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Normalizing Extensions of Semiprime Rings

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Abstract. In this paper we study normalizing extensions of semiprime rings. For an extension S of R we construct the canonical torsion-free S^*, which is a normalizing extension of the symmetric ring of quotients Q of R. We extend results which are known for centralizing extensions and for normalizing bimodules to one-to-one correspondence between closed ideals. Finally we study prime ideals, non-singular prime ideals and (right) strongly prime ideals of intermediate extensions.

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Introduction

Prime ideals in ring extensions $R \subseteq S$ have extensively been studied in recent years. In particular, when the extension is generated by a finite set of R-centralizing elements, S is called a liberal extension ([12], [13]). A normalizing extension is again a finite extension which is generated by a set of R-normalizing generators ([7], [8], [9], [11]). Also strongly normalizing extensions have been considered in [10].

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Prime ideals in (not necessarily finite) centralizing extensions were studied in [1], [2] and [3]. In those papers the results on prime ideals were obtained as applications of the results on closed submodules of centralizing bimodules over prime and semiprime rings.

Recently the results on closed submodules were extended to normalizing bimodules over semiprime rings in [5]. The main result in that paper gives a one-to-one correspondence between closed submodules of a normalizing bimodule M over a semiprime ring R, closed submodules of its extension to a bimodule M^* over the symmetric ring of quotients Q of R and closed submodules of M_0, the set of all the R-normalizing elements of M^* (actually, this last set is not a module, but it can be treated in a very similar way over the set of all the R-normalizing elements of Q).

The purpose of this paper is to extend the results on centred extensions to normalizing extensions, applying the results of [5]. Throughout the paper R is a semiprime ring and S is a normalizing extension of R. Recall that if M is an R-bimodule, then M is said to be an R-normalizing bimodule if there exists $X = (x_i)_{i\in\Omega} \subseteq M$ such that M is generated over R by the set X and $Rx_i = x_iR$, for every $i \in \Omega$. A ring S is said to be a normalizing extension of R if $R \subseteq S$ and S is a normalizing bimodule over R.

In Section 1, we consider some types of normalizing extensions and give examples showing that they are, in general, all different. We show that the torsion submodule is not in general an ideal, but it is an ideal if the extension is of some special type, called essentially normalizing extension. The canonical torsion-free extension S^* of S is constructed in Section 2.

In the next Section 3 we extend the results on closed submodules of [5] to closed ideals. In Section 4 we study intermediate extensions and we show that the one-to-one correspondence can be extended to this context. We also prove that the correspondence preserves prime and semiprime ideals of intermediate extensions.

Finally, in Section 5 we study strongly prime and non-singular prime ideals of intermediate extensions of prime rings. We extend here several results of ([2], Section 6).

In the paper we use freely the terminology and results of [5]. In particular, unless otherwise stated, submodule means sub-bimodule. An ideal H of R is always a two-sided ideal and this will be denoted by $H \triangleleft R$. The set of all essential ideals of R is denoted by $\mathcal{E}(R)$ and we will write simply $\mathcal{E}$ if there is no possibility of misunderstanding. The symmetric Martindale ring of quotients of R will be denoted by Q. The right annihilator of a subset F in R will be denoted by $Ann_{R,r}(F)$. The notations $\subset$ and $\supset$ mean strict inclusions.

1. Normalizing extensions of rings

Let R be a semiprime ring and S a ring extension of R. Recall that an element $x \in S$ is said to be centralizing (resp. normalizing) over R if $rx = xr$, for every $r \in R$ (resp. $Rx = xR$). Also, x is said to be strongly normalizing over R if $Ix = xI$, for any ideal I of R [10].

Another notion we will use in the paper is the following

Definition 1.1. *An element $x \in S$ is said to be essentially normalizing if x is normalizing over R and satisfies the following condition: for any $I \in \mathcal{E}$ there exists $J = J(I) \in \mathcal{E}$ such that $Jx \subseteq xI$ and $xJ \subseteq Ix$.*

The ring extension $S \supseteq R$ is said to be a *centralizing* (*resp. normalizing, strongly normalizing, essentially normalizing*) *extension* if there exists a subset $X = (x_i)_{i\in\Omega}$ of S which is centralizing (resp. normalizing, strongly normalizing, essentially normalizing) over R and $S = \sum_{i\in\Omega} Rx_i$.

Consider the following conditions:

(i) S is a centralizing extension of R.

(ii) S is a strongly normalizing extension of R.

(iii) S is an essentially normalizing extension of R.

(iv) S is a normalizing extension of R.

We immediately have

Lemma 1.2. *The following implications hold:* $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$.

Now we give examples to show that the converse of the above implications do not hold.

Example 1.3. Assume that ϕ is an involution of a field K and put $S = K[x,\phi]$, the skew polynomial ring over K (we write the coefficients in the right). Then S is a strongly normalizing extension of K which is not centralizing over K. This is clear since $s \in S$ is centralizing if and only if $s \in \sum_i x^{2i}K$. □

Example 1.4. Let $\mathbf{Z}$ be the ring of integers, $R = \mathbf{Z} \times \mathbf{Z}$, $\phi : R \to R$ the automorphism defined by $\phi(n,m) = (m,n)$, for $n,m \in \mathbf{Z}$. Put $S = R[x,\phi]$ the skew polynomial ring over R. An element $s \in S$ is R-normalizing if and only if either $s \in \sum_{i=0}^{n} x^{2i}R$ (in this case s is centralizing over R) or $s \in \sum_{i=0}^{n} x^{2i+1}R$. It is easy to see that in the last case s is essentially normalizing, but not strongly normalizing. It follows that S is an essentially normalizing extension of R and it is not strongly normalizing. □

