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On a Conjecture about the Gauss Map of Complete Spacelike Surfaces with Constant Mean Curvature in the Lorentz-Minkowski Space

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Abstract. The Gauss map of complete helicoidal (consequently rotational) surfaces with non-zero constant mean curvature in the Euclidean 3-space contains a maximal circle of the sphere. Observing the Gauss map image for complete spacelike surfaces in the Lorentz-Minkowski 3-space $\mathbb{L}^3$, we propose the following conjecture: "Given a complete spacelike surface in $\mathbb{L}^3$, with non-zero constant mean curvature, its Gauss map image contains an arbitrary maximal geodesic of the hyperboloid contained in $\mathbb{L}^3$". We answer the conjecture for the special class of spacelike rotational surfaces in $\mathbb{L}^3$ and obtain that, in this case, the conjecture is also true, as in the Euclidean space $\mathbb{R}^3$.

1. Introduction

In 1841, Delaunay [5] described the following way of constructing rotational symmetric surfaces of constant mean curvature in the Euclidean 3-space $\mathbb{R}^3$. First, roll a given conic section on a line contained in a plane and then rotate about that line the trace of a focus, which is the profile curve of the rotational surface.

Observing the Delaunay surfaces and the fact that the Gauss map image of a cylinder is a maximal circle of the sphere, in 1981, do Carmo [3] proposed the following conjecture: "Given a complete surface in $\mathbb{R}^3$, with non-zero constant mean curvature, its Gauss map image contains a maximal circle of the sphere S^2, that is a maximal geodesic of the sphere". In 1984, Seaman [13] answered affirmatively the conjecture for the special class of helicoidal surfaces and, as a particular case, for Delaunay surfaces. In 1999, do Espírito-Santo, Frensel and Ripoll [6] obtained also a partial answer for do Carmo's conjecture showing that if the boundary of the image of the Gauss map has at most two components, then the conjecture in $\mathbb{R}^3$ is true. They were motivated by the work of Hoffman, Osserman and Schoen [8] where it is shown that the normals to a complete surface of constant mean curvature in the Euclidean space $\mathbb{R}^3$ cannot lie in a closed hemisphere of the sphere S^2, unless the surface is a plane or a right circular cylinder. As far as we know, do Carmo's conjecture in $\mathbb{R}^3$, without additional conditions, is still open.

In 1984, Hano and Nomizu [7] studied the Delaunay problem in the Lorentz-Minkowski 3-space $\mathbb{L}^3$, restricting themselves to the spacelike surfaces of revolution. In this case, the axis of revolution is either spacelike, timelike or lightlike. In the first two cases, they prove results of the same kind as Delaunay's except that the nature of quadrics needs special attention. In any case, the surfaces were obtained up to a congruence by a Lorentz transformation. The Gauss map of a spacelike surface S can be regarded as a map $N : S \to \mathbb{H}^2$, where $\mathbb{H}^2$ denotes the future directed component of the hyperbolic plane, $\mathbb{H}^2 = \{(x, y, z) \in \mathbb{L}^3 : x^2 + y^2 - z^2 = -1, z > 0\}$. Maximal circles of the sphere S^2 correspond now to maximal geodesics of $\mathbb{H}^2$, which are obtained as the intersection of $\mathbb{H}^2$ and the timelike planes of $\mathbb{L}^3$ passing through the origin. Actually, every spacelike unit vector $a \in \mathbb{L}^3$ determines a maximal geodesic γ_a of $\mathbb{H}^2$ given by $\gamma_a = \{p \in \mathbb{H}^2 : \langle a, p\rangle = 0\}$, where $\langle , \rangle$ is the Lorentzian metric of $\mathbb{L}^3$. In this setting, besides spacelike planes and hyperbolic planes $\mathbb{H}^2(r)$ of radius r, the simplest example of a complete spacelike surface with constant mean curvature in $\mathbb{L}^3$ is a hyperbolic cylinder $\mathbb{H}^1 X\mathbb{R}$, whose Gauss map image is precisely a maximal geodesic of $\mathbb{H}^2$.

In this context we adapted do Carmo's conjecture for the Lorentz-Minkowski 3-space: "Given a complete spacelike surface in $\mathbb{L}^3$, with non-zero constant mean curvature, its Gauss map image contains a maximal geodesic of $\mathbb{H}^2$ ". We found out that, for the special class of complete spacelike rotational surfaces with non-zero constant mean curvature in $\mathbb{L}^3$, the conjecture is also true. In this way, the conjecture proposed in $\mathbb{L}^3$ is still open, as in the Euclidean space $\mathbb{R}^3$. We point out that the case when the mean curvature vanishes identically was studied by Kobayashi [9] and McNertney [10] and are called maximal surfaces. These surfaces do not appear in our conjecture because the plane satisfies that condition and its Gauss map image is just a point (constant), not a maximal geodesic.

Other results about the Gauss map of surfaces in the Lorentz-Minkowski space were obtained, for example, in [1], [2], [4], [12] and [14]. In the first work, Akutagawa and Nishikawa allow us to produce a wealth of spacelike surfaces of constant mean curvature in $\mathbb{L}^3$ and to relate the geometry of these surfaces to the theory of harmonic mappings through their Gauss maps. In the second paper, Aiyama states that spacelike hyperplanes in the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ are the only complete spacelike hypersurfaces with constant mean curvature whose Gauss map image is bounded. This result was also independently proved by Xin in [14] (see also [12] for a previous weaker version of it, given by Palmer). Finally, in [4], Choi and Treibergs interpret properties of spacelike constant mean curvature hypersurfaces in terms of the Gauss map, which is a harmonic mapping.

This paper is organized as follows: in Section 2 we introduce some preliminaries. In

Sections 3, 4 and 5 we study each type of surface of revolution, depending on the causal character of its axis of revolution. In the first part of each section, specially Section 3, we reproduce some computations and results we need from [7] (see Propositions 3.5, 4.2 and 5.1). Hano and Nomizu did not emphasize the completeness of these surfaces and just cited the completeness of one of them. Since this information is very important for our conjecture, we control the completeness of the surfaces by Propositions 3.6, 4.3 and 5.2, using the helpful Lemma 2.2 of [11]. Finally, we point out that Theorem 3.7 and Theorem 5.3 give the positive answer for the conjecture proposed for the class of spacelike surfaces of revolution with spacelike axis and lightlike axis, respectively. According to Proposition 4.3, for the spacelike surfaces of revolution with timelike axis the conjecture is true, since all of them are not complete.

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2. Preliminaries

Let $\mathbb{L}^3$ denote the 3-dimensional Lorentz-Minkowski space, that is the real vector space $\mathbb{R}^3$ endowed with the Lorentzian metric $ds^2 = dx_1^2 + dx_2^2 - dx_3^2$, where $x = (x_1, x_2, x_3)$ are the canonical coordinates in $\mathbb{L}^3$.

As usual, the norm in this space is defined by

$$\|x\| = \sqrt{|\langle x, x\rangle|}.$$

A vector x in $\mathbb{L}^3$ is called timelike, spacelike or lightlike if, respectively $\langle x, x\rangle < 0$; $\langle x, x\rangle > 0$ or $x = 0$; or $\langle x, x\rangle = 0$ and $x \neq 0$.

We can define for any $a, b \in \mathbb{L}^3$ the cross product $a \wedge b \in \mathbb{L}^3$, given by

$$a \wedge b = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_2b_1 - a_1b_2),$$

where $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$. Thus for any $x \in \mathbb{L}^3$ it holds the relation $\langle a \wedge b, x\rangle = det(a, b, x)$.

