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Morphisms of Polar Spaces

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Abstract. Polar spaces are presented from the point of view of paraprojective spaces. Morphisms are introduced and some immediate categorical aspects are reviewed. The morphisms of polar spaces are then studied in more details and are shown to preserve the spaces' structure. Finally, it is shown that a morphism of polar spaces can be split into a morphism of nondegenerate polar spaces and a morphism of projective spaces.

0. Introduction

Let us recall the classical definition of a polar space ensuing from Buekenhout and Shult's work [3]. Let P be a set and $\mathcal{L}$ a collection of subsets of P of cardinality at least two. The elements of P are called *points* and the elements of $\mathcal{L}$ are called *lines*. Two points p and q of P are *collinear* if $p = q$ or if there exists a line containing both points. P is a *polar space* if for any point p and line l, p is collinear with either *one or all* points of l.

The definition we adopt in this paper is a restriction of this one, inasmuch as we require that two distinct lines intersect in at most one point. In this sense the spaces we consider are *partially linear* polar spaces, but we will prefer the name of polar space. This definition is in fact a natural generalization of Veldkamp or Tits' set of axioms ([13] or [12]) combined with Buekenhout and Shult's results. See also [11].

This point of view allows us to regroup polar spaces and projective spaces in the same theory of *paraprojective spaces*. We can then continue Faure and Frölicher's work to obtain a natural definition of a morphism of paraprojective spaces.

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As most proofs in the three first sections are either straightforward or almost identical to proofs given in [6] for similar results, we do not include the details.

In Section 1, we define paraprojective and polar spaces and recall some basic definitions. We also recall some well-known examples of polar spaces in order to get the notations settled for the rest of the article.

In Section 2, we define morphisms of paraprojective spaces and present the resulting category. We then restrict our attention to polar spaces and present a functor between the category of pseudoquadratic spaces with their pseudo-orthogonal maps and the category of polar spaces with their morphisms.

In Section 3, we define the quotient of a paraprojective space by a subspace and mention that the canonical projection is a morphism.

Finally, in Section 4, we show that provided the polar spaces are "not too degenerate" the morphisms do preserve structure (note that the hypotheses are quite weak. In particular, they are satisfied when the polar spaces are generalized quadrangles). We conclude by showing that a morphism of polar spaces induces two particular maps which can be useful to understand the form of the original morphism.

1. Spaces

1.1. Paraprojective and polar spaces

Definition 1.1.1. *Let P be a set. Denote by $\mathcal{P}(P)$ its power set and let*

$$\begin{aligned} * : P \times P &\longrightarrow \mathcal{P}(P) \\ (a, b) &\longmapsto a * b \end{aligned}$$

be an operator satisfying the following axioms:

(P_1) $a * a = \{a\}$;

(P_2) $a * b \neq \emptyset \implies b \in a * b$;

(P_3) $a \in b * p, p \in c * d, b * c \neq \emptyset \neq b * d$ *and* $a \neq c \implies (a * c) \cap (b * d) \neq \emptyset$.

Then $(P, *)$ is a paraprojective space. We will often write P instead of $(P, *)$. An element of P is called a point. If $a * b \neq \emptyset$, the points a and b are said to be collinear. If $a * b \neq \emptyset$ and $a \neq b$, the set $a * b$ is called a line.

Furthermore, P is a polar space if the operator also satisfies:

(P_4) $a * b \neq \emptyset, a \neq b, p \notin a * b \implies$ *there exists $c \in a * b$ such that $p * c \neq \emptyset$.*

Remark 1.1.2. A paraprojective space P in which $a * b \neq \emptyset$ for all points a and b is a projective space, see [6]. In this case, (P_3) is essentially the *Veblen-Young* axiom.

The following proposition puts forth some aspects of the sets $a * b$. In particular, when $a * b$ is non-empty it has the properties one would naturally expect for a line.

Proposition 1.1.3. *An operator satisfying the three axioms (P_1), (P_2) and (P_3) also verifies the following properties:*

(Q_1) $b * a = a * b$;

(Q_2) *$c, d \in a * b, c \neq d \implies c * d = a * b$;*

(Q_3) *$p * a \neq \emptyset, p * b \neq \emptyset$ and $c \in a * b \implies p * c \neq \emptyset$.*

Remark 1.1.4. Note that the conditions (P_2) and (Q_1) yield that when $a * b$ is non-empty, then it contains a and b. The condition (Q_2) implies that two distinct lines intersect in at most one point. Finally, the conditions (P_4) and (Q_3) yield that if p is a point and l a line, then p is collinear with either one or all points of l; this is Buekenhout and Shult's *one-all* axiom.

Definition 1.1.5. *Let P be a paraprojective space. A subset E of P is singular if $a, b \in E$ implies that $a * b \neq \emptyset$. A subset F of P is a subspace of P if $a, b \in F$ implies that $a * b \subseteq F$. A proper subspace H of P is a hyperplane of P if every line of P intersects H in at least one point. Finally, P is thick if every line contains at least three points.*

Remark 1.1.6. Let P be a paraprojective space. As in 1.1.2, we observe that a singular subspace of P is a projective space.

We introduce now a notation which will be used further on. Let P be a paraprojective space, $p \in P$ a point and $E \subseteq P$ a subset. Then $\mathcal{C}(E)$ will denote the smallest subspace of P containing E, and $\mathcal{C}(p, E)$ the smallest subspace of P containing $\{p\} \cup E$.

