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On the Subgroups of the Picard Group

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Abstract. In this paper, the normal closure of the modular group $PSL(2,\mathbb{Z})$ in the Picard group $PSL(2,\mathbb{Z}[i])$ is given. Also, it is given some results about all power subgroups $\mathbf{P}^{6n}$ of the Picard group.

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1. Introduction

It is known that if X is a nonempty subset of a group G, the normal closure of X in G is the intersection of all the normal subgroups of G which contain X. Clearly this is a normal subgroup. So, the notion of "normal closure" is important to find normal subgroups of a given group. Using this notion, in [2] and [4], it was determined some properties of the normal subgroups of the Picard group $\mathbf{P}$ and given a complete classification of the normal subgroups for indices less than 60. The Picard group $\mathbf{P}$ is $PSL(2,\mathbb{Z}[i])$, the group of linear fractional transformations with Gaussian integer coefficients. $\mathbf{P}$ is a free product with amalgamation of the following form, [2]:

$$\mathbf{P} \cong G_1 *_{\mathbf{M}} G_2$$

with $G_1 \cong S_3 *_{Z_3} A_4$, $G_2 \cong S_3 *_{Z_2} D_2$ (S_3 is the symmetric group on three symbols, A_4 is the alternating group on four symbols and D_2 is the Klein 4-group) and $\mathbf{M}$ is the modular group $PSL(2,\mathbb{Z})$. Modular group play a very important role to determine subgroups of the Picard group because of this decomposition. Modular group is a Fuchsian subgroup of $\mathbf{P}$ and is not normal. In [9], the normaliser of $\mathbf{M}$ in $\mathbf{P}$ that is a maximal subgroup of $\mathbf{P}$ in which $\mathbf{M}$ is normal was obtained. Here we determine the group structure of the normal closure of $\mathbf{M}$ in $\mathbf{P}$. Furthermore we obtain some results about the power subgroups $\mathbf{P}^{6n}$ of the Picard group.

2. The normal closure of the modular group

It is known that a presentation for **P** is given by

$$\mathbf{P} = \left\langle x,u,y,r;\ x^3 = u^2 = y^3 = r^2 = (xu)^2 = (xy)^2 = (ry)^2 = (ru)^2 = 1 \right\rangle \tag{2.1}$$

where

$$x(z) = \frac{i}{iz+1}, u(z) = -\frac{1}{z}, y(z) = \frac{z+1}{-z}, r(z) = \frac{i}{iz},$$

[1]. Also a presentation of $\mathbf{M}$ given by $\mathbf{M} \cong \langle u,y;\ u^2 = y^3 = 1\rangle$. Let $N(g_1, g_2, \dots, g_k)$ denote the normal closure of the subgroup generated by $\{g_1, g_2, \dots, g_k\}$. $\mathbf{P}/N(g_1, g_2, \dots, g_k)$ is the group obtained by adding the relations $g_1 = 1, g_2 = 1, \dots, g_k = 1$ to the relations of $\mathbf{P}$, [7]. Now we can determine the $N(u,y)$, the normal closure of $\mathbf{M}$ in $\mathbf{P}$. To do this we use Reidemeister-Schreier method, (see [7] and [3] for more details).

Theorem 2.1. *The normal closure of* $\mathbf{M}$ *in* $\mathbf{P}$ *is*

$$N(u,y) = M_1 *_{\mathbf{M}} M_2$$

where $M_1 \cong M_2 \cong S_3 *_{\mathbb{Z}_3} A_4$. *Further the index of* $N(u,y)$ *in* $\mathbf{P}$ *is two.*

Proof. The proof is straightforward computations. We adjoin the identical relations $u = 1, y = 1$ to the standard presentation (2.1) for $\mathbf{P}$. This gives us a presentation for $\mathbf{P}/N(u,y)$ of which order gives us the index. We have

$$\mathbf{P}/N(u,y) = \left\langle x,u,y,r;\ x^3 = u^2 = y^3 = r^2 = (xu)^2 = (xy)^2 = (ry)^2 = (ru)^2 = 1,\ u = y = 1 \right\rangle.$$

Since $x^3 = x^2 = 1$, this implies that $x = 1$. Therefore

$$\mathbf{P}/N(u,y) = \left\langle r;\ r^2 = 1 \right\rangle \cong \mathbb{Z}_2.$$

Thus $|\mathbf{P} : N(u,y)| = 2$. Let $\{1, r\}$ be a Schreier transversal for $N(u,y)$. Applying the Reidemeister-Schreier process we get all the possible products as follows:

$$\begin{aligned}
S_{1x} &= x.1 = x,\ S_{rx} = rxr \\
S_{1u} &= u.1 = u,\ S_{ru} = rur = u \\
S_{1y} &= y.1 = y,\ S_{ry} = ryr = y^{-1} \\
S_{1r} &= r.r = 1,\ S_{rr} = r^2.1 = 1.
\end{aligned}$$

