla_X \xi = -\varphi X - \varphi h X, \tag{2.4}$$

$$\nabla_\xi \varphi = 0, \tag{2.5}$$

$$Tr\ell = g(Q\xi, \xi) = 2n - Trh^2. \tag{2.6}$$

We denote by D the subbundle of the tangent bundle TM^{2n+1} of M^{2n+1} defined by $\eta = 0$. The restriction $\varphi' = \varphi_{/D}$ of φ to D defines an almost complex structure on D. That means that $\varphi^2_{/D} = -I$ and the Riemannian metric g' defined by $g'(X,Y) = -d\eta\left(X, \varphi_{/D} Y\right)$, for all vector fields X, Y which belong to D, define on D an almost Hermitian structure. The pair $\left(\eta, \varphi_{/D}\right)$ is called the CR-structure associated with the contact metric structure (η, ξ, φ, g) [12]. If the complex distribution $\left\{X - i\varphi_{/D} X \,/\, X \in D\right\}$ is integrable, the contact metric structure (η, ξ, φ, g) is a strongly pseudo-convex integrable CR-structure.

A contact metric structure is a strongly pseudo-convex integrable CR-structure if and only if it satisfies the integrability condition

$$(\nabla_X \varphi) Y - g(X + hX, Y)\xi + \eta(Y)(X + hX) = 0, \ \forall X, Y \in \mathfrak{X}\left(M^{2n+1}\right). \tag{2.7}$$

A K- contact manifold M^{2n+1} is a contact metric manifold such that the characteristic vector field ξ is a Killing vector field with respect to g. M^{2n+1} is K-contact if and only if $h = 0$ or $Q\xi = 2n\xi$. If the almost complex structure J on $M^{2n+1} \times \Re$ defined by the relation

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right)$$

is integrable, M^{2n+1} is said to be Sasakian. A contact metric manifold is Sasakian if and only if it satisfies

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \ \forall X, Y \in \mathfrak{X}\left(M^{2n+1}\right). \tag{2.8}$$

Any Sasakian manifold is K-contact. The converse holds only for three-dimensional spaces. We refer to [1] for more information about contact metric manifolds.

A Riemannian manifold (M^n, g) is called conformally flat if it is conformally equivalent to a Euclidean space. A Riemannian manifold (M^n, g) is conformally flat if and only if it satisfies

$$\begin{aligned} R(X,Y)Z &= \frac{1}{n-2}\left[g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y\right] - \\ &-\frac{r}{(n-1)(n-2)}\left[g(Y,Z)X - g(X,Z)Y\right], \ for\ n > 3, \end{aligned} \tag{2.9}$$

and

$$(\nabla_X P)Y = (\nabla_Y P)X, \ for\ n = 3,$$

where $r = TrQ$ is the scalar curvature of M^n and $P = -Q + \frac{r}{4}Id$.

3. Conformally flat contact metric manifolds with $Q\xi = \varrho\xi$, where ϱ is a smooth function

Let $M^{2n+1}(\eta, \xi, \varphi, g)$ be a contact metric manifold. h is a symmetric operator. Hence it is diagonalizable. That means that there exists an orthonormal frame of eigenvectors of h.

Since $h\xi = 0$, ξ is an eigenvector of h. If $X \in Ker\eta$ is an eigenvector of h with eigenvalue λ then from (2.3) we conclude that φX is also an eigenvector of h with eigenvalue $-\lambda$. Let $\{e_1, e_2, \dots, e_n, e_{n+1} = \varphi e_1, e_{n+2} = \varphi e_2, \dots, e_{2n} = \varphi e_n, \xi\}$ be an orthonormal frame formed by unit eigenvectors e_i of h with eigenvalue λ_i, $(i = 1, 2, \dots, n)$. Then the following relations hold:

$$\nabla_\xi e_i = \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij} e_j + \sum_{j=1}^{n} b_{ij}\varphi e_j, \; i = 1, 2, \dots, n, \tag{3.1}$$

$$\nabla_\xi \varphi e_i = \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}\varphi e_j - \sum_{j=1}^{n} b_{ij} e_j, \; i = 1, 2, \dots, n, \tag{3.2}$$

where

$$a_{ij} = -a_{ji}, \; i, j = 1, 2, \dots, n \tag{3.3}$$

$$b_{ij} = b_{ji}, \; i, j = 1, 2, \dots, n. \tag{3.4}$$

From the relation (2.4) we obtain

$$\nabla_{e_i}\xi = -(1 + \lambda_i)\varphi e_i, \; i = 1, 2, \dots, n, \tag{3.5}$$

$$\nabla_{\varphi e_i}\xi = (1 - \lambda_i) e_i, \; i = 1, 2, \dots, n. \tag{3.6}$$