1.2. Collinearity

Lemma 1.2.1. *Let P be a paraprojective space, $p \in P$ and $E \subseteq P$. Then*

i) *$p^\perp := \{q \in P \mid p * q \neq \emptyset\}$ is a subspace of P;*

ii) *$E^\perp := \bigcap\{p^\perp \mid p \in E\}$ is a subspace of P.*

Remark 1.2.2. Let P be a paraprojective space. We mention that the preceding lemma and Remark 1.1.6 imply that our definition of a paraprojective space is equivalent to the definition given in [4].

Definition 1.2.3. *Let P be a paraprojective space and $E \subseteq P$ a subspace. The radical of E is the subspace*

$$\mathrm{Rad}(E) := \{p \in E \mid p * q \neq \emptyset \text{ for all } q \in E\} = E \cap E^\perp.$$

Furthermore, P is nondegenerate if $\mathrm{Rad}(P) = \emptyset$.

Remark 1.2.4. A nondegenerate polar space P is exactly a nondegenerate polar space in the sense usually found in the literature (see for example [9], [7], [5] or [4]).

Proposition 1.2.5. *Let P be a polar space and $p \in P \setminus \mathrm{Rad}(P)$. Then $p^\perp$ is a hyperplane of P.*

1.3. Examples of polar spaces

The following examples put forth the relation existing between vector or projective spaces and polar spaces. As the classification theorem shows (see [13], [12], [3], [1] and [7]) the first example is generic in the nondegenerate case.

1. Let $\sigma : K \longrightarrow K$ be a field anti-automorphism and ϵ an element of K satisfying $\sigma(\epsilon) = \epsilon^{-1}$ and $\sigma^2(\lambda) = \epsilon^{-1}\lambda\epsilon$ for all $\lambda \in K$. We define:

$$K_{\sigma,\epsilon} := \{\lambda - \epsilon\sigma(\lambda) \mid \lambda \in K\}.$$

Let V be a (left) vector space on K, ϕ a σ-sesquilinear form and q the associated pseudoquadratic form on V relative to σ and ϵ i.e. $q : V \longrightarrow K/K_{\sigma,\epsilon}$ is defined by $q(v) := \phi(v,v) + K_{\sigma,\epsilon}$, see [12] for the original definition). We denote by $(\mathcal{G}(V), \star)$ the projective space associated to V and by $[v]$ the equivalence class of $v \neq 0$ in $\mathcal{G}(V)$. Then the set $P_q := \{[v] \in \mathcal{G}(V) \mid q(v) = K_{\sigma,\epsilon}\}$ coupled with the operator

$$[v] * [w] := ([v] \star [w]) \cap [v]^\perp \cap [w]^\perp$$

(where $[v]^\perp := \{[w] \in P_q \mid \psi(v,w) = 0\}$ and $\psi(v,w) := \phi(v,w) + \epsilon\sigma(\phi(w,v))$ is the hermitian form associated to q) is a polar space. Note that the set $P_\psi := \{[v] \in \mathcal{G}(V) \mid \psi(v,v) = 0\}$ is also a polar space with the same operator (in this case, $[v]^\perp := \{[w] \in P_\psi \mid \psi(v,w) = 0\}$).

2. Let $(G, \star)$ be a projective space and $G^\circledast := G^* \cup \{G\}$, where

$$G^* := \{H \subseteq G \mid H \text{ is a hyperplane of } G\}.$$

Let $\pi : G \longrightarrow G^\circledast$ be a quasipolarity (i.e. for $a, b \in G$, we have that $b \in \pi(a)$ implies $a \in \pi(b)$). We define the set $a^\pi := \pi(a)$ and the symmetric relation $b\,\pi\,a \iff b \in \pi(a)$. Then the set of absolute points $P_\pi := \{a \in G \mid a \in a^\pi\}$ coupled with the operator $a * b := (a \star b) \cap a^\pi \cap b^\pi$ is a polar space.

Remark 1.3.1. We recall that pseudoquadratic forms need only to be introduced in some cases where the underlying field is of characteristic 2; otherwise, sesquilinear reflexive forms suffice to construct polar spaces (see for example [4]). Nonetheless, the precedent approach allows us to regroup most of the significant examples in the same theory, without having to consider particular cases systematically.

Note that when $K_{\sigma,\epsilon} = K$, the pseudoquadratic form q does not determine ψ uniquely; to avoid this problem, we always imply that q is given with ϕ.

2. Categorical aspects

2.1. Morphisms

On the one hand, vector spaces with pseudoquadratic forms are a good model for polar spaces, so natural maps between these vector spaces should induce morphisms on the corresponding polar spaces. On the other hand, there are certain abstract maps that should also be morphisms; some of these require a restriction of the domain to a subspace (see for example Section 3); we chose not to include this aspect in the definition, because of the unnecessary complications it would have led to, particularly in the categorical results.