We get $x_1 = x$, $x_2 = u$, $x_3 = y$ and $x_4 = rxr$ as generators for $N(u,y)$. Using the Reidemeister rewriting process we get the relations

$$\begin{aligned}
\tau(xxx) &= S_{1x}.S_{1x}.S_{1x} = x^3, \\
\tau(uu) &= S_{1u}.S_{1u} = u^2, \\
\tau(yyy) &= S_{1y}.S_{1y}.S_{1y} = y^3, \\
\tau(xuxu) &= S_{1x}.S_{1u}.S_{1x}.S_{1u} = xuxu = (xu)^2, \\
\tau(xyxy) &= S_{1x}.S_{1y}.S_{1x}.S_{1y} = xyxy = (xy)^2, \\
\tau(rxxxr) &= S_{1r}.S_{rx}.S_{rx}.S_{rx}.S_{rr} = 1.rxr.rxr.rxr.1 = (rxr)^3, \\
\tau(rxuxur) &= S_{1r}.S_{rx}.S_{ru}.S_{rx}.S_{ru}.S_{rr} = 1.rxr.u.rxr.u.1 = (rxru)^2, \\
\tau(rxyxyr) &= S_{1r}.S_{rx}.S_{ry}.S_{rx}.S_{ry}.S_{rr} = 1.rxr.y^{-1}.rxr.y^{-1}.1 = (rxry^{-1})^2.
\end{aligned}$$

Hence we obtain

$$\begin{aligned} N(u,y) &= \big\langle x,u,y,rxr; x^3 = u^2 = y^3 = (xu)^2 = (xy)^2 = (rxr)^3 \\ &\quad = (rxru)^2 = (rxry^{-1})^2 = 1 \big\rangle . \end{aligned}$$

Now let

$$M_1 = \big\langle x,u,y; x^3 = u^2 = y^3 = (xu)^2 = (xy)^2 = 1 \big\rangle$$

and

$$M_2 = \big\langle rxr,u,y; (rxr)^3 = u^2 = y^3 = (rxru)^2 = (rxry^{-1})^2 = 1 \big\rangle .$$

Then $N(u,y)$ is generated by M_1 and M_2 with the identifications $u = u$, $y = y$. In M_1, the subgroup generated by u, y is their free product $\mathbb{Z}_2 * \mathbb{Z}_3$ which is the modular group, while this is also true in M_2. Therefore $N(u,y)$ is a free product with the amalgamated subgroup $\mathbf{M}$. In M_1, let

$$\begin{aligned} M_{11} &= \big\langle x,u; x^3 = u^2 = (xu)^2 = 1 \big\rangle , \\ M_{12} &= \big\langle x,y; x^3 = y^3 = (xy)^2 = 1 \big\rangle . \end{aligned}$$

So $M_1 \cong M_{11} * M_{12}$ with the identification $x = x$. This induces a subgroup isomorphism, so $M_1 = S_3 *_{\mathbb{Z}_3} A_4$. Again similarly we get

$$\begin{aligned} M_2 &= \big\langle rxr,u; (rxr)^3 = u^2 = (rxru)^2 = 1 \big\rangle * \big\langle rxr,y; (rxr)^3 = y^3 = (rxry^{-1})^2 = 1 \big\rangle \\ &= S_3 *_{\mathbb{Z}_3} A_4 . \end{aligned}$$

Therefore the normal closure of the modular group in the Picard group is $(S_3 *_{\mathbb{Z}_3} A_4) *_{\mathbf{M}} (S_3 *_{\mathbb{Z}_3} A_4)$. $\square$

In [4], it was proved that, there are exactly three normal subgroups of index 2 in $\mathbf{P}$. So $N(u,y)$ is one of these normal subgroups of index 2 in $\mathbf{P}$. Furthermore $N(u,y)$ is not Fuchsian since $xuyrxr$ is a loxodromic element.

3. Power subgroups

Now we obtain some results about the structure of the power subgroups $\mathbf{P}^{6n}$ of the Picard group. The power subgroups $\mathbf{P}^n$ are the normal subgroups of $\mathbf{P}$ generated by nth powers of elements of $\mathbf{P}$ where n is a positive integer. From the definition one can easily deduce that

$$\mathbf{P}^m \supset \mathbf{P}^{mk} \tag{3.1}$$

and that

$$(\mathbf{P}^m)^k \supset \mathbf{P}^{mk}. \tag{3.2}$$

In the modular group case, it is known that $\mathbf{M}^n = \mathbf{M}$, $\mathbf{M}^2$ or $\mathbf{M}^3$ if $6 \nmid n$ and the exact structure of $\mathbf{M}^{6k}$ is unknown if $k > 1$. $\mathbf{M}^6$ is free of rank 37, $\mathbf{M}^6 \supset \mathbf{M}^{6k}$ and the groups $\mathbf{M}^{6k}$ are free groups, [8]. Similar results hold for $\mathbf{P}$. From [4], we have

1) $\mathbf{P}^2 = \mathbf{P}'$, the commutator subgroup of $\mathbf{P}$,
2) $\mathbf{P}^3 = \mathbf{P}$ and $\mathbf{P}^n = \mathbf{P}$ if $2 \nmid n$,
3) $\mathbf{P}^n = \mathbf{P}^2$ if $2 \mid n$ but $6 \nmid n$,
4) $(\mathbf{P}')^3 = \mathbf{P}''$.