Differentiating the inner products $g(e_i, e_j)$, $g(e_i, \xi)$, $i, j = 1, 2, \dots, 2n$ with respect to e_k, $k = 1, 2, \dots, 2n$ we obtain the following relations:

$$\begin{aligned}
\nabla_{e_i} e_i &= \sum_{\substack{k=1\\ k\neq i}}^{n} A_{ik} e_k + \sum_{\substack{k=1\\ k\neq i}}^{n} \overline{A}_{ik}\varphi e_k + A_i \varphi e_i,\\
\nabla_{\varphi e_i}\varphi e_i &= \sum_{\substack{k=1\\ k\neq i}}^{n} B_{ik} e_k + \sum_{\substack{k=1\\ k\neq i}}^{n} \overline{B}_{ik}\varphi e_k + B_i e_i,\\
\nabla_{e_i} e_j &= -A_{ij} e_i + \sum_{\substack{k=1\\ i\neq k\neq j}}^{n} C^k_{ij} e_k + \sum_{k=1}^{n} \overline{C}^k_{ij}\varphi e_k, \; i \neq j, \qquad (3.7)\\
\nabla_{\varphi e_i}\varphi e_j &= -\overline{B}_{ij}\varphi e_i + \sum_{k=1}^{n} D^k_{ij} e_k + \sum_{\substack{k=1\\ i\neq k\neq j}}^{n} \overline{D}^k_{ij}\varphi e_k, \; i \neq j,
\end{aligned}$$

$$\begin{aligned}
\nabla_{e_i}\varphi e_j &= -\overline{A}_{ij}e_i - \sum_{\substack{k=1\\ i\neq k\neq j}}^{n} \overline{C}^j_{ik}e_k - \overline{C}^j_{ij}e_j - Z_{ij}\varphi e_i + \sum_{\substack{k=1\\ i\neq k\neq j}}^{n} N^k_{ij}\varphi e_k,\ i\neq j,\\
\nabla_{\varphi e_i}e_j &= -E_{ij}e_i - B_{ij}\varphi e_i - D^j_{ij}\varphi e_j - \sum_{\substack{k=1\\ i\neq k\neq j}}^{n} D^j_{ik}\varphi e_k + \sum_{\substack{k=1\\ i\neq k\neq j}}^{n} F^k_{ij}e_k,\ i\neq j,\\
\nabla_{e_i}\varphi e_i &= -A_i e_i - \sum_{\substack{k=1\\ k\neq i}}^{n}\overline{C}^i_{ik}e_k + \sum_{\substack{k=1\\ k\neq i}}^{n} Z_{ik}\varphi e_k + (1+\lambda_i)\,\xi,\\
\nabla_{\varphi e_i}e_i &= -B_i\varphi e_i - \sum_{\substack{k=1\\ k\neq i}}^{n} D^i_{ik}\varphi e_k + \sum_{\substack{k=1\\ k\neq i}}^{n} E_{ik}e_k - (1-\lambda_i)\,\xi,
\end{aligned}$$

where

$$\begin{aligned}
N^k_{ij} &= -N^j_{ik},\ C^k_{ij} = -C^j_{ik},\ F^k_{ij} = -F^j_{ik},\ \overline{D}^k_{ij} = -\overline{D}^j_{ik},\\
i,j,k &\in \{1,2,\dots,n\},\ i\neq k\neq j,\ i\neq j.
\end{aligned} \tag{3.8}$$

From now on we suppose that $M^{2n+1}(\varphi,\xi,\eta,g)$ is a conformally flat contact metric manifold for which the characteristic vector field ξ is an eigenvector field of the Ricci tensor, i.e. $Q\xi = \varrho\xi$, where ϱ is a smooth function on M^{2n+1}. The relations (2.6) and $Q\xi = \varrho\xi$ yield $\varrho = Tr\ell$. Hence

$$Q\xi = (Tr\ell)\,\xi. \tag{3.9}$$

If $n=1$, M^3 is of constant curvature 0 or 1 [5].