Definition 2.1.1. *Let P_1 and P_2 be paraprojective spaces. A partial map $g : P_1 \setminus E \longrightarrow P_2$ is a morphism if the following axioms are verified:*

(M_1) *E is a subspace of P_1;*

(M_2) *$a, b \notin E, c \in E$ and $a \in b * c \implies g(a) = g(b)$;*

(M_3) *$a, b, c \notin E$ and $a \in b * c \implies g(a) \in g(b) * g(c)$.*

The set E is called the kernel of g and is denoted by $\ker(g)$. When the kernel is not specified, we will write $g : P_1 \cdot\!\!\rightarrow P_2$ to designate a partial map. A morphism $g : P_1 \setminus E \longrightarrow P_2$ is rigid if $a * b = \emptyset$ implies $g(a) * g(b) = \emptyset$. An injective morphism with empty kernel $g : P_1 \longrightarrow P_2$ is an embedding if $b * c \neq \emptyset$ and $g(a) \in g(b) * g(c)$ imply $a \in b * c$. Furthermore, an injective map $g : P_1 \longrightarrow P_2$ is a good embedding if $b * c \neq \emptyset$ implies that $g(b * c) = g(b) * g(c)$ (note that a good embedding is also an embedding). If P_1 is a subset of P_2 and the injection map is a good embedding, then P_1 is well-embedded in P_2. Finally, a bijective morphism with empty kernel $g : P_1 \longrightarrow P_2$ is an isomorphism if it satisfies $g(b * c) = g(b) * g(c)$ for all $b, c \in P_1$.

Remark 2.1.2. The condition (M_3) yields in particular that when $a, b \notin E$ are collinear, then $g(a)$ and $g(b)$ are collinear. Furthermore, if $a, b \notin E$ are collinear and $g(a) \neq g(b)$, then (M_2) and (M_3) imply that the restriction of g to the line $a * b$ is an injection.

Definition 2.1.3. *Let $g_1 : P_1 \setminus E \longrightarrow P_2$ and $g_2 : P_2 \setminus F \longrightarrow P_3$ be two morphisms. Then the composite of g_1 and g_2, denoted by $g_2 \circ g_1$, is the partial map defined by $g_2 \circ g_1 : P_1 \setminus (g_1^{-1}(F) \cup E) \longrightarrow P_3$ and $g_2 \circ g_1(p) := g_2(g_1(p))$ for $p \notin g_1^{-1}(F) \cup E$.*

For the paraprojective spaces and their morphisms to form a category, the composite of two morphisms must be a morphism. This follows from the following proposition.

Proposition 2.1.4. *Let $g : P_1 \setminus E \longrightarrow P_2$ be a morphism of paraprojective spaces. Then the set $g^{-1}(F) \cup E$ is a subspace of P_1 for any subspace F of P_2.*

Corollary 2.1.5. *Let $g_1 : P_1 \setminus E \longrightarrow P_2$ and $g_2 : P_2 \setminus F \longrightarrow P_3$ be two morphisms of paraprojective spaces. Then the composite of g_1 and g_2 is a morphism.*

Corollary 2.1.6. *The paraprojective spaces with their morphisms form a category. Polar spaces form a full subcategory of the category of paraprojective spaces. Finally, projective spaces form a full subcategory of the category of polar spaces.*

2.2. Pseudo-orthogonal maps

Definition 2.2.1. *Let V_1 and V_2 be vector spaces over K and L respectively. A map $f : V_1 \longrightarrow V_2$ is semilinear (or τ-semilinear) if:*

(S_1) *$f(v + w) = f(v) + f(w)$ for all $v, w \in V_1$;*

(S_2) *there exists a field homomorphism $\tau : K \longrightarrow L$ such that $f(\lambda v) = \tau(\lambda) f(v)$ for all $v \in V_1$, $\lambda \in K$.*

Let q_i be a pseudoquadratic form (relative to σ_i and ϵ_i) on V_i, and ψ_i its associated hermitian form. A semilinear map $f : V_1 \longrightarrow V_2$ satisfying the following axioms:

(O_1) $q_1(v) = K_{\sigma_1,\epsilon_1} \implies q_2(f(v)) = L_{\sigma_2,\epsilon_2}$ *for all* $v \in V_1$;

(O_2) $\psi_1(v,w) = 0 \implies \psi_2(f(v), f(w)) = 0$ *for all* $v, w \in V_1$;

is called a pseudo-orthogonal map.

If q is a pseudoquadratic form on a vector space V, the couple (V, q) is called a pseudoquadratic space. We will sometimes write V instead of (V, q).

Remark 2.2.2. Because a pseudoquadratic form and its associated hermitian form are closely related, the conditions (O_1) and (O_2) are not independent. In particular, if the characteristic of K is different from 2 then (O_2) implies (O_1) and a pseudo-orthogonal map is simply an orthogonal map.

Proposition 2.2.3. *The pseudoquadratic spaces with their pseudo-orthogonal maps form a category.*

Proposition 2.2.4. *Let $f : V_1 \longrightarrow V_2$ be a pseudo-orthogonal map. Then*

i) *the couple (V_1, q_1) induces a polar space P_{q_1}, and the couple (V_2, q_2) a polar space P_{q_2};*

ii) *the map f induces a morphism $Pf : P_{q_1} \setminus E \longrightarrow P_{q_2}$, where E is the set $\mathcal{G}(\ker(f)) \cap P_{q_1}$.*

Theorem 2.2.5. *The correspondence of the precedent proposition, between the category of pseudoquadratic spaces and the category of polar spaces, yields a functor.*

Remark 2.2.6. In the same way, we can define a quasipolar space as a projective space coupled with a quasipolarity; by considering orthogonal morphisms between these spaces, we get a category. We can then define a functor between the category of quasipolar spaces and the category of polar spaces.

2.3. Examples of morphisms

We have just seen that pseudo-orthogonal maps induce morphisms of polar spaces. We present here some other interesting examples.