Example 1.5. Let $\mathbf{Z}_2$ be the prime field of characteristic 2 and $R = \mathbf{Z}_2 \times \prod_{i=2}^{\infty} Z_i$, where $Z_i = \mathbf{Z}$ for all $i \geq 2$, the product of rings. Consider the epimorphism of R given by $\phi(\bar{a}_1, a_2, a_3, \ldots) = (\bar{a}_2, a_3, \ldots)$, where $\bar{a} = a + 2\mathbf{Z} \in \mathbf{Z}_2$, and put $S = R[x,\phi]$. Then S is a normalizing extension of R with $(x^i)_{i\geq 0}$ as a set of normalizing generators, but is not essentially normalizing. In fact, if S is essentially normalizing over R, then at least one generator of S has to be of the form $s = r_0 + r_1x + \cdots + r_nx^n$, where $r_1 = (\bar{1}, r_{12}, r_{13}, \ldots)$. We have that $I = Z_2 \times \prod_{i=2}^{\infty} 2Z \in \mathcal{E}(R)$ and do not exist $J \in \mathcal{E}(R)$ with $Js \subseteq sI$. □

The torsion submodule $T(M)$ of a normalizing bimodule M over R has been defined and studied in ([5], Sections 2 and 5). The normalizing bimodule M is said to be torsion-free if $T(M) = 0$. When we consider a normalizing extension S of R it would be convenient that torsion submodule $T(S)$ will be an ideal. However, this is not the case, as we will see in the following example.

Let $R = K[x_1, x_2, \ldots]$ be the ring of polynomials in an infinite set of indeterminates $\{x_1, x_2, \ldots\}$ over a field K. Then R is a commutative prime ring. Consider the K-endomorphisms σ and ϕ of R defined by $\sigma(x_i) = x_{i+1}$ for any i, $\phi(x_i) = x_{i-1}$ if $i \geq 2$ and $\phi(x_1) = 0$.

Let W be the free monoid generated by X, Y and consider the ring defined as follows:

$$T = R < X, Y, \sigma, \phi >= \{\sum_w w a_w : a_w \in R, w \in W\},$$

where $\sum_w w a_w$ denotes a finite sum. The addition in T is defined as usual and the multiplication by $x_i X = X\sigma(x_i)$ and $x_i Y = Y\phi(x_i)$.

A monomial is an element of the type $wa \in T$, where a is a monomial in R and $w \in W$. Thus T can be regarded as a graded ring in two different ways:

(Gr_1) The degree of a monomial is the sum of the degrees with respect to X and Y.

(Gr_2) The degree of the monomial wa is the usual total degree of the coefficient a in R with respect to $\{x_1, x_2, \ldots\}$.

Let $I = (Xx_1)$ be the ideal of T generated by Xx_1. Then I is homogeneous in both graduations because the generator is Gr_i-homogeneous, $i = 1, 2$.

Lemma 1.6. *Under the above notation we have:*

(i) $X, Y \notin I$.

(ii) *If* $0 \neq f \in R$, *then* $f \notin I$ *and* $XYf \notin I$.

Proof. (i) Assume that $X \in I$. Then $X = \sum_j f_j X x_1 g_j$, where $f_j, g_j \in T$. Computing the Gr_2-degree we obtain a contradiction. Similarly $Y \notin I$.

(ii) It is enough to show that $XYf \notin I$. Suppose that $0 \neq f \in R$ and $XYf \in I$. Then $XYf = \sum_j f_j X x_1 g_j$, where $f_j, g_j \in T$. Since W is a free monoid there exists at least one j for which f_j contains a summand beginning with XY. Using Gr_1 we obtain a contradiction. □

Put $S = T/I$ and denote by $x = X + I, y = Y + I$. It is clear that S is a ring extension of R. We have

Example 1.7. *S is a normalizing extension of the prime ring R with normalizing generator set $(w + I)_{w \in W}$. Moreover, $x, y \in T(S)$ and $xy \notin T(S)$. In particular, $T(S)$ is not an ideal of S.*

Proof. It is clear that $Rx = xR$ and $Ry = yR$. The first part follows. Since R is a commutative prime ring, $xx_1 = 0$ and $x_1 y = 0$, we obtain $x, y \in T(S)$. Also $x_i xy = xx_{i+1} y = xyx_i$, for all i, so xy is R-centralizing. If $xy \in T(S)$, then for some $0 \neq r \in R$ we have $xyr = 0$, i.e., $XYr \in I$, a contradiction. □

Note that a torsion-free normalizing extension S of R is an essentially normalizing extension ([5] Lemma 2.4). In this case $T(S) = 0$ is, of course, an ideal. We show that the same is true for an essentially normalizing extension.

Assume that S is a normalizing extension of R and $X = (x_i)_{i \in \Omega}$ is a set of R-normalizing generators. We may suppose that $x_{i_0} = 1$, for some $i_0 \in \Omega$.

If $I \subseteq P$ are ideals of S, as in [5] we define the closure of I in P by

$$[I]_P = \{x \in P : \text{there exist } F, H \in \mathcal{E} \text{ such that } FxH \subseteq I\}.$$

The ideal I is said to be closed in P if $[I]_P = I$.

Lemma 1.8. *If S is an essentially normalizing extension of R and $I \subseteq P$ are ideals of S, then $[I]_P$ is also an ideal of S.*

Proof. We already know that $[I]_P$ is a submodule of ${}_RS_R$, by the results in [5]. Let $y \in [I]_P$ and take any generator $x \in X$. Then there exist $F, H \in \mathcal{E}$ such that $FyH \subseteq I$. Also, by Definition 1.1 there exist $F', H' \in \mathcal{E}$ such that $F'x \subseteq xF$ and $xH' \subseteq Hx$. So we have that $F'xyH \subseteq xFyH \subseteq xI \subseteq I$ and $FyxH' \subseteq FyHx \subseteq Ix \subseteq I$. Therefore $xy, yx \in [I]_P$ and the result follows. □

As a particular case of the above we have the following

Corollary 1.9. *Assume that S is an essentially normalizing extension of R. Then the torsion submodule $T(S)$ is an ideal of S.*

Proof. By Lemma 1.8, $T_1(S) = [0]_S$ is an ideal. The result follows since $T(S) = T_2(S) = [T_1(S)]_S$ ([5], Section 5). □

In general, to define the torsion ideal of S we extend the definition of $T(S)$. In this paper we want to study closed ideals. So we may assume that the set of closed ideals of S is not empty. Thus we define the *torsion ideal* $t(S)$ of S as the intersection of all the closed ideals of S, i.e., the smallest closed ideal. Thus any closed ideal of S contains $t(S)$. It is easy to see that there is a one-to-one correspondence between the closed ideals of S and the closed ideals of $S/t(S)$, via the canonical projection (see [5], Lemma 2.2).