We suppose that $n>1$. We compute the curvature tensors $R(e_i,\varphi e_i)\xi$, $R(e_i,e_j)\xi$, $R(\varphi e_i,\varphi e_j)\xi$, $R(e_i,\varphi e_j)\xi$, $i,j=1,2,\dots,n$, $i\neq j$, in two ways, first using (2.9) and (3.9) and secondly through (3.5), (3.6), (3.7) and (3.8) as $R(X,Y) = [\nabla_X,\nabla_Y] - \nabla_{[X,Y]}$. Comparing the resulting expensions we obtain the following relations:

$$(1-\lambda_i)\,A_{ij} + (1+\lambda_i)\,B_{ij} - (1-\lambda_j)\,Z_{ij} - (1-\lambda_j)\,D^i_{ij} = 0, \tag{3.10}$$

$$(1-\lambda_i)\,\overline{A}_{ij} + (1+\lambda_i)\,\overline{B}_{ij} - (1+\lambda_j)\,\overline{C}^i_{ij} - (1+\lambda_j)\,E_{ij} = 0, \tag{3.11}$$

$$(1+\lambda_i)\,A_{ij} - (1+\lambda_j)\,Z_{ij} = e_j\cdot\lambda_i, \tag{3.12}$$

$$(1-\lambda_j)\,\overline{C}^j_{ji} - (1+\lambda_i)\,\overline{A}_{ji} + 2\lambda_j\overline{C}^j_{ij} = 0, \tag{3.13}$$

$$(1+\lambda_j)\,\overline{C}^j_{ik} - (1+\lambda_i)\,\overline{C}^i_{jk} - (1-\lambda_k)\,\overline{C}^k_{ij} + (1-\lambda_k)\,\overline{C}^k_{ji} = 0, \tag{3.14}$$

$$(1+\lambda_i)\,N^k_{ji} - (1+\lambda_j)\,N^k_{ij} + (1+\lambda_k)\,C^k_{ij} - (1+\lambda_k)\,C^k_{ji} = 0, \tag{3.15}$$

$$(1-\lambda_j)\,E_{ij} - (1-\lambda_i)\,\overline{B}_{ij} = \varphi e_j\cdot\lambda_i, \tag{3.16}$$

$$(1+\lambda_i)\,D^i_{ij} - (1-\lambda_j)\,B_{ij} - 2\lambda_i D^i_{ji} = 0, \tag{3.17}$$

$$(1-\lambda_j)\,F^k_{ij} - (1-\lambda_i)\,F^k_{ji} - (1-\lambda_k)\,\overline{D}^k_{ij} + (1-\lambda_k)\,\overline{D}^k_{ji} = 0, \tag{3.18}$$

$$(\lambda_j-1)\,D^j_{ik} + (1-\lambda_i)\,D^i_{jk} + (1+\lambda_k)\,D^k_{ij} - (1+\lambda_k)\,D^k_{ji} = 0, \tag{3.19}$$

$$
\begin{aligned}
(1-\lambda_i) Z_{ij} - (1-\lambda_j) A_{ij} + 2\lambda_i D^i_{ji} &= 0, &(3.20)\\
(1+\lambda_i) \overline{A}_{ij} - (1-\lambda_j) \overline{C}^i_{ij} &= \varphi e_j \cdot \lambda_i, &(3.21)\\
(1+\lambda_i) D^j_{ji} - (1-\lambda_j) B_{ji} &= e_i \cdot \lambda_j, &(3.22)\\
(1+\lambda_j) E_{ji} - (1+\lambda_i) \overline{B}_{ji} - 2\lambda_j \overline{C}^j_{ij} &= 0, &(3.23)\\
(1-\lambda_j) C^k_{ij} + (1+\lambda_i) D^k_{ji} - (1-\lambda_k) N^k_{ij} - (1-\lambda_k) D^i_{jk} &= 0, &(3.24)\\
(1-\lambda_j) \overline{C}^k_{ij} + (1+\lambda_i) \overline{D}^k_{ji} - (1+\lambda_k) \overline{C}^j_{ik} - (1+\lambda_k) F^k_{ji} &= 0, &(3.25)
\end{aligned}
$$

where $i, j, k \in \{1, 2, \ldots, n\}$, $i \neq k \neq j$, $i \neq j$.