The isometries group of $\mathbb{L}^3$ is the semi-direct product of the translations group and the orthogonal Lorentzian group $O(1, 2)$. With respect to the orthogonal group, there are three one-parameter subgroups of isometries of $\mathbb{L}^3$, that fix an axis (line), depending on the causal character of the axis. If the axis is spacelike it is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix}, \ t \in \mathbb{R} \text{ (hyperbolic group).}$$

If the axis is timelike it is given by

$$\begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ 0 < t < 2\pi \text{ (elliptic group),}$$

and if the axis is lightlike it is given by

$$\begin{pmatrix} 1 & -t & t \\ t & 1-t^2/2 & t^2/2 \\ t & -t^2/2 & 1+t^2/2 \end{pmatrix}, \ t \in \mathbb{R} \text{ (parabolic group).}$$

A surface S in $\mathbb{L}^3$ is said spacelike if the induced metric is a Riemannian metric. In [7], Hano and Nomizu obtained parametrizations for the spacelike surfaces of revolution in $\mathbb{L}^3$, using the fact that they must be invariant by the action of one of the 1-parameter subgroups of isometries, cited above. By taking the profile curve Ω in the xz-plane, parametrized by $\Omega(\theta) = (x(\theta), 0, z(\theta))$, they obtained the following parametrizations, for both spacelike and timelike axis:

$$X_S(\theta, t) = (x(\theta),\ z(\theta)\sinh t,\ z(\theta)\cosh t),\ a < \theta < b,\ z(\theta) \neq 0; \tag{2.1}$$

$$X_T(\theta, t) = (x(\theta)\cos t, x(\theta)\sin t, z(\theta)), a < \theta < b, x(\theta) \neq 0. \tag{2.2}$$

For lightlike axis, the profile curve is given by $\Omega(s) = (0, y(s), z(s))$, where s is the arc length parameter, and the parametrization is given by

$$X_L(s,t) = \left(-t\left(y(s) - z(s)\right), y(s) - \left(y(s) - z(s)\right)\frac{t^2}{2}, z(s) - \left(y(s) - z(s)\right)\frac{t^2}{2}\right). \tag{2.3}$$

To study the completeness of these surfaces, we need the following definition and lemma that can be found in [11].

Definition 2.1. *Suppose B and F are semi-Riemannian manifolds and let $f > 0$ be a smooth function on B. The warped product $M = B \times_f F$ is the product manifold $B \times F$ furnished with the metric tensor $g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^*(g_F)$, where g_B and g_F are the metric tensor on B and F, respectively, and π and σ are the projections of $B \times F$ onto B and F, respectively. Explicitly, if x is tangent to $B \times F$ at (p, q), then*

$$< x, x > = < d\pi(x), d\pi(x) > + f^2(p)(d\sigma(x), d\sigma(x)).$$

Lemma 2.2. *If B and F are complete Riemannian manifolds, then $M = B \times_f F$ is complete for every warping function f.*

Remark 2.3. We observe that the spacelike surfaces of revolution with constant mean curvature H in $\mathbb{L}^3$ are warped products whose leaves are the different positions of the rotated curve and whose fibers are the orbits. In fact, after computing the metric for each surface, we get that for X_S the function f is equal to $z(\theta)$, for X_T the function f is equal to $x(\theta)$ and for X_L the function f is equal to $y(s) - z(s)$. Then, to analyse the completeness of these surfaces, it is enough to study the completeness of the profile curve Ω because, clearly, the orbits are complete and the surface can be regarded as the warped product of Ω by the orbit. Moreover, Ω is a maximal geodesic on S and if it is not complete the surface is not geodesically complete and, consequently, S is not complete.

We are going to study the Gauss map image of the above surfaces. For both spacelike and timelike axis, it can be given, locally, by the expression $N(\theta,t) := \pm\dfrac{X_\theta \wedge X_t}{\|X_\theta \wedge X_t\|}$. For lightlike axis, the Gauss map is given by $N(s,t) := \pm\dfrac{X_s \wedge X_t}{\|X_s \wedge X_t\|}$. It is easy to verify that $N(\theta,t)$ and $N(s,t)$ are timelike vectors, as expected. So we can choose a unique timelike unit normal field N, which is future-directed in $\mathbb{L}^3$ and hence we may assume that the surface is oriented by N. By parallel translation to the origin in $\mathbb{L}^3$, we can regard the field N as a map whose image is contained in $\mathbb{H}^2$. We will refer to this image as the Gauss map image of the surface.

Following Hano and Nomizu [7], we are going to define a conic in a two-dimensional Lorentz-Minkowski space. For this, let $\mathbb{L}^2$ be the vector space $\mathbb{R}^2$ provided with the Lorentzian metric $ds^2 = dx_1^2 - dx_2^2$, where $x = (x_1, x_2)$ are the canonical coordinates in $\mathbb{L}^2$.

Definition 2.4. *Let F denote a fix point, D a fix line both in $\mathbb{L}^2$ and $\varepsilon > 0$ a real number. A conic Γ having focus F, directrix D and eccentricity ε is the locus of a point P such that $\dfrac{d(P,F)}{d(P,D)} = \varepsilon$ (d means the Lorentzian distance).*

The conic is called a parabola, an ellipse or a hyperbola if $\varepsilon = 1$, $0 < \varepsilon < 1$ or $\varepsilon > 1$, respectively.

3. Surfaces of revolution with spacelike axis

Let us state a lemma and some computations, obtained by Hano and Nomizu [7], which relates the profile curve Ω_S of the surface with a given conic Γ and is the analogous to the classical characterization given by Delaunay.

Lemma 3.1. *Let Γ be a spacelike curve given in the polar form by the expression $\Gamma(\theta) = (r(\theta)\sinh\theta, r(\theta)\cosh\theta)$, $r(\theta) > 0$ and let Ω_S be the locus of the origin when Γ is rolled along the x-axis. If the curvature of Γ never vanishes, then Ω_S is a spacelike curve for which the center of curvature never lies on the x-axis. Conversely, such a curve Ω_S is obtained as the locus of the origin for the rolling of a certain spacelike curve Γ, which is uniquely determined up to a Lorentz transformation of the xz-plane.*

Observe that $r(\theta) > 0$ and since $\Gamma'(\theta)$ is spacelike, $' = d/d\theta$, we see that $r^2(\theta) - r'^2(\theta) > 0$. In this case, Ω_S is taken as the locus of the origin when Γ is rolled along the x-axis in such a way that Ω_S appears below the x-axis. Then, considering $\xi(\theta)$ the arc length of $\Gamma(\theta)$ starting from $\Gamma(\theta_0)$, Ω_S is written as

$$\Omega_S(\theta) : \begin{cases} x(\theta) = \xi(\theta) - \xi(\theta_0) - r(\theta)\sinh\Phi(\theta); \\ z(\theta) = -r(\theta)\cosh\Phi(\theta), \end{cases} \tag{3.1}$$

where $\Phi = \Phi(\theta)$ is determined by the fact that $r(\theta)\sinh\Phi(\theta)$ is equal to the Lorentz inner product of the position vector of Γ and the unit tangent vector of Γ. Thus,

$$\sinh\Phi(\theta) = \frac{-r'(\theta)}{\sqrt{r(\theta)^2 - r'(\theta)^2}} \quad \text{and} \quad \cosh\Phi(\theta) = \frac{r(\theta)}{\sqrt{r(\theta)^2 - r'(\theta)^2}}. \tag{3.2}$$

We choose an arc length parameter s for Ω, in such a way that $\dot{\Omega}(s) = (\dot{x}(s), 0, \dot{z}(s)) = (\cosh\Phi, 0, \sinh\Phi)$, $(\cdot = d/ds)$. Then, using the parametrization (2.1) we obtain, after some computations, the principal curvatures of the surface given by

$$\frac{\ddot{x}}{\dot{z}} = \dot{\Phi} \quad \text{and} \quad \frac{\dot{x}}{z} = \frac{-1}{r}.$$

By the definition of the mean curvature H we have $2Hr = -1 + r\dot{\Phi}$ and after some computations, Hano and Nomizu found the following result.

