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On Extrinsic Symmetric Cauchy-Riemann Manifolds*

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Abstract. In the present paper we propose to extend the notion of extrinsic symmetric space introduced by D. Ferus [5] to the recently defined symmetric Cauchy-Riemann Manifolds [8]. We show that this is a meaningful extension, by presenting a number of examples that exist in "nature".

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1. Introduction

Cauchy-Riemann manifolds have been a popular subject of study for many years (see for instance [7], [9], [10], [1], [2] and many references in them). Recently (see [8]) the notion of symmetric almost Hermitian CR-manifold (SCR-space for short) has been introduced as a very natural generalization of symmetric spaces (Riemannian or Hermitian). That very interesting paper, which is the motivation of the present one, contains a deep study of these spaces with many examples and a general method of construction.

Roughly, a SCR-space is an almost Hermitian CR-manifold that, at each point $y \in M$, has a CR-diffeomorphism $s_y : M \to M$ (called a symmetry at the point $y \in M$) such that y is a (not necessarily isolated) fixed point of s_y and its derivative $s_{y*}|_y$ restricted to a particular subspace of $T_y(M)$ coincides with $(-Id)$. See Section 2 for the complete definitions.

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Here we propose to extend the classical notion of extrinsic symmetric space, given by D. Ferus in [5], to the recently defined *SCR*-spaces (see Definition 3 below).

The main objective of the present paper is to show that this extended notion is of interest, basically, because many non trivial examples exist "in nature". In fact some of the examples of extrinsic symmetric *CR*-spaces that we may present, for instance odd dimensional spheres S^{2n-1} in C^n or the connected components of the set of tripotents in a Hermitian Positive Jordan Triple System (the former being a particular case of the last) are given as examples of *SCR*-spaces in [8].

Of course the extrinsic symmetric spaces of [5] are included in the larger family, but many new examples arise that, in our opinion, give non-trivial meaning to the extended notion.

The paper is organized as follows. In the next section we present, for the benefit of the reader, the basic material needed for our work. It is divided into three subsections. The first two contain the definitions and results from [8] which are required here. The third one is included to present a few known facts concerning Hermitian Positive Jordan Triple Systems, the basic recent reference here is [4, Part V].

It is interesting to notice that many of the examples presented in [8], particularly those which are relevant for us here, are constructed in connection with Hermitian Jordan Triple Systems. In Section 3 we show that some of the examples in [8] are in fact, extrinsic symmetric.

In Section 4 we present two new families of examples of extrinsic symmetric CR-spaces. This section starts with a subsection describing the method we use to construct them inside Hermitian positive simple Jordan Triple Systems. The possibility of a successful construction seems to depend on the nature of the Hermitian Jordan Triple System under consideration. It is possible to use the method to construct more complicated examples but it does not seem convenient to include them here.

Finally Subsection 4.4 contains a description of all examples that can be constructed by this method in $EIII$ and $MIII\,(2,3)$.

The Appendix contains some notation concerning compact Hermitian symmetric spaces and their canonical embedding as well as some complementary calculations. It goes without saying that, forgetting the embedding, these spaces are also examples of *SCR*-spaces.

2. Necessary facts

In this section, for the convenience of the reader, we give the definitions and facts from [8] and [4] which are needed in the rest of the paper. We have divided it into three subsections.

2.1. *SCR*-spaces

Let M be a connected finite dimensional real manifold (C^∞ or analytic), at each point $x \in M$ the tangent space will be denoted by $T_x\,(M)$. An *almost Cauchy-Riemann structure* or an *almost CR-structure* is the assignation, to every $x \in M$, of a linear subspace $H_x \subset T_x\,(M)$ and a complex structure J_x on H_x in such a way that the subspace H_x and the complex structure J_x depend differentially on x. This dependence means that every point $x \in M$ has a neighborhood $U \subset M$ and for each $y \in U$ a linear endomorphism J_y of $T_y\,(M)$ such that $\left(-J_y^2\right)$ is a projection from $T_y(M)$ onto H_y with $J_y^2 X = -X$ for every $X \in H_y$ and J_y depending smoothly on $y \in U$. Thus all the subspaces H_x have the same dimension. A connected differentiable manifold with an almost CR-structure is called an *almost CR-manifold*.