Lemma 1. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ $(n > 1)$ be a conformally flat contact metric manifold with the characteristic vector field ξ an eigenvector of the Ricci operator Q at every point. Then the following relations hold:*

$$
\begin{aligned}
\overline{C}^i_{kj} - \overline{C}^i_{jk} + \overline{C}^j_{ik} - \overline{C}^j_{ki} + \overline{C}^k_{ji} - \overline{C}^k_{ij} &= 0,\ i \neq k \neq j,\ i \neq j,\\
D^i_{jk} - D^i_{kj} + D^j_{ki} - D^j_{ik} + D^k_{ij} - D^k_{ji} &= 0,\ i \neq k \neq j,\ i \neq j,\\
\overline{C}^i_{jk} - \overline{C}^k_{ji} + \overline{D}^i_{jk} - F^i_{jk} &= 0,\ i \neq k \neq j,\ i \neq j,\\
D^j_{ki} - D^i_{kj} + C^j_{ki} - N^j_{ki} &= 0,\ i \neq k \neq j,\ i \neq j,\\
B_{ji} + A_{ji} - Z_{ji} - D^j_{ji} &= 0,\ i \neq j,\\
\overline{B}_{ji} + \overline{A}_{ji} - \overline{C}^j_{ji} - E_{ji} &= 0,\ i \neq j.
\end{aligned}
$$

Proof. It is well known that on every contact metric manifold M^{2n+1} the following formula holds [9] :

$$d\Phi = d^2\eta = 0.$$

The above formula implies

$$(\nabla_X \Phi)(Y, Z) + (\nabla_Y \Phi)(Z, X) + (\nabla_Z \Phi)(X, Y) = 0, \tag{3.26}$$

where

$$(\nabla_X \Phi)(Y, Z) = X \cdot g(Y, \varphi Z) - g(\nabla_X Y, \varphi Z) - g(Y, \varphi \nabla_X Z), \forall X, Y, Z \in \mathfrak{X}\left(M^{2n+1}\right).$$

Taking $X = e_k$, $Y = e_i$, $Z = e_j$, $i \neq k \neq j$, $i \neq j$, $i, j, k \in \{1, 2, \ldots, n\}$, into (3.26) and using the relations (3.7) we obtain

$$-\overline{C}^j_{ki} + \overline{C}^i_{kj} - \overline{C}^k_{ij} + \overline{C}^j_{ik} - \overline{C}^i_{jk} + \overline{C}^k_{ji} = 0,\ i \neq k \neq j,\ i \neq j. \tag{3.27}$$

Similarly, for $X = \varphi e_k$, $Y = \varphi e_i$, $Z = \varphi e_j$, $i \neq k \neq j$, $i \neq j$, $i, j, k \in \{1, 2, \ldots, n\}$, the relation (3.26) yields, because of (3.7),

$$D^j_{ki} - D^i_{kj} + D^k_{ij} - D^j_{ik} + D^i_{jk} - D^k_{ji} = 0,\ i \neq k \neq j,\ i \neq j. \tag{3.28}$$

Also, putting $X = \varphi e_k$, $Y = e_i$, $Z = \varphi e_j$, $i \neq k \neq j$, $i \neq j$, $i, j, k \in \{1, 2, \dots, n\}$, in the relation (3.26) and taking into account the relations (3.7), (3.8) we have

$$-\overline{C}^k_{ij} + \overline{C}^j_{ik} - F^i_{jk} + F^i_{kj} - \overline{D}^i_{kj} + \overline{D}^i_{jk} = 0, \; i \neq k \neq j, \; i \neq j. \tag{3.29}$$

Replacing in (3.26) X, Y, Z by e_k, φe_j, e_i, $i \neq k \neq j$, $i \neq j$, $i, j, k \in \{1, 2, \dots, n\}$, respectively and taking into account the relations (3.7), (3.8) we have

$$D^k_{ji} - D^i_{jk} + C^j_{ki} - C^j_{ik} + N^j_{ik} - N^j_{ki} = 0, \; i \neq k \neq j, \; i \neq j. \tag{3.30}$$