Proposition 3.2. *The curve $\Gamma(\theta)$ gives rise to a surface of revolution with constant mean curvature H in $\mathbb{L}^3$ if and only if the function $r(\theta)$ satisfies the differential equation*

$$\frac{d^2 \log r}{d\theta^2} = \left[\left(\frac{d\log r}{d\theta}\right)^2 - 1\right] \frac{1 + 2rH}{2 + 2rH}. \tag{3.3}$$

The general solution of (3.3) is given by

$$\frac{1}{r} = a\cosh\theta + b\sinh\theta + c, \ \ r > 0, \text{where} \quad 2Hc = a^2 - b^2 - c^2. \tag{3.4}$$

Remark 3.3. When $c = 0$, $r(\theta) = ae^{\pm\theta}$ are lightlike lines and so are excluded.

Thus a curve $\Gamma(\theta)$ gives rise to a spacelike surface of revolution with constant mean curvature if and only if $r(\theta)$ takes one of the following forms:

(a) $r = \dfrac{1}{c}$, with $H = \dfrac{-c}{2}$, $(c \neq 0)$;

(b) $\dfrac{1}{r} = \pm\lambda \ \cosh(\theta + \mu) + c$, with $\lambda > 0$, $H = \dfrac{(\lambda^2 - c^2)}{2c}$, $(c \neq 0)$;

(c) $\dfrac{1}{r} = \pm\lambda \ \sinh(\theta + \mu) + c$, with $\lambda > 0$, $H = \dfrac{-(\lambda^2 + c^2)}{2c}$, $(c \neq 0)$;

(d) $\dfrac{1}{r} = ae^{\theta} + c$ or $ae^{-\theta} + c$, with $H = \dfrac{-c}{2}$, $(c \neq 0)$.

As observed by Hano and Nomizu, if two curves Γ_1 and Γ_2 simply differ by a Lorentzian transformation of the xz-plane fixing the origin, then the resulting curves Ω_1 and Ω_2 generate congruent surfaces of revolution. Thus in the list above, we may assume $\mu = 0$ in (b) and (c), consider only the $+$ sign in (c) and one or the other (say, $ae^{-\theta} + c$) in (d). Examining the polar equations above, it can be shown that for the case (a) the surface of revolution is an isometric imbedding of the Euclidean plane given by $(s, t) \to (s, \dfrac{-\sinh t}{c}, \dfrac{-\cosh t}{c})$, with constant mean curvature c. This is a Lorentzian cylinder, as mentioned at the Introduction, whose Gauss map image is a particular maximal geodesic of $\mathbb{H}^2$. In order to classify the curves $\Gamma(\theta)$ in case $r(\theta)$ is given by (b), (c) and (d), it is convenient to take $d = \dfrac{1}{\lambda} > 0$ ($z = d$ is the directrix) and $\varepsilon = \dfrac{\lambda}{|c|}$ (ε is the eccentricity). Some of these curves are part of an ellipse, a hyperbola or a parabola, according to Definition 2.4. For more details, see Section 2 of [7].

Remark 3.4. If the center of curvature of Ω_S lies on the x-axis, we get the hyperboloid $x^2+y^2-z^2=\dfrac{-1}{H^2}$, $z<0$.

By summarizing these results, we can write the following

Proposition 3.5. *The spacelike surfaces of revolution, with non-zero constant mean curvature H in $\mathbb{L}^3$, which are obtained by rotating $\Omega_S(\theta)$ along the x-axis, are given by*

$S_1)$ $r(\theta)=\dfrac{\varepsilon d}{1+\varepsilon\cosh(\theta)}$, $\theta\in\mathbb{R}$, $\varepsilon>0, \varepsilon\neq 1$, $H=\dfrac{\varepsilon^2-1}{2\varepsilon d}$;

$S_2)$ $r(\theta)=\dfrac{\varepsilon d}{1-\varepsilon\cosh(\theta)}$, $\log\varepsilon<\theta<-\log\varepsilon$, $0<\varepsilon<1$, $H=\dfrac{\varepsilon^2-1}{2\varepsilon d}$;

$S_3)$ $r(\theta)=\dfrac{\varepsilon d}{-1+\varepsilon\cosh(\theta)}$, $-\log\varepsilon<\theta<\log\varepsilon$, $\varepsilon>1$, $H=\dfrac{1-\varepsilon^2}{2\varepsilon d}$;

$S_4)$ $r=1/c$, $H=-c/2$, $c\neq 0$;

$S_5)$ $x^2+y^2-z^2=-1/H^2$, $z<0$, $H\neq 0$;

$S_6)$ $r(\theta)=\dfrac{\varepsilon d}{1+\varepsilon\sinh(\theta)}$, $\theta>\log\varepsilon$, $\varepsilon>0$, $H=-\dfrac{\varepsilon^2+1}{2\varepsilon d}$;

$S_7)$ $r(\theta)=\dfrac{1}{ae^{-\theta}+c}$, $a>0, c>0, \theta\in\mathbb{R}, H=\dfrac{-c}{2}$;

$S_8)$ $r(\theta)=\dfrac{1}{ae^{-\theta}+c}$, $a<0, c>0, \theta>\log\left(\dfrac{-2a}{c}\right), H=\dfrac{-c}{2}$.

Now we are going to study the completeness of the above surfaces. For this we need Lemma 2.2 and Remark 2.3.

Proposition 3.6. *The only complete spacelike surfaces of revolution in $\mathbb{L}^3$ with non-zero constant mean curvature and spacelike axis are the surfaces labeled as S_2, S_3, S_4, S_5 and S_8.*

Proof. By Remark 2.3 it is enough to study the completeness of the curve $\Omega_S(\theta)$, parametrized by (3.1). The tangent vector of Ω_S is

$$\Omega_S'(\theta)=(x'(\theta),0,z'(\theta))=\left(\frac{r^2}{\sqrt{r^2-r'^2}}\left(1+\frac{d\Phi}{d\theta}\right),0,\frac{-rr'}{\sqrt{r^2-r'^2}}\left(1+\frac{d\Phi}{d\theta}\right)\right). \tag{3.5}$$

It follows that $\|\Omega_S'(\theta)\|=r\left|1+\dfrac{d\Phi}{d\theta}\right|$. By differentiating one of the expressions in (3.2) we get $\dfrac{d\Phi}{d\theta}=\dfrac{r'r''-r'^2}{r^2-r'^2}$. Hence

$$\|\Omega_S'(\theta)\|=\frac{r|r^2-2r'^2+r'r''|}{r^2-r'^2}. \tag{3.6}$$

In the following, we are going to compute, for each case, the limits of $s(\theta)$ for θ tending to the extremes of the correspondent interval given in Proposition 3.5. For this we recall that the arc length parameter for $\Omega_S(\theta)$ is given by $s(\theta)=\displaystyle\int^{\theta}\|\Omega_S'(u)\|du$.