A smooth map $\varphi : M \to N$ between two almost CR-manifolds is called a *CR-map* if for every $y \in M$ the derivative $\varphi_*|_y : T_y(M) \to T_{\varphi(y)}(N)$ maps the complex subspace $H_y(M)$ complex linearly into $H_{\varphi(y)}(N)$. Then it makes sense to consider *CR-diffeomorphisms* between almost CR-manifolds.

Let $\mathfrak{D} = \mathfrak{D}(M)$ be the Lie algebra of smooth vector fields in M and consider the subspace

$$\mathfrak{H} = \mathfrak{H}(M) = \{X \in \mathfrak{D} : X_y \in H_y, \ \forall y \in M\}.$$

We define inductively

$$\begin{aligned} \mathfrak{H}^1 &= \mathfrak{H}, \qquad \mathfrak{H}^k = \mathfrak{H}^{k-1} + \left[\mathfrak{H}, \mathfrak{H}^{k-1}\right] \quad (k \geqq 1), \\ \mathfrak{H}^k &= 0 \quad (k \leqq 0). \end{aligned}$$

Then $[\mathfrak{H}^r, \mathfrak{H}^s] \subset \mathfrak{H}^{r+s}$and $\mathfrak{H}^\infty = \bigcup_k \mathfrak{H}^k$ is the Lie subalgebra of $\mathfrak{D}$ generated by $\mathfrak{H}$. We also denote by $\mathfrak{H}_y^k$ the subspaces of $T_y(M)$

$$\mathfrak{H}_y^k = \left\{X \in T_y(M) : X = U_y, \text{ for some } U \in \mathfrak{H}^k\right\}.$$

Let us assume now that we have on M a Riemannian metric and let $\langle *, * \rangle_y$ denote the corresponding scalar product in $T_y(M)$. We shall say that M is an *almost Hermitian CR-manifold* if for every $y \in M$ and every $X, Z \in H_y$

$$\langle J_y X, J_y Z \rangle_y = \langle X, Z \rangle_y$$

Let us consider now, in $T_y(M)$, the subspace $H_y^1 = H_y$ and define H_y^k and H_y^{-1}, respectively, as the orthogonal complement of $\mathfrak{H}_y^{k-1}$ in $\mathfrak{H}^k$ and that of $\mathfrak{H}_y^\infty$ in $T_y(M)$ with respect to the scalar product $\langle *, * \rangle_y$. Since $H_y \subset \mathfrak{H}_y^\infty$, the orthogonal direct sum $H_y^{-1} \oplus H_y$ is a real subspace of $T_y(M)$ and one has

$$T_y(M) = \bigoplus_{k \geqq -1} H_y^k.$$

Let M be an almost Hermitian CR-manifold and let $\sigma : M \to M$ be a CR-diffeomorphism. The map σ is called a *symmetry* at the point $y \in M$ if y is a (not necessarily isolated) fixed point of σ and its derivative $\sigma_*|_y$ restricted to the subspace $H_y^{-1} \oplus H_y \subset T_y(M)$ coincides with $(-Id)$.

A connected almost Hermitian CR-manifold M is called a *symmetric almost Hermitian CR-manifold* (*SCR-space* for short) if there is a symmetry, denoted by s_y, at each point $y \in M$. It can be proved that there is at most one symmetry ([8, p. 152, 3.3]), at each point of M.

One important observation must be made ([8, p. 153, 3.4]). For every symmetry σ at the point $y \in M$ the derivative $\sigma_*|_y$ of σ at y satisfies

$$\sigma_*|_y X = (-1)^k X$$

for every $X \in H_y^k$ and every $k \geqq -1$. Then if we define

$$T_y^+(M) = \bigoplus_{k\ even} H_y^k,$$
$$T_y^-(M) = \bigoplus_{k\ odd} H_y^k,$$

it is clear that $\sigma_*|_y$ is the $(\pm Id)$ on $T_y^{\pm}(M)$. We also have

Proposition 1. [8, p. 153, 3.6] *Let M be an SCR-space and $I(M)$ the Lie group of all isometric CR-diffeomorphisms of M. Let G be the closed subgroup of $I(M)$ generated by all symmetries s_y, $y \in M$. Let $o \in M$ denote a fixed base point in M and let $K = \{g \in G : g(o) = o\}$. Then G is a Lie group acting transitively and properly on M. The identity component G^o of G has index ≤ 2 in G and coincides with the closed subgroup of $I(M)$ generated by the transvections $s_y \circ s_z$ with $y, z \in M$. The isotropy subgroup K is compact and M can be canonically identified with the homogeneous manifold G/K via $g(o) \mapsto gK$. Furthermore, M is compact if and only if G is a compact Lie group.* □

Remark 1. This implies in particular that the *SCR*-space M has a unique structure of real-analytic *CR*-manifold and the maping $y \mapsto s_y$ from M to G is real-analytic. Also the dimension of the subspace $H_y^k \subset T_y(M)$ does not depend on y for every k.