There is another way to define $t(S)$. We put $t_1(S) = (T(S))$, the ideal generated by $T(S)$. Let $t_2(S)$ be the ideal generated by $t_2'(S)$, where $t_2'(S)$ is the submodule of S such that $t_2'(S)/t_1(S) = T(S/t_1(S))$. We proceed similarly to define $t_\gamma(S)$ for any non-limit ordinal number γ. If α is a limit ordinal we put $t_\alpha(S) = \sum_{\beta<\alpha} t_\beta(S)$. We have a transfinite sequence of ideals. Therefore there exists an ordinal ρ with $t_\rho(S) = t_{\rho+1}(S)$. This means that $T(S/t_\rho(S)) = 0$ and hence $t_\rho(S)$ is a closed ideal of S. It is easy to see that $t(S) = t_\rho(S)$.

Remark 1.10. If S is an essentially normalizing extension of R, then $t(S) = T(S)$, by Corollary 1.9.

Example 1.11. In Example 1.7 the torsion ideal of S is the ideal generated by x and y and the factor ring $S/t(S) = R$.

2. The canonical torsion-free extension

If M is a normalizing bimodule over R, then there exists a canonical torsion-free normalizing Q-bimodule M^* of M, where Q is the symmetric ring of quotients of R ([5], Section 8). The purpose of this section is to show that the same is true for a normalizing extension.

Let $S = \sum_{i\in\Omega} Rx_i$ be a normalizing extension of R with normalizing generators $(x_i)_{i\in\Omega}$. First we assume that S is torsion-free.

Let W denote the free monoid generated by Ω (with the empty word as identity). Take $w = i_1 i_2 \dots i_n \in W$ and consider $x_w = x_{i_1} \cdot \dots \cdot x_{i_n} \in S$, a normalizing element of S. Using similar notation as in ([5], Section 3), we put $A_w = Ann_{R,r}(x_w)$, $B_w = Ann_{R,l}(x_w)$ and let ϕ_w be the isomorphism of rings $\phi_w : R/B_w \to R/A_w$ defined by $\phi_w(r + B_w) = r' + A_w$, where $rx_w = x_w r'$.

For any $w \in W$ we consider a free R/A_w-module $T_w = wR/A_w$, with the unitary basis w, and define a structure of left R/B_w-module on T_w by $(r + B_w)w = w\phi_w(r + A_w)$, $r \in R$. Then T_w is an R-bimodule and $T = \bigoplus_{w \in W} wR/A_w$ is a normalizing bimodule over R with $(w)_{w\in W}$ as a set of normalizing generators. Moreover, we see that T has a ring structure and so is a normalizing extension of R.

In fact, if wr_w and vr_v are monomials in T we put $wr_w \cdot vr_v = wvr^v_w r_v$, where r^v_w is an element of R such that $r_w x_v = x_v r^v_w$. It is easy to see that this is a well-defined multiplication and defines a ring structure on T. Also we may consider $R \subseteq T$ via the application sending r to $r \cdot 1_T$ and so T is a normalizing extension of R. Moreover, the application $\Phi : T \to S$ defined by $\Phi(w) = x_w$ is an epimorphism of normalizing extensions such that $\Phi|_R = id_R$.

Note that T is torsion-free over R. Thus by Lemma 2.2 of [5] there is a canonical one-to-one correspondence between the set of all the closed ideals of S and the set of all the closed ideals of T containing $Ker\Phi$.

Now we define an extension of T to a normalizing Q-bimodule T^*, as in ([5], Section 3). A_w and B_w are closed ideals of R and so there exist closed ideals A^*_w and B^*_w of Q with $A^*_w \cap R = A_w$ and $B^*_w \cap R = B_w$. Thus the isomorphism ϕ_w can be extended to an isomorphism from Q/B^*_w to Q/A^*_w, denoted by ϕ_w again ([5], Corollary 1.2). Put $T^* = \bigoplus_{w\in W} wQ/A^*_w$, the canonical extension of T to a bimodule T^* over Q. Note that $Ann_{Q,r}(w) = A^*_w$ and $Ann_{Q,l}(w) = B^*_w$. It is easy to see that T^* is a ring extension of T and a torsion-free normalizing extension of Q, with $(w)_{w\in W}$ as a set of normalizing generators. Also, for any $x \in T^*$ there exists $H \in \mathcal{E}$ such that $xH \subseteq T$ and $Hx \subseteq T$.

Note that the construction of T^* here is similar to the construction of M^* in ([5], Section 3). In fact, to see this it is enough to consider the extension S as generated over R by $(x_w)_{w\in W}$ instead of $(x_i)_{i\in\Omega}$. Thus we may apply the results of that paper. In particular, by ([5], Theorem 4.9) there is a one-to-one correspondence via contraction between the set of all the R-closed submodules of T and the set of all the Q-closed submodules of T^*. We have

Lemma 2.1. *The one-to-one correspondence above is a one-to-one correspondence between closed ideals.*

Proof. Let I be an R-closed submodule of T and I^* the extension of I to a Q-closed submodule of T^*. If I^* is an ideal so is $I = I^* \cap T$. Conversely, if I is an ideal of T, $s \in I^*$ and $y \in T^*$, then there exist $H, F \in \mathcal{E}$ such that $sH \subseteq I$, $Hs \subseteq I$, $yF \subseteq T$ and $Fy \subseteq T$, by Corollary 4.8 of [5]. Hence $FysH \subseteq I$ and $HsyF \subseteq I$, and so $sy \in I^*$ and $ys \in I^*$, by the above quoted result. □

We consider again the epimorphism $\Phi : T \to S$. Since S is torsion-free over R, 0 is a closed ideal of S. Then $Ker\Phi$ is a closed ideal of T and so there exists a Q-closed ideal K^* of T^* such that $K^* \cap T = Ker\Phi$. We put $S^* = T^*/K^*$ and denote by $j : S \to S^*$ the application defined as follows. Let $x \in S$ and take $y \in T$ such that $\Phi(y) = x$. Thus $y \in T^*$ and we put $j(x) = \pi(y)$, where $\pi : T^* \to S^*$ is the canonical projection. Since K^* is a closed ideal of T^* the ring S^* is a torsion-free normalizing extension of Q and j is an injective ring homomorphism, called the canonical injection (cf. [5], Section 4).