1. Let P_1 be a paraprojective space and $P_2 = \{a\}$ a singleton i.e. a projective space of rank 1). Then for any subspace $E \subseteq P_1$, the constant map $p : P_1 \setminus E \longrightarrow P_2$ is a morphism.
2. Let P_1 be a paraprojective space and P_2 a line (i.e. a projective space of rank 2). Then any injection $i : P_1 \longrightarrow P_2$ is a morphism.
3. Let E be a subspace of a paraprojective space P. Then the inclusion $j : E \longrightarrow P$ is a rigid morphism; note that it is also a good embedding.
4. Let G be a projective space coupled with a quasipolarity π and P_π the associated polar space. Then the inclusion $k : P_\pi \longrightarrow G$ is a morphism which is not rigid in general; nonetheless, it is a good embedding.

5. Let H be a hyperplane of a paraprojective space P and $a \in P \setminus H$. Then the projection $\rho_a : a^\perp \setminus \{a\} \longrightarrow H$ defined by $\rho_a(p) := (a * p) \cap H$ is a morphism.

6. Let V be a vector space over $K = \mathbf{Z}/2\mathbf{Z}$, ϕ a sesquilinear form on V, and ψ a reflexive sesquilinear form given by $\psi(v, w) := \phi(v, w) + \phi(w, v)$. On $V \times K$ we define the reflexive sesquilinear form ψ' by $\psi'((v, x), (w, y)) := \psi(v, w)$. Then the map $f : V \to V \times K$ given by $f(v) = (v, \phi(v, v))$ induces a morphism $g : P_\psi \to P_{\psi'}$.
Indeed, we have $f(v+w) = (v+w, \phi(v, v)+\psi(v, w)+\phi(w, w))$, so $f(v+w) = f(v)+f(w)$ if and only if $\psi(v, w) = 0$. It follows that the induced map g is a morphism; but in general f is not semilinear.
In fact, if there exist three points $[u], [v], [w] \in P_\psi$ such that $\psi(v, w)$ is not equal to 0 and $[u] = [v + w]$, then g cannot be induced by *any* semilinear map.

In a coming article, we will study embeddings in more detail and state a result giving sufficient conditions for a morphism of polar spaces to be induced by a pseudo-orthogonal map.

3. The quotient space

Usually, one takes the quotient of a space by a singular subspace. However, it is possible to define the quotient of a space by any subspace. As most results and proofs do not require much more work in this case, we present the general approach. Moreover, we include a note on the singular case whenever it seems necessary.

3.1. Construction of the quotient space

In this section, $(P, *)$ will denote a paraprojective space and E a subspace.

Definition 3.1.1. *On $E^\perp \setminus E$ we define the following equivalence relation:*

$$p \sim q \iff \mathcal{C}(p, E) = \mathcal{C}(q, E).$$

Remark 3.1.2. If E is a non-empty subspace and $p \in E^\perp$, it is not hard to see that $\mathcal{C}(p, E) = \bigcup\{p * q \mid q \in E\}$. Furthermore, if E is singular, this subspace is also singular.

Proposition 3.1.3. *Let $p \in E^\perp \setminus E$. Then $(\mathcal{C}(p, E) \cap E^\perp) \setminus E$ is the equivalence class of the point p for the relation $\sim$. Furthermore, if E is singular, the equivalence class of p becomes $\mathcal{C}(p, E) \setminus E$.*

Definition 3.1.4. *We denote the quotient set $(E^\perp \setminus E)/{\sim}$ by P/E and the equivalence class of a point p in the quotient by $[\![p]\!]$. On P/E we define the following operator:*

$$[\![a]\!] \circledast [\![b]\!] := \{[\![p]\!] \in P/E \mid \text{there exist } a' \in [\![a]\!], b' \in [\![b]\!] \text{ and } p' \in [\![p]\!] \text{ such that } p' \in a' * b'\}.$$

Lemma 3.1.5. *Let $[\![a]\!], [\![b]\!] \in P/E$. The following statements are equivalent:*

a) *there exist $a' \in [\![a]\!]$ and $b' \in [\![b]\!]$ such that $a' * b' \neq \emptyset$;*

b) *for all $a' \in [\![a]\!]$ and $b' \in [\![b]\!]$, we have $a' * b' \neq \emptyset$.*

Lemma 3.1.6. *Let $[\![a]\!], [\![b]\!], [\![p]\!] \in P/E$. The following statements are equivalent:*

a) *there exist* $a' \in [\![a]\!]$, $b' \in [\![b]\!]$ *and* $p' \in [\![p]\!]$ *such that* $p' \in a' * b'$;

b) *for any* $a' \in [\![a]\!]$ *and* $b' \in [\![b]\!]$, *there exists* $p' \in [\![p]\!]$ *such that* $p' \in a' * b'$.

Proposition 3.1.7. *The set* P/E *coupled with the operator* $\circledast$ *is a paraprojective space. Moreover, if* P *is a polar space then* $(P/E, \circledast)$ *is a polar space.*

Remark 3.1.8. If $E = \mathrm{Rad}(P)$ then P/E is nondegenerate.

3.2. The canonical projection

Definition 3.2.1. *Let* P *be a paraprojective space and* E *a subspace. The map* $\rho : E^{\perp} \backslash E \longrightarrow P/E$ *defined by* $\rho(p) := [\![p]\!]$ *is called the canonical projection of* P *onto* P/E.

Proposition 3.2.2. *Let* P *be a paraprojective space and* E *a subspace. Then the canonical projection* $\rho : E^{\perp} \setminus E \longrightarrow P/E$ *is a rigid morphism.*

We also have the following universal property.