2.2. The construction principle

Kaup and Zaitsev ([8, Sec. 6]) give a way to construct every *SCR*-space by Lie theoretical methods. We need to describe this method here, because we intend to use it to construct some particular examples. Nevertheless, we will try to present the essential aspects restricted to the case of our interest. The general construction can be found in [8, Sec. 6].

Let K be a compact connected Lie group and σ an inner involutive automorphism of K. There is an element $s \in K$ such that $\sigma(g) = sgs$ for $g \in K$. Let $\mathfrak{k}$ denote the Lie algebra of K and $\tau = Ad(s)$ the induced automorphism of $\mathfrak{k}$. Let $\mathfrak{k}_x = F(\tau, \mathfrak{k})$ (the fixed point set of τ in $\mathfrak{k}$) and let $\mathfrak{m}$ denote the orthogonal complement of $\mathfrak{k}_x$ in $\mathfrak{k}$ with respect to the Killing form in $\mathfrak{k}$. Then $\mathfrak{m} = F(-\tau, \mathfrak{k})$. The subgroup $F(\sigma, K)$ is closed in K hence compact. Let us choose an open subgroup $K_x \subset F(\sigma, K)$. It is clear that the Lie algebra of K_x is $\mathfrak{k}_x$. Let $M = K/K_x$ and denote by $x = [K_x] \in M$. Let $T_x(M)$ denote the tangent space to M at x. We may identify naturally $T_x(M)$ with $\mathfrak{m}$. The space M has a unique real analytic structure [6, p. 123, 4.2].

We may take on $\mathfrak{k}$ an $Ad(K_x)$-invariant inner product $\langle *, * \rangle$ (for instance the opposite of the Killing form) and a linear subspace $\mathfrak{h} \subset \mathfrak{m}$ together with a complex structure J on $\mathfrak{h}$ satisfying the following properties:

(i) $\langle JZ, JZ \rangle = \langle Z, Z \rangle\ \forall Z \in \mathfrak{h}$.

(ii) K_x contains the element s.

(iii) For every $g \in K_x$, $Ad(g)\mathfrak{h} \subset \mathfrak{h}$ and $Ad(g)JZ = JAd(g)Z$ for $Z \in \mathfrak{h}$.

We may extend the inner product $\langle *, * \rangle$ restricted to $\mathfrak{m} = T_x(M)$ to a K-invariant Riemannian metric on M. The following proposition from [8, p. 165, 6.2] is what we need to construct our examples in Section 4.

Proposition 2. *Let $\mathfrak{b}$ be the Lie subalgebra of $\mathfrak{k}$ generated by the subspace $\mathfrak{h}$ then M is a minimal symmetric CR-manifold with symmetry $s_x := s$ at x if and only if $\mathfrak{k} = \mathfrak{k}_x + \mathfrak{b}$.* □

Remark 2. That M is minimal means that $H_y^{-1} = \{0\}$ for every $y \in M$ due to Remark 1 (see [8, p. 149]).

2.3. Hermitian Positive Jordan Triple Systems

A *Hermitian Jordan Triple System* [4, p. 429] is a finite dimensional vector space over the complex numbers with a map $L : V \times V \to End_C(V)$ which is C-linear in the first variable, C-antilinear in the second one and satisfies the following identities

i) $L(x, y)z = L(z, y)x \qquad \forall x,\ y,\ z \in V.$

ii) $[L(x, y), L(z, w)] = L(L(x, y)z, w) - L(z, L(y, x)w) \quad \forall x,\ y,\ z,\ w \in V.$

In V we have a naturally defined *sesquilinear* form

$$\langle x, y \rangle = tr L(x, y). \tag{1}$$

The Jordan Triple System is called *positive* if

$$\langle x, x \rangle > 0, \quad \forall x \neq 0 \textit{ in } V. \tag{2}$$

This defines in V a Hermitian inner product which has the following property:

$$\langle L(x, y) z, w \rangle = \langle z, L(y, x) w \rangle \qquad \forall x,\ y \in V.$$

An *ideal* in a Hermitian Jordan Triple System V is a vector subspace $I \subset V$ such that

$$L(I, V)V \subset I$$
$$L(V, I)V \subset I.$$

A Hermitian Jordan Triple System V is called *simple* if $V \neq \{0\}$ and has no ideals except $\{0\}$ and V.