Now let S be any normalizing extension of R. We consider the torsion-free normalizing extension of $S/t(S)$, where $t(S)$ is the torsion ideal of S. Then there exists the torsion-free extension of $S/t(S)$, denoted again by S^*. We define $j : S \to S^*$ as the composition of the canonical mappings $S \to S/t(S)$ and $S/t(S) \to S^*$, where the second application is the canonical injection.

Definition 2.2. *The pair (S^*, j) is called the canonical torsion-free extension of S to a normalizing extension of Q.*

It is clear that $Ker(j) = t(S)$ and we may consider $S \subseteq S^*$ if and only if S is torsion-free. If $(x_i)_{i\in\Omega}$ is a set of R-normalizing generators of S, then $(j(x_i))_{i\in\Omega}$ is a set of Q-normalizing generators of S^*. Finally, the pair (S^*, j) satisfies a universal property, as in ([5], Section 8), so is unique up to isomorphisms.

Note that since S^* is torsion-free, then S^* is always an essentially normalizing extension of Q.

Example 2.3. Let S be the normalizing extension given in Example 1.7. Then $S^* = Q$, since $S/t(S) = R$. We can modify the example in such a way that the canonical extension is not trivial. For example, if $S_1 = S[Y]$, where Y is a set of indeterminates, then $S_1^* = Q[Y]$ is a polynomial ring over Q.

3. The one-to-one correspondence

A one-to-one correspondence between closed submodules is obtained in ([5], Theorem 8.3). The purpose of this section is to show that the same is true for ideals of normalizing extensions.

In [5], the set Z of all the R-normalizing elements of Q was considered (Section 1). This is a multiplicative semigroup with an identity and has an addition partially defined, but is not in general a ring. Also in Sections 6–8 of that paper, the subset M_0 of all the elements of M^* which are R-normalizing plays an important role. This subset has also partially defined addition and a multiplication by the elements of Z, and the operations have natural properties. We extended the terminology by saying that M_0 is a Z-module.

We consider here the corresponding sets. Let $S_0 = \{x \in S^* : Rx = xR\}$. Hence S_0 is a semigroup with identity element and is a Z-module in the above sense. Also the operations have natural properties such as associativity and distributivity when addition is defined. We say here that S_0 is a Z-semigroup. Note that by definition $Z \subseteq S_0$.

We can consider semigroup ideals of S_0. A semigroup ideal I is a subset with the property: for any $x \in I$ and $s \in S_0$ we have $sx, xs \in I$. Actually a more restrictive concept is of interest here:

Definition 3.1. *A semigroup (submodule, semigroup ideal) I of S_0 is said to be saturated if the following holds: if $a_1, a_2, \ldots, a_n \in I$ and the addition $a_1 + a_2 + \ldots + a_n$ is defined in S_0, then $a_1 + a_2 + \ldots + a_n \in I$.*

If A and B are saturated ideals of S_0, we define the product of A and B by

$$AB = \{\sum_{i=1}^{n} a_i b_i : a_i \in A, b_i \in B \text{ and } \sum_{i=1}^{n} a_i b_i \in S_0\}.$$

It is easy to see that AB is also a saturated ideal of S_0.

Recall that an ideal H of Z is said to be essential if $Ann_Z(H) = 0$. The set of all the essential ideals of Z is denoted by $\mathcal{E}(Z)$. An ideal I of S_0 is said to be closed if $s \in S_0$ and $sH \subseteq I$, for some $H \in \mathcal{E}(Z)$, implies that $s \in I$.

Note that S, S^* and S_0 are defined as in ([5], Theorem 8.3). The one-to-one correspondence in that theorem also shows that any closed submodule of S_0 is saturated. Hence any closed ideal of S_0 is a saturated ideal.

As an easy consequence we have

Theorem 3.2. *Let S be a normalizing extension of a semiprime ring R, (S^*, j) the canonical torsion-free extension of S and S_0 the normalizer of R in S^*. Then there is a one-to-one correspondence between the set of all the R-closed ideals of S, the set of all the Q-closed ideals of S^* and the set of all the Z-closed ideals of S_0. Moreover, the correspondence associates the closed ideal I of S with the closed ideal I^* of S^* and the closed ideal I_0 of S_0 if $j^{-1}(I^*) = I$ and $I_0 = I^* \cap S_0$ (equivalently, $I^* = QI_0$).*

Proof. Recall that $I^* = \{x \in S^* :$ there exists $H \in \mathcal{E}$ such that $xH \subseteq I\} = \{x \in S^* :$ there exists $H \in \mathcal{E}$ such that $Hx \subseteq I\}$. Then if I is an ideal so is I^*. In fact, if $x \in I^*$ and $y \in S^*$, then there exist $F, H \in \mathcal{E}$ such that $xH \subseteq I$ and $Fy \subseteq S$. Thus $FyxH \subseteq I$ and it follows that $yx \in I^*$. Similarly, $xy \in I^*$ and so I^* is an ideal. The rest is clear. □

4. Intermediate extensions

In this section we consider intermediate extensions. Since closed ideals always contain $t(S)$ we restrict ourselves to the torsion-free case. So we assume that S is a torsion-free normalizing extension of R.