(a) For the surface S_1 we observe that $r(\theta) = \dfrac{\varepsilon d}{1+\varepsilon\cosh\theta}$, $\theta \in \mathbb{R}$, $\varepsilon > 0$, $\varepsilon \neq 1$. Then, we have

$$r^2 - r'^2 = \frac{\varepsilon^2 d^2(1+\varepsilon^2+2\varepsilon\cosh\theta)}{(1+\varepsilon\cosh\theta)^4} > 0 \quad \text{and by (3.6),}$$

$$\|\Omega_S'(\theta)\| = \frac{\varepsilon d}{1+\varepsilon^2+2\varepsilon\cosh\theta} = \frac{\varepsilon d e^\theta}{(e^\theta+\varepsilon)(\varepsilon e^\theta+1)}.$$

After integrating we obtain that $s(\theta) = \dfrac{\varepsilon d}{\varepsilon^2-1}\log\left(\dfrac{1+\varepsilon e^\theta}{\varepsilon+e^\theta}\right)$ and the limits

$$\lim_{\theta\to-\infty} s(\theta) = \frac{\varepsilon d}{1-\varepsilon^2}\log\varepsilon \text{ and } \lim_{\theta\to\infty} s(\theta) = \frac{\varepsilon d}{-1+\varepsilon^2}\log\varepsilon$$

are both finite. Then the curve $\Omega_S(\theta)$ is not complete and, by Remark 2.3, S_1 is not complete.

(b) For the surfaces S_2 and S_3 we observe that $r(\theta) = \pm\dfrac{\varepsilon d}{1-\varepsilon\cosh\theta}$. In this case, we have

$$r^2 - r'^2 = \frac{\varepsilon^2 d^2(1+\varepsilon^2-2\varepsilon\cosh\theta)}{(1-\varepsilon\cosh\theta)^4} > 0 \text{ and by (3.6),}$$

$$\|\Omega_S'(\theta)\| = \frac{\varepsilon d}{1+\varepsilon^2-2\varepsilon\cosh\theta} = \frac{\varepsilon d e^\theta}{(e^\theta-\varepsilon)(1-\varepsilon e^\theta)}.$$

After integrating we obtain that $s(\theta) = \dfrac{\varepsilon d}{\varepsilon^2-1}\log\left(\dfrac{1-\varepsilon e^\theta}{e^\theta-\varepsilon}\right)$. For S_2 we have $\log\varepsilon < \theta < -\log\varepsilon$, $0<\varepsilon<1$, and we get

$$\lim_{\theta\to\log\varepsilon} s(\theta) = -\infty \text{ and } \lim_{\theta\to-\log\varepsilon} s(\theta) = \infty.$$

For S_3 we have $-\log\varepsilon < \theta < \log\varepsilon$, $\varepsilon > 1$ and we get

$$\lim_{\theta\to-\log\varepsilon} s(\theta) = -\infty \text{ and } \lim_{\theta\to\log\varepsilon} s(\theta) = \infty.$$

In both cases $\Omega_S(\theta)$ is complete and, consequently, S_2 and S_3 are complete.

(c) For S_6, we have $r(\theta) = \dfrac{\varepsilon d}{1+\varepsilon\sinh(\theta)}$, $\theta > \log\varepsilon$, $\varepsilon > 0$. Hence

$$r^2 - r'^2 = \frac{\varepsilon^2 d^2(1-\varepsilon^2+2\varepsilon\sinh\theta)}{(1+\varepsilon\sinh\theta)^4} > 0 \text{ and by (3.6)}$$

$$\|\Omega_S'(\theta)\| = \frac{\varepsilon d}{1-\varepsilon^2+2\varepsilon\sinh\theta} = \frac{\varepsilon d e^\theta}{(e^\theta-\varepsilon)(\varepsilon e^\theta+1)}.$$

Hence $s(\theta) = \dfrac{\varepsilon d}{1+\varepsilon^2}\log\left(\dfrac{\varepsilon e^\theta-1}{e^\theta+\varepsilon}\right)$, $-\log\varepsilon < \theta < \log\left(\dfrac{1+\sqrt{1+\varepsilon^2}}{\varepsilon}\right)$, $\varepsilon > 0$, and we have

$$\lim_{\theta\to\log\left(\frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon}\right)} s(\theta) = \frac{\varepsilon d}{1+\varepsilon^2}\log\left(\frac{\varepsilon}{1+\sqrt{1+\varepsilon^2}}\right).$$

Since this limit is finite, the surface S_6 is not complete.

(d) For S_7 and S_8 we have

$$r(\theta) = \frac{1}{ae^{-\theta}+c},\ r^2 - r'^2 = \frac{ce^{-\theta}(2a+ce^{\theta})}{(ae^{-\theta}+c)^4} > 0 \text{ and } \|\Omega'_S(\theta)\| = \frac{e^{\theta}}{2a+ce^{\theta}}.$$

The arc lengths of these two surfaces are given by $s(\theta) = \frac{1}{c}\log(2a+ce^{\theta})$, $\theta < \log\left(\frac{a}{b}\right)$.

For the surface S_7 , $a > 0, c > 0, \theta \in \mathbb{R}$ and we get

$$\lim_{\theta\to-\infty} s(\theta) = \frac{1}{c}\log(2a)$$

which is finite and, hence, S_7 is not complete.

For S_8 , $a < 0, c > 0, \theta > \log\left(\frac{-2a}{c}\right)$. In this case

$$\lim_{\theta\to\log\left(\frac{-2a}{c}\right)} s(\theta) = -\infty \text{ and } \lim_{\theta\to+\infty} s(\theta) = +\infty.$$

We conclude that the corresponding profile curve is complete and, hence, S_8 is complete.

(e) The surfaces S_4 and S_5 are clearly complete, since they are the Lorentzian cylinder and the hyperboloid, respectively, which finishes the proof of the proposition.

Considering the fact that N was chosen to be contained in $\mathbb{H}^2$ and the parametrizations we are considering for the rotational surfaces, let us denote by $\mathcal{H}$ the convenient maximal geodesic of $\mathbb{H}^2$, contained in the plane $x = 0$. This maximal geodesic is contained in the Gauss map image of surfaces S_2, S_3, S_4, and S_5, but it is not contained in the Gauss map image of the surface S_8. However, for this surface, the Gauss map image contains other maximal geodesics of $\mathbb{H}^2$, obtained by intersecting this sheet with timelike planes through the origin.

Theorem 3.7. *The Gauss map image of the complete spacelike surfaces of revolution in $\mathbb{L}^3$ with non-zero constant mean curvature and spacelike axis (surfaces labeled as S_2, S_3, S_4, S_5, and S_8), contains a maximal geodesic of $\mathbb{H}^2$. Actually, the Gauss map image of S_2, S_3 and S_5 is $\mathbb{H}^2$. The Gauss map image of S_4 is the maximal geodesic $\mathcal{H}$ of $\mathbb{H}^2$, contained in the plane $x = 0$ and the Gauss map image of S_8 is the open half-sheet of $\mathbb{H}^2$ with $x > 0$.*

Proof. The surface S_4 is the Lorentzian cylinder and the Gauss map image is exactly the geodesic $\mathcal{H}$. The surface S_5 is the hyperboloid, whose Gauss map image is $\mathbb{H}^2$. We recall that the remaining complete surfaces can be parametrized by $X_S(\theta,t) = (x(\theta), z(\theta)\sinh t, z(\theta)\cosh t)$, where $t \in \mathbb{R}$, θ lies in one of the intervals of Proposition 3.5. Here $x(\theta)$ and $z(\theta)$ are the coordinates of the profile curve given by (3.1) and using (3.2) we get

$$x(\theta) = \xi(\theta) - \xi(\theta_0) + \frac{r(\theta)r'(\theta)}{\sqrt{r(\theta)^2 - r'(\theta)^2}};\quad z(\theta) = \frac{-r(\theta)^2}{\sqrt{r(\theta)^2 - r'(\theta)^2}}.$$

Since $\xi'(\theta) = \sqrt{r^2 - r'^2}$, we obtain

$$x'(\theta) = \frac{r^2\left(r^2 - 2r'^2 + rr''\right)}{\left(r^2 - r'^2\right)^{\frac{3}{2}}},\ z'(\theta) = \frac{-rr'\left(r^2 - 2r'^2 + rr''\right)}{\left(r^2 - r'^2\right)^{\frac{3}{2}}} \text{ and}$$

$$x'(\theta)^2 - z'(\theta)^2 = \frac{r^2\left(r^2 - 2r'^2 + rr''\right)^2}{(r^2 - r'^2)^2}.$$