We will assume that our Hermitian Jordan triple system V is positive and simple. This is not always necessary but will simplify our statements.

An element $e \in V$ is called a *tripotent* if $L(e, e)e = 2e$, (this not the canonical definition [8, p. 172] but is the one used in [4] and it is more convenient for our purposes). If $V \neq \{0\}$ then it contains a non-zero tripotent [4, p. 513]. Two tripotent elements e_1 and e_2 are called *orthogonal* if $L(e_1, e_2) = 0$ (this is equivalent to $L(e_2, e_1) = 0$).

If e_1 and e_2 are orthogonal then $L(e_1, e_1)$ and $L(e_2, e_2)$ commute, $e = e_1 + e_2$ is a tripotent and

$$L(e, e) = L(e_1, e_1) + L(e_2, e_2).$$

Furthermore
$$L(e_j, e_j)\, e_k = 2\delta_k^j e_j.$$
A tripotent e is called *primitive* if it can not be written as a sum of two orthogonal tripotents.

Let e be a tripotent in V. The operator $L(e,e)$ is diagonalizable with eigenvalues $\{0,1,2\}$. The eigenspaces are

$$V_\alpha(e) = \{Z \in V : L(e,e)\, Z = \alpha Z\} \qquad \alpha = 0,1,2.$$

and we have the *Pierce decomposition of V relative to e* :

$$V = V_0 \oplus V_1 \oplus V_2.$$

If $e \neq 0$ then $dim V_2 \neq 0$. It is important to notice that under our hypothesis, a non-zero tripotent e is primitive if and only if $V_2(e) = Ce$.

Let $\{e_1, \ldots, e_p\}$ be a set of mutually orthogonal tripotents in V. The operators $L(e_j, e_j)$ commute and are simultaneously diagonalizable with eigenvalues $0,1,2$. The simultaneous eigenspaces which may be different from $\{0\}$ are

$$V_{\alpha\beta} = \left\{z \in V : L(e_j, e_j)\, z = \left(\delta_\alpha^j + \delta_\beta^j\right) z, \quad 1 \leq j \leq p\right\}$$

for $0 \leq \alpha \leq \beta \leq p$. The decomposition

$$V = \bigoplus_{0 \leq \alpha \leq \beta \leq p} V_{\alpha\beta}$$

is the Pierce decomposition of V with respect to $\{e_1, \ldots, e_p\}$. The spaces $V_{\alpha\beta}$ are Jordan triple subsystems of V and invariant by the operators $L(e_j, e_j)$.

A *frame* on V is a maximal set of mutually orthogonal, primitive tripotents. It is known that the number of elements in a frame is a constant. This is the rank of V. A frame of V is also a basis (over the reals) for a *maximal flat subspace* S of V. A real subspace $S \subset V$ is called a *flat subspace* if it is a real triple subsystem of V and $L(x,y) = L(y,x) \ \ \forall x, y \in S$. Every Hermitian positive JTS has a frame.

An *automorphism* of V is a complex linear isomorphism f of V such that

$$L(fx, fy) = f \circ L(x,y) \circ f^{-1},$$

hence it satisfies
$$\langle fx, fy\rangle = \langle x, y\rangle.$$
The automorphisms form a group, $Aut(V)$, which is a closed subgroup of the unitary group of V (with the Hermitian product $\langle x, y\rangle$). We denote by $(Aut(V))_0$ the connected component of the identity in $Aut(V)$. The Lie algebra $\mathfrak{D}$ of $(Aut(V))_0$ is the Lie algebra of derivations of V. $T \in End_C(V)$ is a derivation if it satisfies

$$T(L(x,y)\, z) = L(Tx, y)\, z + L(x, Ty)\, z + L(x,y)\, Tz.$$

Notice that $\mathfrak{D}$ is a real Lie algebra.