Theorem 3.2.3. *Let* $g : P_1 \setminus F \longrightarrow P_2$ *be a morphism of paraprojective spaces,* E *a subspace of* P_1 *and* $\rho : E^{\perp} \setminus E \longrightarrow P_1/E$ *the canonical projection. Then there exists a morphism* $\overline{g} : P_1/E \cdot\!\rightarrow P_2$ *such that* $g = \overline{g} \circ \rho$ *if and only if* $\mathrm{Rad}(E) \subseteq F$ *and* $P_1 \setminus F \subseteq E^{\perp} \setminus E$.

Furthermore, if $\overline{g}$ *exists, it is unique and* $\ker(g) = F/E$ *(where* F/E *is the image by* ρ *of the set* $(F \cap E^{\perp}) \setminus E$*). In particular, if* $E = F$ *then the kernel of* $\overline{g}$ *is empty.*

In the case where E is a subspace contained in the radical of P_1, the theorem takes a more familiar form.

Corollary 3.2.4. *Let* $g : P_1 \setminus F \longrightarrow P_2$ *be a morphism of paraprojective spaces,* E *a subspace of* $\mathrm{Rad}(P_1)$ *and* $\rho : P_1 \setminus E \longrightarrow P_1/E$ *the canonical projection. Then there exists a morphism* $\overline{g} : P_1/E \cdot\!\rightarrow P_2$ *such that* $g = \overline{g} \circ \rho$ *if and only if* $E \subseteq F$.

4. Morphisms of polar spaces

In this section, we are going to investigate the properties of morphisms in the setting of polar spaces. Surprisingly, although the definition of a morphism is weak, the results we obtain reveal a very rigid structure.

4.1. Two properties

When considering morphisms of polar spaces, natural conditions that appear are that non-collinear points should be mapped onto non-collinear points (see for example [8]), or that the kernel must be a subset of the radical. Including such conditions in the definition would either complicate it or render the categorical aspects rather awkward. The two following propositions show that under some weak hypotheses the mentioned properties are consequences of the definition.

Proposition 4.1.1. *Let* $g : P_1 \setminus E \longrightarrow P_2$ *be a morphism of polar spaces. Suppose that* P_1 *is thick,* $P_1/\mathrm{Rad}(P_1)$ *contains a line and* $g(P_1 \setminus E)$ *is non-singular. Then* $E \subseteq \mathrm{Rad}(P_1)$.

Proposition 4.1.2. *Let $g : P_1 \setminus E \longrightarrow P_2$ be a morphism of polar spaces. Suppose that P_1 is thick, $P_1/\mathrm{Rad}(P_1)$ contains a line and $g(P_1 \setminus E)$ is non-singular. Then $p * q = \emptyset$ implies $g(p) * g(q) = \emptyset$.*

Both proofs use a particular construction, which we decompose into the following lemmas.

Lemma 4.1.3. *Let P be a polar space and $a, p \in P$ such that $\emptyset \neq a * p \subseteq P \setminus \mathrm{Rad}(P)$. Then there exists $r \in P$ such that $r^\perp \cap (a * p) = \{a\}$.*

Proof. See [7] Proposition 3.1. □

Lemma 4.1.4. *Let P be a polar space such that $P/\mathrm{Rad}(P)$ contains a line and $a, b \in P$ such that $a * b = \emptyset$. Then there exist $p, q \in \{a, b\}^\perp$ such that $p * q = \emptyset$.*

Proof. We remark that we must have $a \notin \mathrm{Rad}(P)$. We first show that there exists a line l such that $a \in l \subseteq P \setminus \mathrm{Rad}(P)$. Indeed, by h ypothesis there exists a line $l' \subseteq P \setminus \mathrm{Rad}(P)$. If $a \in l'$, we are done; if not, there exists $q' \in l'$ such that $a * q' \neq \emptyset$. If $(a * q') \cap \mathrm{Rad}(P) = \emptyset$, we are done; if not, let $\{r'\} = (a * q') \cap \mathrm{Rad}(P)$ and $p' \in l'$ such that $p' \neq q'$. As $p' * q' \neq \emptyset$ and $p' * r' \neq \emptyset$, we have $p' * a \neq \emptyset$; $l' \subseteq P \setminus \mathrm{Rad}(P)$ implies that $(p' * a) \cap \mathrm{Rad}(P) = \emptyset$, so we can set $l = a * p'$.

We show now the existence of p and q. There exists $p \in l$, $p \neq a$ such that $b * p \neq \emptyset$. By the preceding lemma, there exists $r \in P$ such that $r^\perp \cap l = \{a\}$, so there exists $q \in a * r$ such that $b * q \neq \emptyset$. Furthermore, we must have that $p * q = \emptyset$ or else we could conclude $r * p \neq \emptyset$, a contradiction. □

Lemma 4.1.5. *Let $g : P_1 \setminus E \longrightarrow P_2$ be a morphism of polar spaces. Let $a, b \in P_1 \setminus E$ be such that $g(a) * g(b) = \emptyset$ and $p \in \{a, b\}^\perp \setminus E$. Then neither $a * p$ nor $b * p$ intersects E.*

Proof. Let c be a third point on $a * p$. If $c \in E$ then $g(a) = g(p)$ by (M_2) and $g(a) * g(b) = g(p) * g(b) \neq \emptyset$ by (M_3), a contradiction. The case where $c \in b * p$ is symmetrical. □

Lemma 4.1.6. *Let $g : P_1 \setminus E \longrightarrow P_2$ be a morphism of polar spaces. Suppose that P_1 is thick. Let $a, b \in P_1 \setminus E$ be such that $g(a) * g(b) = \emptyset$ and $p, q \in \{a, b\}^\perp$ such that $p * q = \emptyset$. Then none of the lines $a * p$, $b * p$, $a * q$, $b * q$ intersects E.*

Proof. We recall that $g(a) * g(b) = \emptyset$ implies $a * b = \emptyset$ by (M_3). By the preceding lemma, we only have to show that $p, q \notin E$. As $a * b = p * q = \emptyset$ and $p \in \{a, b\}^\perp$, the points a, b, p, q are all distinct. By hypothesis, there exists a third point $c \in a * p$, so there exists $d \in b * q$ such that $c * d \neq \emptyset$ and $d \neq q$.