Now

$$\begin{aligned}
\frac{\partial X_S}{\partial \theta}(\theta, t) &= (x'(\theta), z'(\theta)\sinh t, z'(\theta)\cosh t),\\
\frac{\partial X_S}{\partial t}(\theta, t) &= (0, z(\theta)\cosh t, z(\theta)\sinh t), \text{ and}\\
\frac{\partial X_S}{\partial \theta} \wedge \frac{\partial X_S}{\partial t} &= (-z(\theta)z'(\theta), -z(\theta)x'(\theta)\sinh t, -z(\theta)x'(\theta)\cosh t).
\end{aligned}$$

We have also the coeficients of the first fundamental form of these surfaces given by

$$E(\theta, t) = \langle \frac{\partial X_S}{\partial \theta}, \frac{\partial X_S}{\partial \theta} \rangle = \frac{r^2\left(r^2 - 2r'^2 + rr''\right)^2}{(r^2 - r'^2)^2},$$

$$F(\theta, t) = 0 \text{ and } G(\theta, t) = \langle \frac{\partial X_S}{\partial t}, \frac{\partial X_S}{\partial t} \rangle = \frac{r^4}{r^2 - r'^2}.$$

Finally,

$$\langle \frac{\partial X_S}{\partial \theta} \wedge \frac{\partial X_S}{\partial t}, \frac{\partial X_S}{\partial \theta} \wedge \frac{\partial X_S}{\partial t} \rangle = F^2 - EG = \frac{-r^6\left(r^2 - 2r'^2 + rr''\right)^2}{(r^2 - r'^2)^3} < 0.$$

Since $N(\theta, t) = \pm \frac{\frac{\partial X_S}{\partial \theta} \wedge \frac{\partial X_S}{\partial t}}{||\frac{\partial X_S}{\partial \theta} \wedge \frac{\partial X_S}{\partial t}||}$, by choosing the sign in such a way that the third component of N is positive, we can write

$$N(\theta, t) = \left(\frac{-r'(\theta)}{\sqrt{r^2(\theta) - r'^2(\theta)}}, \frac{r(\theta)\sinh t}{\sqrt{r^2(\theta) - r'^2(\theta)}}, \frac{r(\theta)\cosh t}{\sqrt{r^2(\theta) - r'^2(\theta)}} \right). \tag{3.7}$$

a) For the surfaces labeled as S_2 and S_3, it is possible to unify the reasoning by taking $r(\theta) = \pm \frac{\varepsilon d}{1 - \varepsilon \cosh \theta}$, where the plus sign corresponds to S_2 and the minus sign corresponds to S_3. Hence,

$$r'(\theta) = \pm \frac{\varepsilon^2 d \sinh \theta}{(1 - \varepsilon \cosh \theta)^2}, \quad \sqrt{r^2(\theta) - r'^2(\theta)} = \frac{\varepsilon d \sqrt{1 + \varepsilon^2 - 2\varepsilon \cosh \theta}}{(1 - \varepsilon \cosh \theta)^2},$$

and by (3.7)

$$N(\theta, t) = \pm \frac{1}{\sqrt{1 + \varepsilon^2 - 2\varepsilon \cosh \theta}} \left(-\varepsilon \sinh \theta, (1 - \varepsilon \cosh \theta)\sinh t, (1 - \varepsilon \cosh \theta)\cosh t\right).$$

Now, it is easy to show that the first component $N_1(\theta, t)$ of $N(\theta, t)$, regarded as a function $N_1 : I_\varepsilon \to \mathbb{R}$ is bijective, where $I_\varepsilon = (\log \varepsilon, -\log \varepsilon)$, if $0 < \varepsilon < 1$ (surface S_2) and $I_\varepsilon = (-\log \varepsilon, \log \varepsilon)$, if $\varepsilon > 1$ (surface S_3). In fact, given $(x, y, z) \in \mathbb{H}^2$, there exists a unique $\theta \in I_\varepsilon$ such that $x = N_1(\theta)$. For this θ, there exists a unique $t \in \mathbb{R}$ such that

$$y = \frac{|1 - \varepsilon \cosh \theta| \sinh t}{\sqrt{1 + \varepsilon^2 - 2\varepsilon \cosh \theta}} \quad \text{and} \quad z = \frac{|1 - \varepsilon \cosh \theta| \cosh t}{\sqrt{1 + \varepsilon^2 - 2\varepsilon \cosh \theta}},$$

which implies that, for the surfaces S_2 and S_3, the function $N : I_\varepsilon \times \mathbb{R} \to \mathbb{H}^2$ is a bijection.

b) For the surface S_8,

$$r(\theta) = \frac{1}{ae^{-\theta}+c}, \quad r'(\theta) = \frac{ae^{-\theta}}{(ae^{-\theta}+c)^2} \quad \text{and} \quad \sqrt{r^2(\theta) - r'^2(\theta)} = \frac{\sqrt{c}\sqrt{c+2ae^{-\theta}}}{(ae^{-\theta}+c)^2}.$$

Then

$$N(\theta,t) = \frac{1}{\sqrt{c}\sqrt{c+2ae^{-\theta}}}\left(-ae^{-\theta}, (ae^{-\theta}+c)\sinh t, (ae^{-\theta}+c)\cosh t\right).$$

In this case, $N_1(\theta) = \dfrac{-ae^{-\theta}}{\sqrt{c}\sqrt{c+2ae^{-\theta}}}$, with $\theta > \log(\frac{-2a}{c})$ and, after computing the limits, we get

$$\lim_{\theta\to\infty} N_1(\theta) = 0 \text{ and } \lim_{\theta\to\log(\frac{-2a}{c})^+} N_1(\theta) = +\infty.$$

This implies that $N_1 : \left(\log(\frac{-2a}{c}), \infty\right) \to (0,\infty)$ is a bijection and, consequently,

$$N : \left(\log(\frac{-2a}{c}), \infty\right) \times \mathbb{R} \to \mathbb{H}^2{}_+ = \left\{(x,y,z) \in \mathbb{H}^2 : x > 0\right\}$$

is a bijection, which implies that $ImN = \mathbb{H}^2{}_+$.

4. Surfaces of revolution with timelike axis

The results in this section are not written in [7], but there it is observed this case is quite similar to the previous one.

Let Γ a plane curve in the xz-plane given by polar coordinate graph $r = r(\theta)$. If one rolls Γ on the z-axis, the trace of the origin O of the polar coordinate system attached to Γ plots another curve Ω_T. The following result states the analytical relationship between Γ and Ω_T, where prime denotes $d/d\theta$.