The relation (3.29) because of the relation (3.27) can be written in the form

$$\overline{C}^i_{jk} - \overline{C}^i_{kj} - \overline{C}^k_{ji} + \overline{C}^j_{ki} + F^i_{kj} - F^i_{jk} + \overline{D}^i_{jk} - \overline{D}^i_{kj} = 0, \; i \neq k \neq j, \; i \neq j. \tag{3.31}$$

We alternate the indices i, k in the relation (3.29) and we add the result to (3.29). We obtain in this way the relation

$$\overline{C}^j_{ik} + \overline{C}^j_{ki} - \overline{C}^k_{ij} - \overline{C}^i_{kj} - F^j_{ik} - F^j_{ki} + \overline{D}^j_{ki} + \overline{D}^j_{ik} = 0, \; i \neq k \neq j, \; i \neq j.$$

We alternate the indices i, j in the above relation and we add the result to (3.31). We obtain then

$$\overline{C}^i_{jk} - \overline{C}^k_{ji} + \overline{D}^i_{jk} - F^i_{jk} = 0, \; i \neq k \neq j, \; i \neq j. \tag{3.32}$$

The relation (3.30) because of the relation (3.28) can be written in the form

$$D^j_{ki} - D^j_{ik} + D^k_{ij} - D^i_{kj} + N^j_{ik} - N^j_{ki} + C^j_{ki} - C^j_{ik} = 0, \; i \neq k \neq j, \; i \neq j. \tag{3.33}$$

We alternate the indices i, j in the relation (3.30) and we add the result to (3.30). We obtain in this way the relation

$$D^k_{ij} + D^k_{ji} - D^i_{jk} - D^j_{ik} + C^k_{ij} + C^k_{ji} - N^k_{ij} - N^k_{ji} = 0, \; i \neq k \neq j, \; i \neq j.$$

We alternate the indices j, k in the above relation and we add the result to (3.33). We obtain then

$$D^j_{ki} - D^i_{kj} + C^j_{ki} - N^j_{ki} = 0, \; i \neq k \neq j, \; i \neq j. \tag{3.34}$$

We alternate the indices i, j in the relation (3.12) and we subtract (3.22) from the result. We obtain then the following relation

$$(1 - \lambda_j)\, B_{ji} - (1 + \lambda_i)\, D^j_{ji} - (1 + \lambda_i)\, Z_{ji} + (1 + \lambda_j)\, A_{ji} = 0, \; i \neq j.$$

Adding the above relation to the relation obtained from (3.10) alternating the indices i, j we have

$$B_{ji} + A_{ji} - Z_{ji} - D^j_{ji} = 0, \; i \neq j. \tag{3.35}$$

Similarly, alternating the indices i, j in the relations (3.16) and (3.21) and subtracting the results we obtain

$$(1+\lambda_j)\,\overline{A}_{ji} - (1-\lambda_i)\,\overline{C}^j_{ji} - (1-\lambda_i)\,E_{ji} + (1-\lambda_j)\,\overline{B}_{ji} = 0,\ i \neq j.$$

Adding the above relation to the relation obtained from (3.11) alternating the indices i, j we have

$$\overline{A}_{ji} + \overline{B}_{ji} - \overline{C}^j_{ji} - E_{ji} = 0, i \neq j. \tag{3.36}$$

□

We suppose now that there exists an open subset U of M^{2n+1} where $h \neq 0$ and let m a point of U. Then there exists a local orthonormal frame

$$\{e_1, e_2, \ldots, e_n, e_{n+1} = \varphi e_1, e_{n+2} = \varphi e_2, \ldots, e_{2n} = \varphi e_n, \xi\}$$

of smooth eigenvectors e_i of h in an open neighborhood $V \subset U$ of m with eigenvalue $\lambda_i, (i = 1, 2, \ldots, n)$ and $\lambda_i \neq 0$ for $i = 1, 2, \ldots, \nu$, $1 \leq \nu \leq n$.