For each $u \in V$, $iL(x,x)$ is a derivation of V due to the identity 2.3 (ii). Furthermore the subspace $Span_R\{iL(x,x) : x \in V\}$ contains all the differences $L(x,y) - L(y,x)$ and since $[iL(x,x), iL(y,y)]$ is one of these differences it is clear that $Span_R\{iL(x,x) : x \in V\}$ is a subalgebra of $\mathfrak{D}$ which is called the *algebra of inner derivations of* V and denoted by $Int(V)$.

Let $\{e_1, \ldots, e_r\}$ be a frame in V and consider an element

$$x = \sum_{j=1}^{r} \lambda_j e_j, \qquad \lambda_j \in C;\ 1 \le j \le r.$$

Set $\lambda_0 = 0$. Then for $y \in V_{\alpha\beta}$, $0 \le \alpha \le \beta \le r$, the following relation holds

$$L(x,x)\, y = \left(|\lambda_\alpha|^2 + |\lambda_\beta|^2\right) y. \tag{3}$$

3. Extrinsic symmetric *CR*-spaces exist in nature

It is quite natural to try to extend the definition of extrinsic symmetric manifold, given by D. Ferus in [5], to symmetric almost Hermitian *CR*-manifolds by giving the following definition.

Let M be a compact symmetric almost Hermitian *CR*-manifold and let $h : M \to R^n$ be an isometric embedding (taking in R^n the usual Euclidean metric).

Definition 3. *We shall say that h is an extrinsic symmetric embedding or that M is an extrinsic symmetric almost Hermitian CR-submanifold of R^n if for each $y \in M$ there exists an isometry θ_y of R^n such that*

(i) $\theta_y(h(M)) \subset h(M)$, $\theta_y(h(y)) = h(y)$ *(this means in particular that θ_y is linear).*

(ii) *For every $X \in T_y(M)$*

$$\theta_y \circ h_*|_y X = h_*|_y \circ s_{y*}|_y X.$$

(iii) *θ_y is the identity on the normal space $T_y(M)^\perp$ at the point $h(y)$.*

That extrinsic symmetric almost Hermitian *CR*-submanifolds of R^n exist is a trivial fact since symmetric *R*-spaces are totally real and they obviously satisfy our definition. But it is also obvious that there are some where the *CR*-structure is not trivial, for instance odd dimensional spheres S^{2n-1} which have a natural and not trivial *CR*-structure [8, p.146] are extrinsic symmetric submanifolds of $R^{2n} = C^n$ with the extended definition.

The extrinsic symmetric spheres are a particular case of the family given in the next proposition. This is a consequence of [8, p. 176, 8.14].

Proposition 4. *Let V be a positive simple Hermitian JTS and let M be a connected component of the set $Tri(V)$ of tripotent elements in V. Then M is an extrinsic symmetric almost Hermitian CR-submanifold of V.*

Our notation from [4] makes it necessary to rephrase this proposition because our tripotents satisfy $L(e,e)e = 2e$ instead of $L(e,e)e = e$ as in [8]. However if e is a tripotent for our definition then $x = \frac{1}{\sqrt{2}}e$ is clearly a tripotent for [8] and vice versa. Then the proposition we want to prove is the following.

Proposition 5. *Let V be a positive simple Hermitian JTS and let M be a connected component of the set*

$$\frac{1}{\sqrt{2}}Tri(V) = \left\{\frac{1}{\sqrt{2}}e : e \in Tri(V)\right\} \subset V.$$

Then M is an extrinsic symmetric almost Hermitian CR-submanifold of V.

Proof. Let $e \in Tri(V)$ and set $h = \frac{1}{\sqrt{2}}e$. Let M be the connected component of h in $\frac{1}{\sqrt{2}}Tri(V)$. It is known that M is an orbit of the connected component of the identity $(Aut(V))_0$. We take in $V \simeq R^{2n}$ the real scalar product $\langle x, y\rangle_R = \mathrm{Re}\langle x, y\rangle$ which induces in M a Riemannian metric so that the inclusion $i : M \hookrightarrow V$ is an isometry. Associated to the tripotent h there is the triple multiplication operator $\mu_h(x) = L(h, h)x$ which is Hermitian and splits V into an orthogonal direct sum $V = V_0 \oplus V_{\frac{1}{2}} \oplus V_1$ of eigenspaces of μ_h, called the Pierce spaces of h, corresponding to the eigenvalues $0, 1/2, 1$.