Lemma 4.1. *Let Γ be a timelike curve given by polar coordinate graph $r = r(\theta)$ with $r(\theta) > 0$. Let Ω_T be the locus of the origin when Γ is rolled along the z-axis. If the curvature of Γ never vanishes, then Ω_T is a spacelike curve for which the center of curvature never lies on the z-axis. Conversely, such a curve Ω_T is obtained as the locus of the origin for the rolling of a certain timelike curve Γ, which is uniquely determined up to a Lorentz transformation of the xz-plane.*

Proof. Now we are going to take Γ a smooth curve given in the xz-plane by the timelike vector-valued function

$$\Gamma(\theta) = (r(\theta)\sinh\theta, r(\theta)\cosh\theta). \tag{4.1}$$

Since Γ is a timelike curve, that is, the tangent vector of Γ is always timelike, we get $r'^2(\theta) - r^2(\theta) > 0$. Consequently, $r'(\theta)$ never vanishes and we can assume $r'(\theta) > 0, \forall\theta$. Considering $\Phi(\theta)$ the angle between Γ and Γ', we obtain

$$\sinh\Phi(\theta) = \frac{r(\theta)}{\sqrt{r'(\theta)^2 - r(\theta)^2}} \quad \text{and} \quad \cosh\Phi(\theta) = \frac{r'(\theta)}{\sqrt{r'(\theta)^2 - r(\theta)^2}}. \tag{4.2}$$

The angle between Γ' and the polar axis is given by $\overline{\Phi}(\theta) = \Phi(\theta) + \theta$ and we can write

$$\frac{\Gamma'(\theta)}{||\Gamma'(\theta)||} = (\sinh \overline{\Phi}(\theta), \cosh \overline{\Phi}(\theta)). \tag{4.3}$$

After differentiating both sides of (4.3) we get

$$\frac{\Gamma''(\theta)}{||\Gamma'(\theta)||} + \Gamma'(\theta)\frac{d}{d\theta}\frac{1}{||\Gamma'(\theta)||} = \left(\cosh \overline{\Phi}(\theta), \sinh \overline{\Phi}(\theta)\right)\frac{d\overline{\Phi}}{d\theta}. \tag{4.4}$$

By taking the inner product, in both sides of (4.4), with the normal vector to $\Gamma(\theta)$ we obtain

$$\frac{d\overline{\Phi}}{d\theta} = ||\Gamma'(\theta)|| k_\Gamma(\theta), \tag{4.5}$$

where $k_\Gamma(\theta)$ is the curvature of the curve $\Gamma(\theta)$. We point out that formula (4.5) is similar to the formula related to a plane curve in $\mathbb{R}^3$.

By assumption, the curvature $k_\Gamma(\theta) \neq 0$, $\forall \theta$, which implies

$$\frac{d\overline{\Phi}}{d\theta} = 1 + \frac{d\Phi}{d\theta} \neq 0, \forall \theta. \tag{4.6}$$

Now let Ω_T be the locus of the origin O, when Γ is rolled along the z-axis, in such a way that Ω_T appears on the half-plane $x > 0$.

Following the same reasoning of Section 3, Ω_T can be written as

$$\Omega_T(\theta) : \begin{cases} x(\theta) = r(\theta)\,\sinh \Phi(\theta); \\ z(\theta) = \xi(\theta) - \xi(\theta_0) - r(\theta)\,\cosh \Phi(\theta), \end{cases} \tag{4.7}$$

where ξ is the arc length parameter of Γ, that is $\xi' = \sqrt{r'^2 - r^2}$.
The tangent vector of Ω_T is

$$\Omega'_T(\theta) = (x'(\theta), 0, z'(\theta)) = \left(\frac{rr'}{\sqrt{r'^2 - r^2}}\left(1 + \frac{d\Phi}{d\theta}\right), 0, \frac{-r^2}{\sqrt{r'^2 - r^2}}\left(1 + \frac{d\Phi}{d\theta}\right)\right). \tag{4.8}$$

Since $\langle \Omega'_T(\theta), \Omega'_T(\theta)\rangle = r^2(1 + \frac{d\Phi}{d\theta})^2$, (4.6) implies that Ω_T is regular and spacelike. We also get $\langle \Omega'_T(\theta), \Gamma(\theta)\rangle = 0$ and, assuming these conditions, we can choose an arc length parameter s for Ω_T, in such a way that $\dot{\Omega}_T(s) = (\cosh \Phi(s), \sinh \Phi(s))$, where the dot denotes d/ds. Then $\frac{ds}{d\theta} = ||\Omega'_T(\theta)|| \neq 0$, and hence $\dot{\theta}(s) = \frac{d\theta}{ds} \neq 0, \forall s$.

By the first equation in (4.7) and some computations, we obtain

$$\dot{r}(s) = \frac{d}{ds}\left(\frac{x(s)}{\sinh \Phi(s)}\right) = \coth \Phi(s)(1 - r(s)\dot{\Phi}(s)), \tag{4.9}$$

and from the second equation in (4.7) we get

$$\dot{\xi}(s) = \frac{1 - r(s)\dot{\Phi}(s)}{\sinh \Phi(s)}. \tag{4.10}$$

From $\xi' = \sqrt{r'^2 - r^2}$, it follows that $r^2\dot{\theta}(s) = \dot{r}^2(s) - \dot{\xi}^2(s)$ and using (4.9) and (4.10) we obtain

$$\dot{\theta}(s) = \pm\frac{1 - r(s)\dot{\Phi}(s)}{r(s)}. \tag{4.11}$$

By choosing directions of the curve Γ and Ω_T, we can choose the + sign in (4.11). Finally, since $\dot{\theta}(s) \neq 0, \forall s$, we have

$$1 - r(s)\dot{\Phi}(s) \neq 0, \forall s. \tag{4.12}$$

The center of curvature of $\Omega_T(s)$ is given by

$$C(s) = \Omega_T(s) - \frac{1}{k_{\Omega_T}(s)}(\sinh\Phi(s), -\cosh\Phi(s)),$$

where $k_{\Omega_T}(s) = \dot{\Phi}(s)$ is the curvature of $\Omega_T(s)$. After computations

$$C(s) = \left(\frac{-1 + r(s)\dot{\Phi}(s)}{\dot{\Phi}(s)}\sinh\Phi(s), \xi(s) - \xi(s_0) + \frac{1 - r(s)\dot{\Phi}(s)}{\dot{\Phi}(s)}\cosh\Phi(s)\right)$$

and by (4.12) the first coordinate of $C(s)$ never vanishes. Then, the center of curvature of $\Omega_T(s)$ never lies on the z-axis.

Conversely, under the assumptions of the Lemma and the above computations for $C(s)$, one always has $1 - r(s)\dot{\Phi}(s) \neq 0, \forall s$. Choose a starting point (x_0, y_0) and assign the corresponding values of $s = 0$, $\theta = 0$, $r = r_0 = \dfrac{x_0}{\sinh\Phi_0}$. Then $\theta = \theta(s) = \displaystyle\int^s \frac{1 - r(u)\dot{\Phi}(u)}{r(u)}du$ is clearly a strictly monotone function of s. Hence, one may solve for s in terms of θ and substitute into $r = r(s)$ to obtain the polar equation $r = r(\theta)$, which defines a plane curve $\Gamma(\theta)$. If one rolls this curve on the z-axis, it is possible to check that the trace of the origin of its attached polar coordinate system is exactly the given curve Ω_T, which finishes the proof of the lemma.

Now using parametrization (2.2) we obtain the principal curvatures of the surface given by

$$\frac{\ddot{z}}{\dot{x}} = -\dot{\Phi} \text{ and } \frac{\dot{z}}{x} = \frac{-1}{r}.$$

By the definition of the mean curvature H, we get $2Hr = -1 - r\dot{\Phi}$ and easily find that the surface S has constant mean curvature H if and only if the function $r(\theta)$ satisfies the same differential equation (3.3) which, of course, has the same solution (3.4).

The result below follows the same steps of Proposition 3.5.