If p and q are in E, (M_2) implies that $g(a) = g(c)$ and $g(b) = g(d)$, so $g(a) * g(b) = g(c) * g(d) \neq \emptyset$ by (M_3), a contradiction.

If $p \in E$ and $q \notin E$, we have that $d \notin E$ by the preceding lemma. So we get $g(a) = g(c)$ by (M_2), and $b \in d * q$ implies $g(b) \in g(d) * g(q)$ by (M_3). But $g(a) * g(d) = g(c) * g(d) \neq \emptyset$ and $g(a) * g(q) \neq \emptyset$ then imply $g(a) * g(b) \neq \emptyset$, again a contradiction.

If $q \in E$ and $p \notin E$, we proceed by symmetry and get a contradiction. We conclude that we must have $p, q \notin E$. □

Lemma 4.1.7. *Let $g : P_1 \setminus E \longrightarrow P_2$ be a morphism of polar spaces. Suppose that P_1 is thick. Let $a, b \in P_1 \setminus E$ be such that $g(a) * g(b) = \emptyset$ and $p, q \in \{a, b\}^\perp$ such that $p * q = \emptyset$. Then $g(p) * g(q) = \emptyset$.*

Proof. Let c be a third point on $a * p$. There exists $d \in b * q$ such that $c * d \neq \emptyset$ and $d \neq q$. We remark that $c, d, p, q \notin E$ by the preceding lemma.

Suppose that $g(p) * g(q) \neq \emptyset$. As $g(a) * g(q) \neq \emptyset$ and $g(c) \in g(a) * g(p)$, we get $g(c) * g(q) \neq \emptyset$. As $g(c) * g(d) \neq \emptyset$ and $g(b) \in g(d) * g(q)$, we get $g(c) * g(b) \neq \emptyset$. As $g(p) * g(b) \neq \emptyset$ and $g(a) \in g(c) * g(p)$, we get $g(a) * g(b) \neq \emptyset$, a contradiction. We conclude that $g(p) * g(q) = \emptyset$. □

Lemma 4.1.8. *Let $g : P_1 \setminus E \longrightarrow P_2$ be a morphism of polar spaces. Suppose that P_1 is thick, $P_1/\mathrm{Rad}(P_1)$ contains a line and $g(P_1 \setminus E)$ is non-singular. Then for all $x \in P_1$, there exist $u, v \in x^\perp \setminus E$ such that $g(u) * g(v) = \emptyset$.*

Proof. As $g(P_1 \setminus E)$ is non-singular, there exist $a, b \in P_1 \setminus E$ such that $g(a) * g(b) = \emptyset$, so in particular $a * b = \emptyset$. We can suppose that $x^\perp \neq P_1$ (if $x^\perp = P_1$, take $u = a$ and $v = b$). By 4.1.4, there exist $p, q \in \{a, b\}^\perp$ such that $p * q = \emptyset$.

We first consider the case where $p \notin x^\perp$. As $a \neq p \neq b$ and $x^\perp$ is a hyperplane, there exist $u \in (a * p) \cap x^\perp$ and $v \in (b * p) \cap x^\perp$. By 4.1.6, $u, v, p \notin E$ and $g(u) \in g(a) * g(p)$, $g(v) \in g(b) * g(p)$ by (M_3). As $g(b) * g(p) \neq \emptyset$, we have $g(a) \neq g(p)$, so $g(u) \neq g(p)$ by Remark 2.1.2, and $g(a) \in g(u) * g(p)$. In the same way, we get $g(b) \in g(v) * g(p)$. Suppose that $g(u) * g(v) \neq \emptyset$. As $g(p) * g(v) \neq \emptyset$, we get $g(a) * g(v) \neq \emptyset$. In the same way, $g(a) * g(v) \neq \emptyset$ and $g(a) * g(p) \neq \emptyset$ imply $g(a) * g(b) \neq \emptyset$, a contradiction. We conclude that $g(u) * g(v) = \emptyset$.

We can now consider the case where $p \in x^\perp$ and $q \in x^\perp$ by symmetry. By 4.1.6 we have that $p, q \notin E$. Setting $u = p$ and $v = q$, 4.1.7 allows us to conclude. □

We can now prove the propositions.

Proof of 4.1.1. Let $p \in E$. By 4.1.8, there exist $u, v \in p^\perp \setminus E$ such that $g(u) * g(v) = \emptyset$. If $p^\perp \neq P_1$, there exists $q \in P_1$ such that $p * q = \emptyset$. There also exist $a \in (p * u) \cap q^\perp$ and $b \in (p * v) \cap q^\perp$. We remark that $a, b \notin E$. (M_2) implies $g(a) = g(u)$ and $g(b) = g(v)$, so $g(a) * g(b) = \emptyset$. We are then in the situation of 4.1.6; but $p \in E$, a contradiction. This proves that we must have $p^\perp = P_1$. □

Proof of 4.1.2. Let $p, q \in P_1$ be such that $p * q = \emptyset$. By the preceding proposition, we have that $p, q \notin E$. By 4.1.8 there exist $u, v \in p^\perp \setminus E$ such that $g(u) * g(v) = \emptyset$. This implies that $u * v = \emptyset$ and $u \neq p \neq v$; so there exist $a \in (p * u) \cap q^\perp$ and $b \in (p * v) \cap q^\perp$ with $a \neq p \neq b$. We remark that $a, b \notin E$ by 4.1.5.