From (3.2), we get

$$\mathbf{P}'' \supset \mathbf{P}^6 \tag{3.3}$$

since $\mathbf{P}'' = (\mathbf{P}')^3 = (\mathbf{P}^2)^3$. Also from (3.1), we get

$$\mathbf{P}^6 \supset \mathbf{P}^{6n} \text{ so } \mathbf{P}'' \supset \mathbf{P}^{6n}. \tag{3.4}$$

Therefore we get the following corollary:

Corollary 3.1. *The power subgroups $\mathbf{P}^{6n}$ of the Picard group are the subgroups of the second commutator subgroup $\mathbf{P}''$.*

In [4], it was proved that $\mathbf{P}'' = K_1 *_K K_2$ where $K_1 \simeq K_2 = D_2 * D_2$ and $K = \mathbb{Z} * \mathbb{Z}$, $|\mathbf{P} : \mathbf{P}''| = 12$. Also $\mathbf{P}''$ is the only subgroup of index 12 and $\mathbf{P}'' = N(ltu)$ where l, t and u are the generators in the another presentation of $\mathbf{P}$ given in [4]. Since $\mathbf{P}''$ is a free product with amalgamation, $\mathbf{P}^{6n}$ is an HNN group. This follows from the Karrass-Solitar subgroup theorems, [6]. We then have the following result.

Theorem 3.2. *The groups $\mathbf{P}^{6n}$ are HNN groups.*

Now we are going to determine the structure of the quotient groups $\mathbf{P}/\mathbf{P}^{6n}$. Let us consider the following presentation of $\mathbf{P}$ given in [1]:

$$\mathbf{P} = \left\langle a, w, b; b{=}aw^2a^{-1}w^{-2}aw^2, (a^2waw^{-1})^2{=}(awaw^{-1})^3{=}(wb)^2{=}(ab)^2{=}b^2{=}1 \right\rangle$$

where $a = xr$ and $w = ury$. If we write $awaw^{-1} = v$, we have

$$\mathbf{P} = \left\langle a, w, b, v; (av)^2 = v^3 = (wb)^2 = (ab)^2 = b^2 = 1 \right\rangle.$$

Firstly, to find the factor group $\mathbf{P}/\mathbf{P}^6$, we adjoin the identical relation $X^6 = 1$ to this presentation. Then we have

$$\mathbf{P}/\mathbf{P}^6 = \left\langle a, w, b, v; (av)^2 = v^3 = (wb)^2 = (ab)^2 = b^2 = 1, a^6 = w^6 = 1 \right\rangle.$$

Hence we get

$$\begin{aligned}
\mathbf{P}/\mathbf{P}^6 &= \left\langle a, b, v; a^6 = v^3 = b^2 = (av)^2 = (ab)^2 = 1 \right\rangle * \left\langle b, w; w^6 = b^2 = (wb)^2 = 1 \right\rangle \\
&= \left(\left\langle a, b; a^6 = b^2 = (ab)^2 = 1 \right\rangle * \left\langle a, v; a^6 = v^3 = (av)^2 = 1 \right\rangle \right) *_{\mathbb{Z}_2} D_6 \\
&= (D_6 *_{\mathbb{Z}_6} D(6,3,2)) *_{\mathbb{Z}_2} D_6.
\end{aligned}$$

and similarly

$$\begin{aligned}
\mathbf{P}/\mathbf{P}^{6n} &= \langle a, w, b, v; (av)^2 = v^3 = (wb)^2 = (ab)^2 = b^2 = 1, a^{6n} = w^{6n} = 1\rangle \\
&= \langle a, b, v; a^{6n} = v^3 = b^2 = (av)^2 = (ab)^2 = 1\rangle * \langle b, w; w^{6n} = b^2 = (wb)^2 = 1\rangle \\
&= (D_{6n} *_{\mathbb{Z}_{6n}} D(6n, 3, 2)) *_{\mathbb{Z}_2} D_{6n}
\end{aligned}$$

where $D(6n, 3, 2)$ is the von Dyck group. It is known that the von Dyck group $D(l, m, n)$ is finite if and only if $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} > 1$, [5]. In our case, we conclude that the von Dyck groups $D(6n, 3, 2)$ are of infinite order since $\frac{1}{6n} + \frac{1}{3} + \frac{1}{2} = \frac{5n+1}{6n} \leq 1$. Therefore the power subgroups $\mathbf{P}^{6n}$ are of infinite index in the Picard group.

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