Lemma 2. *On V the following formulas hold:*

$$A_{ij} = Z_{ij},\ E_{ij} = \overline{B}_{ij},\ B_{ij} = D^i_{ij},\ \overline{A}_{ij} = \overline{C}^i_{ij},\ \forall i, j \in \{1, 2, \ldots, n\},\ i \neq j.$$

Proof. Replacing in (2.9) X, Y, Z by ξ, X, Y respectively, where $X, Y \in \{e_1, e_2, \ldots, e_n, e_{n+1} = \varphi e_1, e_{n+2} = \varphi e_2, \ldots, e_{2n} = \varphi e_n\}$, we have

$$R(\xi, X)\,Y = \frac{1}{2n-1}\left[g(X,Y)\,Q\xi + g(QX,Y)\,\xi\right] - \frac{r}{2n(2n-1)}g(X,Y)\,\xi.$$

The above relation because of the relation (3.9) can be written in the form

$$R(\xi, X)\,Y = \frac{1}{2n-1}\left[g(X,Y)\,Tr\ell + g(QX,Y) - \frac{r}{2n}g(X,Y)\right]\xi.$$

Hence $R(\xi, X)\,Y = \kappa\xi$, where $\kappa = \frac{1}{2n-1}\left[g(QX,Y) + \left(Tr\ell - \frac{r}{2n}\right)g(X,Y)\right]$ and $X,\ Y \in \{e_1, e_2, \ldots, e_n, e_{n+1} = \varphi e_1, e_{n+2} = \varphi e_2, \ldots, e_{2n} = \varphi e_n\}$.

It is well known that on every contact metric manifold M^{2n+1} the following formula holds [9] :

$$\begin{aligned} & g(R(\xi,X)\,Y,Z) - g(R(\xi,X)\,\varphi Y,\varphi Z) + \\ & + g(R(\xi,\varphi X)\,Y,\varphi Z) + g(R(\xi,\varphi X)\,\varphi Y,Z) \\ = \ & 2(\nabla_{hX}\Phi)(Y,Z) - 2\eta(Y)\,g(X+hX,Z) + 2\eta(Z)\,g(X+hX,Y), \end{aligned} \tag{3.37}$$

$$\forall X, Y, Z \in \mathfrak{X}\left(M^{2n+1}\right).$$

The relation (3.37) for $X, Y, Z \in \{e_1, e_2, \dots, e_n, e_{n+1} = \varphi e_1, \dots, e_{2n} = \varphi e_n\}$, because of the relation $R(\xi, X) Y = \kappa\xi$, becomes

$$(\nabla_{hX}\Phi)(Y, Z) = 0, \forall X, Y, Z \in \{e_1, e_2, \dots, e_n, e_{n+1} = \varphi e_1, e_{n+2} = \varphi e_2, \dots, e_{2n} = \varphi e_n\}. \tag{3.38}$$

We have the following cases:

Case 1. Let $i \in \{1, 2, \dots, \nu\}$, $j \in \{1, 2, \dots, n\}, 1 \leq \nu \leq n, i \neq j$. Taking $X = Y = e_i$, $Z = e_j$, into (3.38) and using the relations (3.7) we obtain

$$\lambda_i \left(\overline{A}_{ij} - \overline{C}^i_{ij}\right) = 0.$$

Since $\lambda_i \neq 0$ on V, $\forall i \in \{1, 2, \dots, \nu\}$, $1 \leq \nu \leq n$, the above relation yields

$$\overline{A}_{ij} = \overline{C}^i_{ij}. \tag{3.39}$$

Also, setting $X = Y = e_i$, $Z = \varphi e_j$, in (3.38) and taking into account the relations (3.7) we have

$$\begin{aligned} \lambda_i (A_{ij} - Z_{ij}) &= 0, \text{ or} \\ A_{ij} &= Z_{ij}, \end{aligned} \tag{3.40}$$

since $\lambda_i \neq 0$ on V.