So $u \in a * p$ implies $g(u) \in g(a) * g(p)$ and in the same way, $g(v) \in g(b) * g(p)$. Suppose that $g(a) * g(b) \neq \emptyset$. As $g(p) * g(b) \neq \emptyset$, we get $g(u) * g(b) \neq \emptyset$. Using that $g(u) * g(p) \neq \emptyset$, we conclude that $g(u) * g(v) \neq \emptyset$, a contradiction. So we must have $g(a) * g(b) = \emptyset$. Finally, 4.1.7 allows us to conclude. □

4.2. Splitting

We are now going to study the consequences of the preceding results and split a morphism of polar spaces into a morphism of nondegenerate polar spaces and a morphism of projective spaces.

Proposition 4.2.1. *Let $g : P_1 \setminus E \longrightarrow P_2$ be a morphism of polar spaces. Suppose that P_1 is thick, $P_1/\mathrm{Rad}(P_1)$ contains a line and $g(P_1 \setminus E)$ is non-singular. Then:*

i) $a \notin \mathrm{Rad}(P_1)$, $a \neq b$ *and* $g(a) = g(b) \implies (a * b) \cap E \neq \emptyset$;

ii) $(b * c) \cap \mathrm{Rad}(P_1) = \emptyset$, $a \notin b * c$ *and* $g(a) \in g(b) * g(c) \implies$ *there exists* $d \in (b * c)$ *such that* $(a * d) \cap E \neq \emptyset$;

iii) $a \notin \mathrm{Rad}(P_1) \implies g(a) \notin \mathrm{Rad}(P_2)$; *on the other hand,* $a \in \mathrm{Rad}(P_1) \setminus E \implies g(a) \in \mathrm{Rad}(F)$, *where* $F := \mathcal{C}(g(P_1 \setminus E))$;

iv) $(b * c) \cap \mathrm{Rad}(P_1) = \emptyset \implies (g(b) * g(c)) \cap \mathrm{Rad}(P_2) = \emptyset$.

Proof. i) Let $a \notin \mathrm{Rad}(P_1)$, $a \neq b$ and $g(a) = g(b)$. By 4.1.2, we have $a * b \neq \emptyset$. As $a \notin \mathrm{Rad}(P_1)$, there exist $c \in P_1$ and $d \in a * b$ such that $a * c = \emptyset$ and $c * d \neq \emptyset$; 4.1.1 yields that $c \notin E$. Suppose that $d \notin E$; $g(a) = g(b)$ would imply that $g(d) = g(a)$, so $g(a) * g(c) = g(d) * g(c) \neq \emptyset$, contradicting 4.1.2.

ii) Let $(b * c) \cap \mathrm{Rad}(P_1) = \emptyset$, $a \notin b * c$ and $g(a) \in g(b) * g(c)$. By 4.1.2, we have $b * c \neq \emptyset$; by i) we can suppose that $g(a) \neq g(b)$, so in particular $b \neq c$. As $(b * c) \cap \mathrm{Rad}(P_1) = \emptyset$, there exists a point $p \in b * c$ such that $\emptyset \neq a * p \subseteq P_1 \setminus \mathrm{Rad}(P_1)$. We can then apply 4.1.3 to get a point $r \in P_1$ such that $(a * p) \cap r^\perp = \{a\}$. There exists a unique point $d \in b * c$, $d \neq p$ such that $r * d \neq \emptyset$. If $g(a) \neq g(d)$, we would have $g(r) * g(a) \neq \emptyset$ and $g(r) * g(d) \neq \emptyset$, which would imply $g(r) * g(p) \neq \emptyset$, contradicting 4.1.2. So $g(a) = g(d)$ and i) allows us to conclude.

iii) Suppose that $a \notin \mathrm{Rad}(P_1)$. There exists $b \in P_1$ such that $a * b = \emptyset$; this implies by 4.1.1 that $b \notin E$. We then have $g(a) * g(b) = \emptyset$ by 4.1.2, so $g(a) \notin \mathrm{Rad}(P_2)$.

Suppose now that $a \in \mathrm{Rad}(P_1) \setminus E$. We have $g(P_1 \setminus E) = g(a^\perp \setminus E) \subseteq g(a)^\perp \cap F$. As $g(a)^\perp \cap F$ is a subspace of P_2, $\mathcal{C}(g(P_1 \setminus E)) = F$ implies that $g(a)^\perp \cap F = F$.

iv) Let $(b*c) \cap \mathrm{Rad}(P_1) = \emptyset$. We can suppose that $b*c \neq \emptyset$ (if not, 4.1.2 allows us to conclude) and $b \neq c$ by the preceding point. As $c \notin \mathrm{Rad}(P_1)$, there exists $a \in P_1$ such that $a * c = \emptyset$. There also exists $d \in b*c$, $d \neq c$, such that $a*d \neq \emptyset$. We remark that $g(c)*g(d) = g(b)*g(c)$ by i). If $(g(c) * g(d)) \cap \mathrm{Rad}(P_2) = \{q\}$, we would have $g(c) \in g(d) * q$ (as $g(d) \neq q$ by the preceding point), and $g(a) * q \neq \emptyset$ would imply $g(a) * g(c) \neq \emptyset$, contradicting 4.1.2. □

This proposition takes on a much more pleasant form when the radical of P_1 is empty. The following corollary summarizes this.