Associated to μ_h, there is the "Pierce reflection" [8, p. 172]

$$\rho_h = \exp(2\pi i\mu_h) = P_1 - P_{\frac{1}{2}} + P_0$$

were P_j is the canonical projection $P_j : V \to V_j$ for $j = 0, 1/2, 1$.

V_1 is a unital complex Jordan algebra with identity element h ([8, p. 172]) and has a conjugate linear, algebra involution $z \mapsto z^* = L(h, z)h$. The selfadjoint part of V_1, $A = \{z \in V_1 : z^* = z\}$ is a formally real Jordan algebra and $V_1 = A \oplus iA$. This is an orthogonal direct sum, since they are eigenspaces of the involutive operator $*$ which is selfadjoint.

It is indicated in [8, p. 175] that $T_h(M) = iA \oplus V_{\frac{1}{2}}$ and hence $T_h(M)^\perp = A \oplus V_0$. The holomorphic tangent space at h is $H_h(M) = V_{\frac{1}{2}}$.

Since the Pierce reflection is contained in $(Aut(V))_0$, it fixes h and leaves $Tri(V)$ invariant, then ρ_h leaves M invariant. It is proved in [8, p. 176, (8.14,15)] that ρ_h is a symmetry of M at h and it is clear that ρ_h is the identity on $T_h(M)^\perp = A \oplus V_2$. This proves the proposition. □

The requirement of simplicity for V is clearly not necessary, we include it to be coherent with our previous convention (see Subsection 2.3).

It is interesting to notice that these CR-manifolds are not the only ones that have the property of being extrinsic symmetric almost Hermitian CR-submanifold of a positive simple Hermitian JTS V. Our next section is devoted to the construction of other examples.

4. New examples

These examples are also contained in a Hermitian positive simple Jordan triple system. For that reason we start with a subsection in which we make explicit the construction principle for the case of orbits in a HJTS.

4.1. The construction principle applied to HJTS

Let V be a Hermitian positive simple Jordan triple system and let K be a closed connected subgroup of the connected component of the identity of $Aut(V)$, then K is compact. Let S

be maximal flat subspace of V and let $x \in S$. We want to study the orbit $M = K(x) \subset V$. Let K_x denote the isotropy group of K at x, then $M = K/K_x$. Let $\mathfrak{k}$ and $\mathfrak{k}_x$ denote the Lie algebras of K and K_x respectively. Let $\mathfrak{m}$ denote the orthogonal complement of $\mathfrak{k}_x$ in $\mathfrak{k}$ with respect to the Killing form in $\mathfrak{k}$, then $\mathfrak{k} = \mathfrak{k}_x \oplus \mathfrak{m}$. The subspace $\mathfrak{m}$ can be naturally "identified", in the usual manner, with the tangent space $T_x(M)$ to M at x i.e. every $Z \in \mathfrak{m}$ is identified with the vector tangent to the curve $\exp(tZ)x$ at the point x that is $\frac{d}{dt}\exp(tZ)x\big|_{t=0} = Z.x$. Under this identification the isotropy representation of K_x on $T_x(M)$ becomes $g_*|_x(Z.x) = (Ad(g)Z).x$. We may write then $T_x(M) = \mathfrak{m}.x$.

We have on V the Hermitian inner product $\langle *, * \rangle$. The real part of this Hermitian inner product $\langle x,, y\rangle_R = \mathrm{Re}\langle x, y\rangle$ yields on V^R a real scalar product which, in turn, induces a Riemannian metric on M. Let ρ be an involutive automorphism i.e. $\rho^2 = Id$, of V which leaves invariant the subspace $T_x(M)$ (hence it also leaves invariant $T_x(M)^{\perp}$). Furthermore, we will assume that $\rho \in K$ and it is the identity on $T_x(M)^{\perp}$.

We have an induced automorphism of the group K defined by $\sigma(g) = \rho g \rho$ and this in turn, induces a map s_x of M by $s_x(g(x)) = \sigma(g)x = \rho g(x)$ i.e. s_x is just the restriction of ρ to M.