Proposition 4.2. *The spacelike surfaces of revolution with non-zero constant mean curvature H in $\mathbb{L}^3$, which are obtained by rotating $\Omega_T(\theta)$ along the z-axis, are given by*

$T_1)$ $r(\theta) = \dfrac{\varepsilon d}{1 - \varepsilon\cosh\theta}$, $-\log\varepsilon < \theta < \log(\dfrac{1 + \sqrt{1 - \varepsilon^2}}{\varepsilon}), 0 < \varepsilon < 1, H = \dfrac{\varepsilon^2 - 1}{2\varepsilon d}$;

$T_2)$ $r(\theta) = \dfrac{\varepsilon d}{-1 + \varepsilon\cosh\theta}$, $\theta < \log(\dfrac{1 - \sqrt{1 - \varepsilon^2}}{\varepsilon})$, $0 < \varepsilon < 1$, $H = \dfrac{1 - \varepsilon^2}{2\varepsilon d}$;

$T_3)\quad r(\theta) = \dfrac{\varepsilon d}{-1+\varepsilon\cosh\theta},\ \theta < -\log\varepsilon,\ \varepsilon > 1,\ H = \dfrac{1-\varepsilon^2}{2\varepsilon d};$

$T_4)\quad r(\theta) = \dfrac{\varepsilon d}{1-\varepsilon\sinh\theta},\ -\log\varepsilon < \theta < \log(\dfrac{1+\sqrt{1+\varepsilon^2}}{\varepsilon}),\ \varepsilon > 0,\ H = \dfrac{-1-\varepsilon^2}{2\varepsilon d};$

$T_5)\quad r(\theta) = \dfrac{\varepsilon d}{-1-\varepsilon\sinh\theta},\ \theta < \log(\dfrac{-1+\sqrt{1+\varepsilon^2}}{\varepsilon}),\ \varepsilon > 0,\ H = \dfrac{1+\varepsilon^2}{2\varepsilon d};$

$T_6)\quad r(\theta) = \dfrac{1}{ae^{-\theta}-b},\ \theta < \log(\dfrac{a}{b}),\ \varepsilon > 0,\ \textit{where } a > 0 \textit{ and } b > 0,\quad H = \dfrac{b}{2}.$

As before, it is very important to control the completeness of the surfaces given above. In this direction we can state

Proposition 4.3. *There exists no complete spacelike surface of revolution in $\mathbb{L}^3$ with non-zero constant mean curvature and timelike axis.*

Proof. By Remark 2.3 it is enough to show that the curve $\Omega_T(\theta)$, parametrized by (4.7) is not complete. It remains to show that, in each case, one of the limits of the arc length parameter $s(\theta) = \int^\theta \|\Omega_T'(u)\|du$ (for θ tending to the extremes of the correspondent interval given in Proposition 4.2), is a finite number. Then $\Omega_T(\theta)$ is not complete because its arc length parameter is not onto $\mathbb{R}$.

From (4.8), $\|\Omega_T'(\theta)\| = r\left|1+\dfrac{d\Phi}{d\theta}\right|$. By differentiating one of the expressions in (4.2) we get $\dfrac{d\Phi}{d\theta} = \dfrac{r'^2-rr''}{r'^2-r^2}$. Hence

$$\|\Omega_T'(\theta)\| = \frac{r|2r'^2-r^2-rr''|}{r'^2-r^2}. \tag{4.13}$$

We will compute $\|\Omega_T'(\theta)\|$ and $s(\theta)$ for each $r(\theta)$ given in Proposition 4.2:

(a) For the surfaces T_1, T_2 and T_3 we observe that $r(\theta) = \pm\dfrac{\varepsilon d}{1-\varepsilon\cosh\theta}$. Then, for these three surfaces, we have

$$r'^2 - r^2 = \frac{\varepsilon^2 d^2(-1-\varepsilon^2+2\varepsilon\cosh\theta)}{(1-\varepsilon\cosh\theta)^4} > 0 \text{ and}$$

$$\|\Omega_T'(\theta)\| = \frac{\varepsilon d}{-1-\varepsilon^2+2\varepsilon\cosh\theta} = \frac{\varepsilon d e^\theta}{(e^\theta-\varepsilon)(\varepsilon e^\theta - 1)}.$$

By integrating we obtain that $s(\theta) = \dfrac{\varepsilon d}{1-\varepsilon^2}\log\left(\dfrac{\varepsilon e^\theta - 1}{e^\theta - \varepsilon}\right)$.

For T_1, we have $-\log\varepsilon < \theta < \log\left(\dfrac{1+\sqrt{1-\varepsilon^2}}{\varepsilon}\right)$, $0 < \varepsilon < 1$, and we get

$$\lim_{\theta\to\log(\frac{1+\sqrt{1-\varepsilon^2}}{\varepsilon})} s(\theta) = \frac{\varepsilon d}{1-\varepsilon^2}\log\left(\frac{\varepsilon}{1+\sqrt{1-\varepsilon^2}}\right).$$

For T_2, $\theta < \log\left(\dfrac{1-\sqrt{1-\varepsilon^2}}{\varepsilon}\right)$, $0<\varepsilon<1$. We obtain

$$\lim_{\theta\to\log(\frac{1-\sqrt{1-\varepsilon^2}}{\varepsilon})} s(\theta) = \frac{\varepsilon d}{1-\varepsilon^2}\log\left(\frac{\varepsilon}{1-\sqrt{1-\varepsilon^2}}\right).$$

For T_3, $\theta < -\log\varepsilon, \varepsilon > 1$, and we obtain

$$\lim_{\theta\to-\infty} s(\theta) = \frac{d\varepsilon\log\varepsilon}{\varepsilon^2-1}.$$

(b) For T_4, we get $r'^2 - r^2 = \dfrac{\varepsilon^2 d^2(-1+\varepsilon^2+2\varepsilon\sinh\theta)}{(1-\varepsilon\sinh\theta)^4} > 0$ and

$$\|\Omega_T'(\theta)\| = \frac{\varepsilon d}{-1+\varepsilon^2+2\varepsilon\sinh\theta} = \frac{\varepsilon d e^\theta}{(e^\theta+\varepsilon)(\varepsilon e^\theta-1)}.$$

In this case $s(\theta) = \dfrac{\varepsilon d}{1+\varepsilon^2}\log\left(\dfrac{\varepsilon e^\theta - 1}{e^\theta+\varepsilon}\right)$, $-\log\varepsilon < \theta < \log\left(\dfrac{1+\sqrt{1+\varepsilon^2}}{\varepsilon}\right)$, $\varepsilon > 0$, and we obtain

$$\lim_{\theta\to\log(\frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon})} s(\theta) = \frac{\varepsilon d}{1+\varepsilon^2}\log\left(\frac{\varepsilon}{1+\sqrt{1+\varepsilon^2}}\right).$$

(c) For T_5, we have $r'^2 - r^2 = \dfrac{\varepsilon^2 d^2(-1+\varepsilon^2-2\varepsilon\sinh\theta)}{(1+\varepsilon\sinh\theta)^4} > 0$ and

$$\|\Omega_T'(\theta)\| = \frac{\varepsilon d}{-1+\varepsilon^2-2\varepsilon\sinh\theta} = \frac{\varepsilon d e^\theta}{(\varepsilon-e^\theta)(\varepsilon e^\theta+1)}.$$

It follows that $s(\theta) = \dfrac{\varepsilon d}{1+\varepsilon^2}\log\left(\dfrac{\varepsilon e^\theta+1}{\varepsilon-e^\theta}\right)$, $\theta < \log\left(\dfrac{-1+\sqrt{1+\varepsilon^2}}{\varepsilon}\right)$, $\varepsilon > 0$, and we obtain

$$\lim_{\theta\to-\infty} s(\theta) = -\frac{d\varepsilon\log\varepsilon}{1+\varepsilon^2}.$$

(d) Similarly, for T_6 we have $r'^2 - r^2 = \dfrac{be^{-\theta}(2a-be^\theta)}{(ae^{-\theta}-b)^4} > 0$ and $\|\Omega_T'(\theta)\| = \dfrac{e^\theta}{2a-be^\theta}$.

By integrating we obtain $s(\theta) = \dfrac{-1}{b}\log(2a-be^\theta)$, $\theta < \log\left(\dfrac{a}{b}\right)$, and

$$\lim_{\theta\to-\infty} s(\theta) = \frac{-1}{b}\log(2a).$$

Since all the limits calculated above are finite, we conclude that there exists no complete spacelike surface of revolution in $\mathbb{L}^3$ with non-zero constant mean curvature and timelike axis.