Corollary 4.2.2. *Let $g : P_1 \setminus E \longrightarrow P_2$ be a morphism of polar spaces. Suppose that P_1 is thick, contains a line, is nondegenerate and $g(P_1 \setminus E)$ is non-singular. Then g is a rigid embedding. Furthermore, if g is surjective then g is an isomorphism.*

For the following results, we will denote by $\rho_i : P_i \setminus \mathrm{Rad}(P_i) \longrightarrow P_i/\mathrm{Rad}(P_i)$ the canonical projections, where $i = 1, 2$.

Lemma 4.2.3. *Let $g : P_1 \setminus E \longrightarrow P_2$ be a morphism of polar spaces such that $P_2 = \mathcal{C}(g(P_1 \setminus E))$. Suppose that P_1 is thick, $P_1/\mathrm{Rad}(P_1)$ contains a line and $g(P_1 \setminus E)$ is non-singular. Let $a, b \notin \mathrm{Rad}(P_1)$ and $c \in \mathrm{Rad}(P_1)$ be such that $a \in b * c$. Then $\rho_2 \circ g(a) = \rho_2 \circ g(b)$.*

Proof. By 4.2.1, we have that $g(a), g(b) \notin \mathrm{Rad}(P_2)$, so $\rho_2 \circ g(a)$ and $\rho_2 \circ g(b)$ are well-defined. Considering the two cases $c \in E$ and $c \notin E$, it is easy to check that the equality is verified. □

Theorem 4.2.4. *Let $g : P_1 \setminus E \longrightarrow P_2$ be a morphism of polar spaces such that $P_2 = \mathcal{C}(g(P_1 \setminus E))$. Suppose that P_1 is thick, $P_1/\mathrm{Rad}(P_1)$ contains a line and $g(P_1 \setminus E)$ is non-singular. Then there exists a morphism $\overline{g} : P_1/\mathrm{Rad}(P_1) \longrightarrow P_2/\mathrm{Rad}(P_2)$ such that $\overline{g} \circ \rho_1 = \rho_2 \circ g$. Moreover, $\overline{g}$ is unique.*

Furthermore, $\overline{g}$ is a rigid embedding and the restriction $\underline{g}$ of g to the radical of P_1 ($\underline{g} : \mathrm{Rad}(P_1) \setminus E \longrightarrow \mathrm{Rad}(P_2)$) is a morphism of projective spaces.

Proof. We first verify that $\overline{g}$ is well-defined by $\overline{g} \circ \rho_1 := \rho_2 \circ g$. As in the proof of the lemma, we remark that $\rho_2 \circ g(p)$ exists for any point $p \in P_1 \setminus \mathrm{Rad}(P_1)$; so we can consider the case where $\mathrm{Rad}(P_1) \neq \emptyset$. Let $a, b \in P_1 \setminus \mathrm{Rad}(P_1)$ be such that $\rho_1(a) = \rho_1(b)$. This implies that there exists $c \in \mathrm{Rad}(P_1)$ such that $a \in b * c$, and the preceding lemma yields that $\rho_2 \circ g(a) = \rho_2 \circ g(b)$. The existence and unicity of the map $\overline{g} : P_1/\mathrm{Rad}(P_1) \longrightarrow P_2/\mathrm{Rad}(P_2)$ follows.

We verify now that $\overline{g}$ is a morphism; as its kernel is empty, we only have to check (M_3). Let $a, b, c \in P_1 \setminus \mathrm{Rad}(P_1)$ be such that $\rho_1(a) \in \rho_1(b) \circledast \rho_1(c)$. By 3.1.6, this means that there exists $a' \in \rho_1(a)$ such that $a' \in b * c$, so we have $\rho_2(g(a')) = \rho_2(g(b)) \circledast \rho_2(g(c))$. On the other hand, there exists a point $c' \in \mathrm{Rad}(P_1)$ such that $a \in a' * c'$ and the lemma allows us to conclude.

Corollary 4.2.2 then yields that $\overline{g}$ is a rigid embedding.

Finally, by 4.2.1 $\underline{g}$ is well-defined and the statement follows from the fact that the radicals are projective spaces. □

Remark 4.2.5. The preceding theorem states that a morphism of polar spaces is, up to a morphism between the radicals, essentially a rigid embedding. So the present approach does not yield anything new on the polar space level (contrary to the projective space case, see [6]). However, if we consider maps from polar to projective spaces, the situation is much richer, as any semilinear map induces a morphism.

Note that the converse of the theorem is not true in general. This can be seen by considering a morphism $\overline{g} : P_1/\mathrm{Rad}(P_1) \longrightarrow P_2/\mathrm{Rad}(P_2)$ induced by a linear map f_1, and $\underline{g} : \mathrm{Rad}(P_1) \setminus E \longrightarrow \mathrm{Rad}(P_2)$ induced by a τ-semilinear map f_2 with $\tau \neq \mathrm{id}$. Under suitable hypotheses, the existence of a morphism $g : P_1 \setminus E \longrightarrow P_2$ would imply the existence of a semilinear map f inducing g (see [10] Theorem 5.1.1). By [6] Proposition 6.3.6, the map f determines f_1 and f_2 up to scalar multiplication, which is impossible because $\tau \neq \mathrm{id}$.

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