Recall that if $N \subseteq P$ are submodules of a torsion-free normalizing bimodule, then N is said to be dense in P if $[N]_P = P$. In this case there is a one-to-one correspondence, via contraction, between the set of all the closed submodules of P and the set of all the closed submodules of N ([5], Lemma 2.1). Also, in the torsion-free case we have that

$$[N]_P = \{x \in P : \text{there exists } H \in \mathcal{E} \text{ such that } xH \subseteq N\} =$$

$$\{x \in P : \text{there exists } F \in \mathcal{E} \text{ such that } Fx \subseteq N\},$$

([5], Corollary 4.2).

An intermediate extension is a subring of S containing R. Assume that $U \subseteq V$ are intermediate extensions such that U is dense in V.

If I is an ideal of U, then $[I]_V$ is an ideal of V, as is easy to see. Also, if I is a closed R-submodule of U we have $I = [I]_V \cap U$. So if $[I]_V$ is an ideal of V, then I is an ideal of U. Thus the following is an obvious extension of Lemma 2.1 from [5].

Lemma 4.1. *If U is dense in V, then there is a one-to-one correspondence between the set of all closed ideals of V and the set of all closed ideals of U.*

Now we have the following

Proposition 4.2. *Assume that $U \subseteq V$ are intermediate extensions and U is dense in V. Then the correspondence of Lemma 4.1 preserves prime and semiprime ideals.*

Proof. Let P be a closed submodule of V and put $\mathcal{P} = P \cap U$.

Assume that $\mathcal{P}$ is prime and let A,B be ideals of V with $AB \subseteq P$. It is easy to see that $[A]_V[B]_V \subseteq P$, since P is closed. Hence we may suppose that A and B are closed. We have $(A \cap U)(B \cap U) \subseteq \mathcal{P}$, consequently either $(A \cap U) \subseteq \mathcal{P}$ or $(B \cap U) \subseteq \mathcal{P}$ and it follows that $A = [A]_V = [A \cap U]_V \subseteq [\mathcal{P}]_V = P$ or $B = [B]_V = [B \cap U]_V \subseteq [\mathcal{P}]_V = P$. Therefore P is prime.

Conversely, assume that P is prime and A, B are ideals of U with $AB \subseteq \mathcal{P}$. We have that $[A]_V[B]_V \subseteq P$ and so either $A \subseteq [A]_V \subseteq P$ or $B \subseteq [B]_V \subseteq P$. Consequently, $A \subseteq P \cap U = \mathcal{P}$ or $B \subseteq P \cap U = \mathcal{P}$ and so $\mathcal{P}$ is prime.

The proof of the semiprime case is the same with $A = B$. □

If V is an intermediate extension, then $[V]_S$ is also an intermediate extension which is closed as an R-submodule of S and V is dense in $[V]_S$. Thus by ([5], Theorem 8.3) there exist a Q-closed submodule V^* of S^* such that $V^* \cap S = [V]_S$ and a Z-closed submodule V_0 of S_0 with $V_0 = V^* \cap S_0$ and $V^* = V_0Q$. Hence V^* is a subring of S^* containing Q and V_0 is a saturated subsemigroup of S_0 containing Z. By Corollary 8.4 in [5] there is a one-to-one correspondence between the set of all R-closed submodules of V, the set of all Q-closed submodules of V^* and the set of all Z-closed submodules of V_0.

A saturated ideal P_0 of V_0 is said to be prime (resp. semiprime) if the following holds: $AB \subseteq P_0$ (resp. $A^2 \subseteq P$), for ideals A, B of V_0 (resp. A of P_0), implies that either $A \subseteq P_0$ or $B \subseteq P_0$. (resp. $A \subseteq P_0$). The semigroup V_0 is said to be prime (semiprime) if 0 is a prime (semiprime) ideal of V_0.

We have the following extension of Theorem 3.2.

Theorem 4.3. *Let V be an intermediate extension of R, V^* and V_0 as above. Then the one-to-one correspondence between closed submodules gives a one-to-one correspondence between closed ideals (resp. closed prime ideals, closed semiprime ideals).*

Proof. By Proposition 4.2 we may assume that V is closed. Let P be a closed submodule of V, P^* the extension of P to V^* and $P_0 = P^* \cap V_0$. As in Theorem 3.2 it follows that when one of the submodules P, P^*, P_0 is an ideal, so are the others.

Assume that P is a prime ideal of V and A_0, B_0 are ideals of V_0 with $A_0B_0 \subseteq P_0$. As in the proof of Proposition 4.2 we may assume that A_0 and B_0 are closed. Note that $(QA_0 \cap V)(QB_0 \cap V) \subseteq QA_0B_0 \cap V \subseteq QP_0 \cap V = P$. Then either $A = QA_0 \cap V \subseteq P$ or $B = QB_0 \cap V \subseteq P$ and therefore either $A_0 = A^* \cap V_0 \subseteq P^* \cap V_0 = P_0$ or $B_0 \subseteq P_0$. Thus P_0 is prime.

Suppose that P_0 is a prime ideal of V_0 and $AB \subseteq P^*$, A, B ideals of V^*. As above we may assume that A, B are Q-closed. Then we have $(A \cap V_0)(B \cap V_0) \subseteq AB \cap V_0 \subseteq P^* \cap V_0 = P_0$

and so either $A_0 = A \cap V_0 \subseteq P_0$ or $B_0 \subseteq P_0$. It follows that either $A = QA_0 \subseteq QP_0 = P^*$ or $B \subseteq P^*$, and thus P^* is prime.

Finally, assume that P^* is a prime ideal of V^* and $AB \subseteq P$, where A, B are ideals of V. Since for $x \in A^*$, $y \in B^*$ there exist $F, H \in \mathcal{E}$ such that $Fx \subseteq A$ and $yH \subseteq B$, where A^* (resp. B^*) denotes the extension of $[A]_V$ (resp. $[B]_V$) to a closed ideal of V^* (Corollary 4.8 in [5]), it easily follows that $A^*B^* \subseteq P^*$. Hence either $A^* \subseteq P^*$ or $B^* \subseteq P^*$. Therefore either $A \subseteq A^* \cap V \subseteq P^* \cap V = P$ or $B = B^* \cap V \subseteq P^* \cap V = P$ and so P is prime.