5. Surfaces of revolution with lightlike axis

A surface of revolution with lightlike axis $x = 0$, $y = z$ can be parametrized by (2.3). The profile curve $\Omega_L(s) = (0, y(s), z(s))$, parametrized by arc length, is spacelike, that is, $\dot{y}^2(s) - \dot{z}^2(s) = 1$. The principal curvatures are given by $\dfrac{\ddot{y}(s)}{z(s)}$ and $\dfrac{-(\dot{y}(s) - \dot{z}(s))}{y(s) - z(s)}$. For convenience, Hano and Nomizu make a change of coordinates in [7], using the null coordinates (u, v) and then Ω_L is written as $u = \dfrac{y+z}{\sqrt{2}}$, $v = \dfrac{-y+z}{\sqrt{2}}$, with $v > 0$. Therefore the tangent vector is of the form $(\dot{u}, \dot{v}) = (\frac{e^{\Phi}}{\sqrt{2}}, \frac{-e^{-\Phi}}{\sqrt{2}})$, by choosing an arc length parameter such that $\dot{v} < 0$. In those coordinates the principal curvatures of the surface are expressed as $\dfrac{-\ddot{v}}{\dot{v}}$ and $\dfrac{-\dot{v}}{v}$.

The surface has constant mean curvature H if and only if

$$2Hv\dot{v} = -v\ddot{v} - \dot{v}^2, \;\; v > 0, \dot{v} < 0.$$

Solving this equation and using the fact that $-2\dot{u}\dot{v} = 1$, we get

$$\frac{du}{dv} = \frac{-v^2}{2(c - Hv^2)^2}, \tag{5.1}$$

where c is a constant.

Integrating (5.1) and taking $H \neq 0$, we get the possibilities for $u = u(v)$. In this case, the profile curve takes the form

$$\Omega_L(v) = \frac{1}{\sqrt{2}}(0, u(v) - v, u(v) + v), \tag{5.2}$$

and the surface can be parametrized by

$$X_L(v,t) = \sqrt{2}\,\left(tv, \left(\frac{-1+t^2}{2}\right)v + \frac{u(v)}{2}, \left(\frac{1+t^2}{2}\right)v + \frac{u(v)}{2}\right). \tag{5.3}$$

Proposition 5.1. *The spacelike surfaces of revolution with non-zero constant mean curvature H in $\mathbb{L}^3$, which are obtained by rotating $\Omega_L(v)$ along the lightlike axis $x = 0$, $y = z$, are given by* (5.3) *where $u = u(v)$ takes one of the forms*

(L_1) If $H \neq 0$ and $\frac{C}{H} = a^2 > 0$, then $u(v) = \frac{1}{4H^2}\left(\frac{v}{v^2-a^2} - \frac{1}{2a}\log\frac{v-a}{v+a} + b\right)$, *where b is an arbitrary constant and $v \in (a, \infty), a > 0$.*

(L_2) If $H \neq 0$ and $\frac{C}{H} = -a^2 < 0$, then $u(v) = \frac{1}{4H^2}\left(\frac{-1}{a}arctan\frac{v}{a} + \frac{v}{v^2+a^2} + b\right)$, *where b is an arbitrary constant and $v \in (0, \infty)$.*

(L_3) If $H \neq 0$ and $c = 0$, then $u(v) = \frac{1}{2H^2}\left(\frac{1}{v} + b\right)$, *where b is an arbitrary constant and $v \in (0, \infty)$.*

The completeness of these surfaces is studied below.

Proposition 5.2. *The only complete spacelike surfaces of revolution in the list above are L_1 and L_3 .*

Proof. By (5.2), $\Omega_L'(v) = (0, y'(v), z'(v)) = \frac{1}{\sqrt{2}}(0, u'(v)-1, u'(v)+1)$. Since $y'(v)^2 - z'(v)^2 = \dfrac{v^2}{(c-Hv^2)^2}$, the arc-length parameter for $\Omega_L(v)$ is given by

$$s(v) = \int^v ds(t)dt = \int^v \sqrt{y'(t)^2 - z'(t)^2}dt = \int^v \frac{t}{|c-Ht^2|}dt, \tag{5.4}$$

where c is the constant that appeared in (5.1).

By replacing each value of H in (5.4) we can compute the limits of $s(v)$ for v tending to the extremes of the correspondent intervals.

For the surface labeled as L_1, $s(v) = \dfrac{1}{|H|}\log\sqrt{v^2-a^2}$, $v \in (a,\infty)$. Hence $\lim\limits_{v\to a^+} s(v) = -\infty$ and $\lim\limits_{v\to\infty} s(v) = +\infty$, which implies that the surface L_1 is complete.

For the surface labeled as L_2, $s(v) = \dfrac{1}{|H|}\log\sqrt{v^2+a^2}$, $v \in (0,\infty)$. Now $\lim\limits_{v\to 0^+} s(v) = \dfrac{1}{|H|}\log a$ and $\lim\limits_{v\to\infty} s(v) = \infty$, which implies that the surface L_2 is not complete.

Finally, for L_3, $s(v) = \dfrac{1}{|H|}\log v$, $v \in (0,\infty)$. Then $\lim\limits_{v\to 0^+} s(v) = -\infty$, $\lim\limits_{v\to\infty} s(v) = +\infty$ and the surface L_3 is also complete.

Now we are going to show that, for lightlike surfaces, the answer for the conjecture, as mentioned at the Introduction, is also affirmative.

Theorem 5.3. *The Gauss map image of the complete spacelike surfaces of revolution in $\mathbb{L}^3$ with non-zero constant mean curvature and lightlike axis (surfaces labeled as L_1 and L_3) contains a maximal geodesic of $\mathbb{H}^2$. Actually, their Gauss map image is $\mathbb{H}^2$.*

Proof. Since the orientation of these surfaces, parametrized by formula (5.3), was chosen such that N is a future directed timelike vector, an easy computation shows that

$$N(v,t) = \frac{1}{2\sqrt{|u'|}}(2t, -1+t^2-u', 1+t^2-u') \subset \mathbb{H}^2, \tag{5.5}$$

where $u' = u'(v)$.

In order to show that the image of N, for the surfaces L_1 and L_3, is the set $\mathbb{H}^2$, we are going to prove that the function $N : I_\varepsilon \times \mathbb{R} \to \mathbb{H}^2$ is a bijection.

In fact, given $(x,y,z) \in \mathbb{H}^2$, there exists a unique $t \in \mathbb{R}$ such that $t = \dfrac{x}{z-y}$ and a unique $v \in I_\varepsilon$, given by the equation

$$\frac{1}{\sqrt{|u'(v)|}} = z - y. \tag{5.6}$$

More clearly, we can find, exactly, the value of v in each case, as follows:

a) For the surface labeled as L_1, $I_\varepsilon = (a,\infty)$, $a > 0$ and $u'(v) = \dfrac{-v^2}{2H^2(a^2-v^2)^2}$. Then, by replacing this expression of $u'(v)$ in (5.6) we get

$$\frac{\sqrt{2}|H|(v^2-a^2)}{v} = z - y. \tag{5.7}$$

After solving equation (5.7) for v, we get $v = \dfrac{z-y+\sqrt{(z-y)^2+8a^2H^2}}{2\sqrt{2}|H|}$.

b) For the surface labeled as L_3, $I_\varepsilon = (0,\infty)$ and $u'(v) = \dfrac{-1}{2H^2v^2}$. Now, solving (5.6) for v, we obtain $v = \dfrac{z-y}{\sqrt{2}|H|}$.

For the values of v, obtained in a) and b), and the value of t written above, we have that $N(v,t) = (x,y,z)$ and hence $ImN = \mathbb{H}^2$.

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