Taking into account the relations (3.39), (3.40), (3.35) and (3.36) we obtain

$$B_{ij} = D^i_{ij} \quad \text{and} \quad \overline{B}_{ij} = E_{ij}.$$

Case 2. Let $i, j \in \{\nu + 1, \dots, n\}$, $1 \leq \nu \leq n$, $i \neq j$. Then we have on V that $\lambda_i = \lambda_j = 0$. Alternating the indices i, j in the relation (3.22) we have

$$e_j \cdot \lambda_i - (1 + \lambda_j) D^i_{ij} + (1 - \lambda_i) B_{ij} = 0.$$

This implies that

$$B_{ij} = D^i_{ij},$$

since $\lambda_i = \lambda_j = 0$. Similarly, the relation (3.21) yields

$$\overline{A}_{ij} = \overline{C}^i_{ij}.$$

Hence taking into account the relations (3.35) and (3.36) we obtain

$$A_{ij} = Z_{ij} \quad \text{and} \quad \overline{B}_{ij} = E_{ij}.$$

Case 3. Let $i \in \{\nu + 1, \dots, n\}, j \in \{1, 2, \dots, \nu\}, 1 \leq \nu \leq n$. In this case the relation (3.22) takes the form

$$B_{ij} - (1 + \lambda_j) D^i_{ij} = 0, \tag{3.41}$$

since $\lambda_i = 0$. Similarly the relation (3.17) takes the form

$$-\left(1-\lambda_j\right) B_{ij} + D^i_{ij} = 0. \tag{3.42}$$

The relations (3.41) and (3.42) form at every point of V a homogeneous system. Its determinant is equal to $\lambda_j^2 \neq 0$, since $j \in \{1, 2, \ldots, \nu\}, 1 \leq \nu \leq n$. Hence the only solution is

$$B_{ij} = D^i_{ij} = 0,$$

and the relation (3.35) yields

$$A_{ij} = Z_{ij}.$$

Working in a similar way as before we can obtain from the relations (3.13) and (3.21)

$$\overline{A}_{ij} = \overline{C}^i_{ij} = 0.$$

The above relations and (3.36) yield

$$\overline{B}_{ij} = E_{ij}.$$

This completes the proof. □

Lemma 3. *On V the following formulas hold:*

$$\overline{C}^k_{ij} = \overline{C}^j_{ik}, N^k_{ij} = C^k_{ij}, \overline{D}^k_{ij} = F^k_{ij}, D^k_{ij} = D^j_{ik}, \forall i, j, k \in \{1, 2, \ldots, n\}, i \neq k \neq j, i \neq j.$$

Proof. We have the following cases:
Case 4. Let $i, j \in \{1, 2, \ldots, n\}, k \in \{1, 2, \ldots, \nu\}, 1 \leq \nu \leq n, i \neq k \neq j, i \neq j$. We apply the relation (3.37) for $X = e_k$, $Y = e_i$, $Z = e_j$ and taking into account that $R(\xi, X) Y = \kappa\xi$ for $X, Y \in \{e_1, e_2, \ldots, e_n, e_{n+1} = \varphi e_1, e_{n+2} = \varphi e_2, \ldots, e_{2n} = \varphi e_n\}$ we obtain

$$\begin{aligned} \left(\nabla_{he_k}\Phi\right)(e_i, e_j) &= 0, \text{ or} \\ \lambda_k \left(\nabla_{e_k}\Phi\right)(e_i, e_j) &= 0, \text{ or} \\ \left(\nabla_{e_k}\Phi\right)(e_i, e_j) &= 0, \end{aligned} \tag{3.43}$$

since $\lambda_k \neq 0$.

The relation (3.43) because of the relations (3.7) gives

$$\overline{C}^j_{ki} = \overline{C}^i_{kj}.$$

Taking into account the above relation and (3.32) we obtain

$$\overline{D}^j_{ki} = F^j_{ki}.$$

Similarly, setting $X = \varphi e_k$, $Y = \varphi e_i$, $Z = \varphi e_j$ in (3.37) we have for the same reasons

$$D^j_{ki} = D^i_{kj}.$$

This last relation and (3.34) give

$$N^j_{ki} = C^j_{ki}.$$

Case 5. Let $i, j \in \{1, 2, \ldots, \nu\}$, $k \in \{\nu+1, \ldots, n\}$, $1 \leq \nu \leq n$, $i \neq j$. In this case $\lambda_i \neq 0$, $\lambda_j \neq 0$. Then from Case 1 we have that

$$\overline{C}^j_{ik} = \overline{C}^k_{ij} \text{ and } \overline{C}^k_{ji} = \overline{C}^i_{jk},$$