The semiprime case is the same taking above $A_0 = B_0$ (resp. $A = B$). □

As a direct consequence of the former results we have the following corollary which holds for any intermediate extension (in particular, for $V = S$).

Corollary 4.4. *Let V be an intermediate extension. Then the following conditions are equivalent*:

(i) *V is a prime (resp. semiprime) ring.*

(ii) *V^* is a prime (resp. semiprime) ring.*

(iii) *V_0 is a prime (resp. semiprime) semigroup.*

5. Special types of prime ideals

In this section we study prime ideals of torsion-free normalizing extensions of prime rings.

First, assume that R is semiprime and M is a torsion-free R-normalizing bimodule. In ([5], Theorem 4.5) a closed submodule was characterized as a complement submodule. Theorem 2.1 of [4] gives a stronger result for centralizing bimodules. We now give an extension of this result. Note that our proof here is simpler than the proof of [4] for the centralizing case.

Let T be the canonical torsion-free bimodule associated to M ([5], Section 3). Any element $x \in T$ can be written as a finite sum $x = \sum_{i\in\Omega} e_i a_i$, where $(e_i)_{i\in\Omega}$ is the set of normalizing generators of L and $a_i \in R$ are uniquely determined modulo $Ann_{R,r}(e_i)$, for all i. The support $supp(x)$ of x is defined as the set of all e_i such that $e_i a_i \neq 0$.

Proposition 5.1. *Assume that M is a torsion-free normalizing bimodule over a semiprime ring R and let $N \subseteq P$ submodules of M. Then N is closed in P if and only if for any right submodule K of P with $N \subset K$ there exists $0 \neq x \in K$ such that $RxR \cap N = 0$.*

Proof. One implication is immediate from Theorem 4.5 in [5]. For the other assume that N is closed in P. First we prove the result for $M = T$, T as above. Take an element $x \in K \setminus N$ of minimal support $\Gamma = \{e_1, \dots, e_n\}$, say $x = e_1a_1 + e_2a_2 + \ldots + e_na_n$. If there exists a nonzero element $y \in N$ with $supp(y) \subseteq \Gamma$ we may assume that $y(e_1) \neq 0$, where $y(e_1)$ denotes the e_1-coefficient of y. Let I be the ideal of R of all the elements a such that there exists $z \in N$ with $supp(z) \subseteq \Gamma$ and $z(e_1) = a$. Then I is a nonzero ideal of R and $H = I \oplus Ann(I)$ is an essential ideal.

For any $0 \neq b \in I$ there exists $z = e_1a_1b + e_2b_2 + \ldots + e_nb_n \in N$ and we have $xb - z \in K$ and $supp(xb - z) \subset \Gamma$. By the minimality of $supp(x)$ we have $xb - z \in N$ and so $xb \in N$. Since $x \notin N$ and N is closed, $xH \not\subseteq N$. Therefore there exists $c \in Ann(I)$ such that $xc \neq 0$. Hence $xc \in K$ and $RxcR \cap N = 0$.

Now the general case can be proved in a canonical way using the epimorphism $\Phi : T \to M$. □

Note that if R is prime $\mathcal{E}$ is the set of all nonzero ideals of R.

Lemma 5.2. *If R is prime and M is a normalizing torsion-free bimodule over R, then there exists a submodule L of M which has a normalizing free basis and is dense in M.*

Proof. If $x \in M$ is a normalizing element and $xa = 0$, $a \in R$, then $xRaR = RxaR = 0$ and so $x = 0$, since M is torsion-free. Let $(x_i)_{i\in\Omega}$ be the set of normalizing generators of M. Then there exists a maximal right R-independent set of generators $E = (x_i)_{i\in\Lambda}$. If L is the free submodule of M generated by E, then it is easy to see that L is dense in M. □

The free submodule L of M will be called a *free dense submodule* of M.

Corollary 5.3. *If R is prime and M is a normalizing bimodule over R, then the canonical torsion-free extension M^* of M is free over Q.*

Proof. Let L be a free dense submodule of $M/T(M)$. It is easy to see that L^* is free over Q and that $M^* = (M/T(M))^* = L^*$. The result follows. □

In the rest of the paper S is always a torsion-free normalizing extension of a prime ring R and V is an intermediate extension. If I is an R-disjoint ideal of V, then $[I]_V$ is also R-disjoint. Moreover, if $I \cap R \neq 0$, then $[I]_V = V$. Hereafter we denote by $[I]$ the closure $[I]_V$ of I in V.

Now we extend and improve results of ([2], Section 6). Recall that a ring T is said to be (right) strongly prime if any nonzero ideal J of T contains a (right) insulator, i.e., a finite set $F \subseteq J$ such that $Ann_{T,r}(F) = 0$.

Also the (right) singular ideal $Z(T)$ of a T is the set of all the elements $x \in T$ such that $Ann_{T,r}(x)$ is an essential right ideal of T ([6], pag. 30–36). The ring T is said to be (right) non-singular if $Z(T) = 0$.

In the following strongly prime (non-singular) means right strongly prime (right non-singular). An ideal P of T is said to be strongly prime (non-singular prime) if the factor ring T/P is strongly prime (non-singular prime).

Proposition 5.4. *Let R be prime ring and V be intermediate extension, as above. Then P is a closed prime ideal of V provided that one of the following conditions is fulfilled:*

(i) *P is an ideal of V which is maximal with respect to $P \cap R = 0$.*

(ii) *P is a strongly prime R-disjoint ideal of V.*

Proof. (i) Since R is prime it follows easily that P is prime. Also, since $P \cap R = 0$ we have $[P] \cap R = 0$ and maximality of P implies that $P = [P]$. Hence P is closed.

(ii) Suppose that $[P] \supset P$. Then there exists a finite set $F \subseteq [P]$ such that $Fx \subseteq P$, $x \in V$, implies $x \in P$. However, since F is finite, there exists $H \in \mathcal{E}$ with $FH \subseteq P$ and $H \not\subseteq P$. This is a contradiction and the result follows. □

The following result is an extension of Theorem 6.2 in [2].

Theorem 5.5. *Assume that R is a non-singular prime ring and V an intermediate extension. If P is an ideal of V which is maximal with respect to $P \cap R = 0$, then P is a non-singular prime ideal.*

Proof. Assume, by contradiction, that $Z(V/P) = I/P \neq 0$, where I is an ideal of V. By the maximality of P there exists $0 \neq a \in I \cap R$. We show that $a \in Z(R)$, which is a contradiction.

Take a free dense submodule L of S with basis $(e_i)_{i\in\Lambda}$ over R. For a nonzero right ideal J of R the right ideal $N = JV + P$ of V properly contains P. Thus $K = \{y \in N : ay \in P\}$ is a right ideal of V with $P \subset K$ because $a + P \in Z(V/P)$. Since P is closed we easily get $P \cap L \subset K \cap L$.

Note that if $x \in JV$ we can write $x = \sum_i a_i v_i$, where $a_i \in J$ and $v_i \in V$. Take a nonzero ideal H of R such that $v_i H \subseteq L$, for any i. We can easily see that for any $h \in H$ we have $xh \in \sum \oplus_j J e_j$. It follows that xh can uniquely be represented in the basis $(e_i)_{i\in\Lambda}$ with coefficient in J.

By the above there exists an element $z \in JV \cap L \setminus P$ such that $az \in P$. Since P is closed, changing z by zh we may assume that $z \notin P$ and it can be represented as an element of $\sum \oplus_j J e_j$, say $z = \sum_{j=1}^n b_j e_j$ with $0 \neq b_j \in J$, for $1 \leq j \leq n$, and $\{e_1, \ldots, e_n\}$ is minimal.

We claim that do not exist $0 \neq y \in P \cap L$ such that $supp(y) \subseteq \{e_1, \ldots, e_n\}$. In fact, let $0 \neq y = a_1 e_1 + \ldots + a_n e_n \in P \cap L$. We may suppose that $a_1 \neq 0$. For any $r \in R$ there exists $r' \in R$ with $e_1 r = r' e_1$. Also there exists $0 \neq a' \in R$ with $e_1 a' = a_1 e_1$. Thus $zr'a' - b_1 ry \in \sum \oplus_j J e_j$, $a(zr'a' - b_1 ry) \in P$ and $supp(zr'a' - b_1 ry) \subset \{e_1, \ldots, e_n\}$. Therefore $zRa'R \subseteq P$, which is a contradiction since P is closed.

By the claim we have that $az = 0$, since $az \in P$ and $supp(az) \subseteq \{e_1, \ldots, e_n\}$. Consequently $Ann_{J,r}(a) \neq 0$ and the proof is complete. □

Now we prove the converse of Theorem 5.5.

Proposition 5.6. *Let R be a prime ring and V an intermediate extension. If P is an R-disjoint non-singular closed prime ideal of V, then R is non-singular.*

Proof. Suppose that $0 \neq a \in Z(R)$ and let K be a right ideal of V with $K \supset P$. Take a free dense submodule L of S with basis $(e_i)_{i\in\Lambda}$ over R. Since P is closed, $K \cap L \supset P \cap L$. Then there exists $x = \sum_{j=1}^n a_j e_j \in K \cap L \setminus P \cap L$ of minimal support with this property, i.e., for any element $y \in K \cap L$ such that $supp(y) \subset supp(x)$ we have $y \in P \cap L$. As in the proof of Theorem 5.5 we show that for any $y \in P \cap L$ with $supp(y) \subseteq supp(x)$ we have that $y = 0$.

Since $a_1 R \neq 0$ there exists $r \in R$ such that $a_1 r \neq 0$ and $aa_1 r = 0$. Also, let $r' \in R$ be with $e_1 r' = re_1$. Thus $0 \neq xr' \in K \cap L$ and so $axr' = 0$, since $supp(axr') \subset supp(x)$. Hence $Ann_{K/P,r}(a) \neq 0$ and we have $a + P \in Z(V/P) = 0$. Consequently $a \in P \cap R = 0$, a contradiction. This shows that $Z(R) = 0$. □

Recall that if S is a strongly normalizing extension of R and P is a prime ideal of S, then $\mathcal{P} = P \cap R$ is a prime ideal of R ([10], Proposition 1.5). Also, if I is an ideal of R, then IS is an ideal of S with $IS \cap R = I$. Thus, by factoring out the ideals $\mathcal{P}$ and $\mathcal{P}S$ from R and S, respectively, we immediately have the following

Corollary 5.7. *Let R be a prime ring, S a strongly normalizing extension of R and V an intermediate extension. If $\mathcal{P}$ is a prime ideal of R and P is an ideal of V which is maximal with respect to $P \cap R = \mathcal{P}$, then $\mathcal{P}$ is non-singular in R if and only if P is non-singular in V.*

Now we consider strongly prime rings and ideals. The following result is an extension of Theorem 6.1 of [2].

Theorem 5.8. *Let R be a strongly prime ring, V an intermediate extension and P an ideal of V which is maximal with respect to $P \cap R = 0$. Then P is a strongly prime ideal.*

Proof. Suppose that I is an ideal of V with $I \supset P$. Then $I \cap R \neq 0$ and so there exists a finite set $F \subseteq I \cap R$ such that $Ann_{R,r}(F) = 0$. Put $K = \{y \in V : Fy \subseteq P\}$, a right ideal of V containing P. We prove that $K = P$ and this shows that F is an insulator in V/P.