since $i, j \in \{1, 2, \ldots, \nu\}$. The above relations and (3.27) give

$$\overline{C}^i_{kj} = \overline{C}^j_{ki}.$$

The last relation and (3.32) give

$$\overline{D}^j_{ki} = F^j_{ki}.$$

Similarly, using the result of Case 1 and the relation (3.28) we can prove that

$$D^j_{ki} = D^i_{kj}.$$

Using this last relation in (3.34) we have

$$N^j_{ki} = C^j_{ki}.$$

Case 6. Let $i, j, k \in \{\nu+1, \ldots, n\}, 1 \leq \nu \leq n$, $i \neq k \neq j$, $i \neq j$. In this case the relations (3.14) and (3.27) yield

$$\overline{C}^i_{kj} = \overline{C}^j_{ki}.$$

This relation and (3.32) give

$$\overline{D}^j_{ki} = F^j_{ki}.$$

Similarly, using the relations (3.19) and (3.28) we obtain

$$D^j_{ki} = D^i_{kj}.$$

The last relation and (3.34) yield

$$N^j_{ki} = C^j_{ki}.$$

Case 7. Let $i \in \{\nu+1, \ldots, n\}$, $j \in \{1, 2, \ldots, \nu\}$, $k \in \{\nu+1, \ldots, n\}, 1 \leq \nu \leq n$, $k \neq i$. Then from Case 1 we have that

$$\overline{C}^k_{ji} = \overline{C}^i_{jk},$$

since $j \in \{1, 2, \dots, \nu\}$. The above relation and (3.27) give

$$\overline{C}^i_{kj} - \overline{C}^j_{ki} + \overline{C}^j_{ik} - \overline{C}^k_{ij} = 0. \tag{3.44}$$

Alternating the indices i, j in the relation (3.25) and adding the result to (3.18) we obtain, because of (3.32), the relation

$$\lambda_j \left(\overline{D}^k_{ij} - F^k_{ij} \right) = 0.$$

The above relation gives

$$\overline{D}^k_{ij} = F^k_{ij},$$

since $j \in \{1, 2, \dots, \nu\}$. The last relation and (3.32) yield

$$\overline{C}^k_{ij} = \overline{C}^j_{ik}.$$

Using this relation and (3.44) we obtain

$$\overline{C}^i_{kj} = \overline{C}^j_{ki}.$$

Similarly, using the result of Case 1 and the relations (3.28), (3.24), (3.15) and (3.34) we can prove that

$$N^j_{ki} = C^j_{ki} \quad \text{and} \quad D^j_{ki} = D^i_{kj}. \qquad \square$$

Finally, we prove

Theorem 1. *Let M^{2n+1} be a conformally flat contact metric manifold with the characteristic vector field an eigenvector of the Ricci operator Q at every point. Then M^{2n+1} is of constant curvature 1 if $n > 1$ and 1 or 0 if $n = 1$.*

Proof. If $n = 1$ then M^3 has constant sectional curvature 0 or 1 [5] . Let $n > 1$. If $h \equiv 0$, then M^{2n+1} is K-contact. S.Tanno proved [11] that a conformally flat K-contact manifold has constant sectional curvature. Z.Olszak proved [9] that any contact metric manifold of constant sectional curvature and of dimension ≥ 5 is Sasakian of constant curvature 1. Hence in this case M^{2n+1} is Sasakian of constant curvature 1. We suppose now that there exists an open subset U of M^{2n+1} where $h \neq 0$ and let m a point of U. Then there exists a local orthonormal frame

$$\{e_1, e_2, \dots, e_n, e_{n+1} = \varphi e_1, e_{n+2} = \varphi e_2, \dots, e_{2n} = \varphi e_n, \xi\}$$

of smooth eigenvectors e_i of h in an open neighborhood $V \subset U$ of m with eigenvalue $\lambda_i, (i = 1, 2, \dots, n)$ and $\lambda_i \neq 0$ for $i = 1, 2, \dots, \nu$, $1 \leq \nu \leq n$. Then from Lemmas 3.2, 3.3 and the relations (2.1), (2.2), (2.5), (3.5), (3.6), (3.7) and (3.8) we have that on V holds the integrability condition (2.7). Hence V is a strongly pseudo-convex integrable CR-manifold. Then, since V is conformally flat and $n > 1$, we have from [7] that V has constant curvature 1. Hence $h = 0$ on V. This is a contradiction. $\square$

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