Assume, by contradiction, that $K \supset P$. By Propositions 5.4 and 5.1, there exists $0 \neq x \in K$ such that $RxR \cap P = 0$. Let L be a free dense submodule of S. Then there exists a nonzero ideal H of R such that $xH \subseteq L$ and we have $FxH \subseteq RxR \cap P = 0$. We easily obtain $xH = 0$, since L is free. This is a contradiction because S is torsion-free. □

The converse of Theorem 5.8 holds if we assume that S is a strongly normalizing extension of R.

Proposition 5.9. *Let S be a torsion-free strongly normalizing extension of R and V an intermediate extension. If P is a strongly prime R-disjoint ideal of V, then R is strongly prime and P is closed.*

Proof. Let H be a nonzero ideal of R. Then VHV is a nonzero ideal of V and $VHV \not\subseteq P$. Thus there exists a finite set $F \subseteq VHV$ such that $Fx \subseteq P$, $x \in V$, implies $x \in P$. Moreover, any $y_j \in F \subseteq VHV \subseteq S$ can be written as $y_j = \sum_i x_i a_{ij}$, for $a_{ij} \in H$, since S is a strongly normalizing extension, where $(x_i)_i$ are strongly normalizing generators. Therefore, $\{a_{ij}\} \subseteq H$ is an insulator in R and so R is strongly prime. Finally, P is closed by Proposition 5.4. □

As in Corollary 5.7 we immediately have the following.

Corollary 5.10. *Let S be a strongly normalizing extension of R and V an intermediate extension. If $\mathcal{P}$ is an ideal of R and P is an ideal of V which is maximal with respect to $P \cap R = \mathcal{P}$, then $\mathcal{P}$ is strongly prime if and only if P is strongly prime.*

Now we relate strongly primeness between S, S^* and S_0. We say that a subsemigroup V_0 of S_0 is strongly prime if any nonzero ideal of V_0 contains an insulator, i.e. a finite subset F_0 with $Ann_{V_0,r}(F_0) = 0$.

Note that any ideal of V_0 is closed and so is saturated. In fact, a nonzero ideal of Z has a nonzero R-normalizing element of Q, so contains an invertible element of Q. Thus, if I is a nonzero ideal of V_0 and for $z \in V_0$ we have $zH \subseteq I$, where $0 \neq H \lhd Z$, it follows that $z \in I$.

First we prove the following

Lemma 5.11. *Let S be a torsion-free strongly normalizing extension of R and assume that $U \subseteq V$ are intermediate extensions such that U is dense in V. Then U is strongly prime if and only if V is strongly prime.*

Proof. If U is strongly prime and I is a nonzero ideal of V, then $J = I \cap U \neq 0$. Thus there exists a finite set $F \subseteq J$ such that $Ann_{U,r}(F) = 0$. We can easily see that $Ann_{V,r} = 0$ and so V is strongly prime.

Conversely, assume that V is strongly prime and let I be a nonzero ideal of U. Then $[I]_V$ is a nonzero ideal of V. Thus there exists a finite set $F \subseteq [I]_V$ such that $Ann_{V,r}(F) = 0$. Also there exists a nonzero ideal H of R such that $HF \subseteq I$. By Proposition 5.9, R is strongly prime. Hence there exists a finite set $F' \subseteq H$ with $Ann_{R,r}(F') = 0$. Consequently $F'F \subseteq I$ is a finite set which is an insulator in U. In fact, take $x \in U$ such that $F'Fx = 0$. By Corollary 5.3 the canonical torsion-free normalizing extension S^* of S is free over Q. Using this we easily see that $Fx = 0$. Therefore $x = 0$ and we are done. □

Remark 5.12. A similar argument as in Lemma 5.11 shows that if S is a normalizing extension, R is strongly prime and V is an intermediate extension, for an ideal I of V there exists an insulator in I if and only if there is an insulator in $[I]_V$. Thus to show that V is strongly prime it is enough to find an insulator in any nonzero closed ideal of V.

To end the paper we prove the following

Theorem 5.13. *Let R be a prime ring, S a torsion-free strongly normalizing extension of R and V an intermediate extension. Let V^* and V_0 be the corresponding closed subrings of S^* and S_0 with $V^* \cap S = [V]_S$ and $V_0 = V^* \cap S_0$. The following conditions are equivalent:*

(i) *V is strongly prime.*

(ii) *R and V^* are strongly prime.*

(iii) *R and V_0 are strongly prime.*

Proof. We may assume that V is closed in S, by Lemma 5.11.

(i) $\to$ (iii) Suppose that V is strongly prime. Then R is strongly prime by Proposition 5.9. Let I_0 be a nonzero ideal of V_0. Then we have that $I = QI_0 \cap V$ is a nonzero ideal of V and so there exists a finite set $F = \{y_1, \ldots, y_n\} \subseteq I$ such that $Ann_{V,r}(F) = 0$. For any i we can write $y_i = \sum_j q_{ij} m_{ij}$, $q_{ij} \in Q$, $m_{ij} \in I_0$. We can easily see that $F_0 = \{m_{ij}\} \subseteq I_0$ is an insulator in V_0, since S^* is free as right R-module.

(iii) $\to$ (ii) Let I be a nonzero ideal of V^*. We show that I contains an insulator. Since R is strongly prime we have that Q is also strongly prime. So by Remark 5.12 we may assume that I is closed in V^*. Put $I_0 = I \cap V_0$, a nonzero ideal of V_0. So there exists a finite set $F \subseteq I_0$ such that $Ann_{V_0,r}(F) = 0$. It is not hard to show that $F \subseteq I$ is an insulator in V^*.

(ii) $\to$ (i) Let I be a nonzero closed ideal of V. Then there exists a closed ideal I^* of V^* with $I = I^* \cap V$. By assumption there exists a finite set $F \subseteq I^*$ such that $Ann_{V^*,r}(F) = 0$. Also, since F is finite $FH \subseteq I$, for some $0 \neq H \lhd R$. Take a finite set $F' \subseteq H$ such that $Ann_{R,r}(F') = 0$. Then $FF' \subseteq I$ is an insulator in V. □

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