Let $T_x^{\pm}(M) \simeq \mathfrak{m}^{\pm}.x$ denote the subspaces corresponding to the eigenvalues (± 1) of ρ on $T_x(M) \simeq \mathfrak{m}.x$ respectively.

Let us assume now that there exists a complex subspace $\mathfrak{h}.x$ of $\mathfrak{m}^{-}.x \simeq T_x^{-}(M)$ which satisfies the following requirements (then $\mathfrak{h} \subset \mathfrak{m} \subset \mathfrak{k}$).

(i) $\langle iZ.x, iZ.x\rangle_R = \langle Z.x, Z.x\rangle$ for all $Z.x \in \mathfrak{h}.x$.

(ii) K_x contains the element s_x.

(iii) Every element $g \in K_x$ leaves the subspace $\mathfrak{h}$ invariant and commutes, there, with multiplication by i, i.e. $Z \in \mathfrak{h}$ implies $Ad(g)Z \in \mathfrak{h}$ and $Ad(g)iZ = iAd(g)Z$, $\forall g \in K_x$

In this particular situation, Proposition 2 gives the necessary and sufficient condition which makes M a *minimal symmetric CR-manifold with symmetry s_x at x*. Namely the Lie subalgebra $\mathfrak{b}$ of $\mathfrak{k}$ generated by the subspace $\mathfrak{h}$ satisfies

$$\mathfrak{k} = \mathfrak{k}_x + \mathfrak{b}.$$

We use this procedure to construct the following examples.

For the sake of simplicity our examples are constructed in rank two. It would be clear that this can be reproduced in larger ranks.

4.2. Example in *EIII*

We shall take V as the exceptional JTS of dimension 16 over C. Then V is the tangent space, at some point, to the compact irreducible Hermitian symmetric space $EIII = E_6/Spin(10).U(1)$. To introduce the structure of Jordan triple system in V we shall use the canonical isometric Euclidean embedding of $EIII$ in $\mathfrak{e}_6$, the compact Lie algebra of E_6. We take in $\mathfrak{e}_6$ the opposite of the Killing form as real scalar product and the induced Riemannian metric on $EIII$. A way to associate the JTS to our Hermitian Symmetric Space $EIII$ is well known. For the canonical isometric Euclidean embedding mentioned above, we have its second fundamental form $\alpha(x, y)$, the corresponding shape operator A_ξ and the Riemannian curvature tensor $R(x, y)$ for the Riemannian connection ∇. Let E denote our basic point in $EIII$ which we will determine below. We define $V = T_E(M)$ and

$$L(x, y)z = A_{\alpha(x,y)}z + R(x, y)z \qquad \forall x,\ y,\ z \in V.$$

The induced real scalar product on $T_E\,(EIII) = V$ is a real positive multiple of $\langle x, y\rangle_R$ on V due to irreducibility. Then we may use on V, $\langle X, Y\rangle_R$ or $\langle X, Y\rangle$, keeping as a scalar product on $\mathfrak{e}_6$ the opposite of the Killing form.

The maximal root of $\mathfrak{e}_6$ is $\eta = \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$; we have to take $\Phi = \pi-\{\alpha_1\}$ (or equivalently $\pi - \{\alpha_6\}$).

The set Φ^u (of tangent roots) consists of those positive roots which contain α_1. $|\Phi^u| = 16$. They are:

$$\begin{array}{ll}
\eta = \beta_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 & \beta_9 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 \\
\beta_2 = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 & \beta_{10} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \\
\beta_3 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 & \beta_{11} = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 \\
\beta_4 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 & \beta_{12} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\
\beta_5 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 & \beta_{13} = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 \\
\beta_6 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 & \beta_{14} = \alpha_1 + \alpha_3 + \alpha_4 \\
\beta_7 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 & \beta_{15} = \alpha_1 + \alpha_3 \\
\beta_8 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 & \beta_{16} = \alpha_1.
\end{array}$$

In $\mathfrak{h}_R = \sum_{j=1}^{6} RH_{\alpha_j}$ we define vectors v_i as it is indicated in the Appendix. Then

$$\alpha_1\,(v_1) = \alpha_6\,(v_6) = 1,\ \ \alpha_2\,(v_2) = \alpha_3\,(v_3) = \alpha_5\,(v_5) = 1/2,\ \ \alpha_4\,(v_4) = 1/3.$$

Let us take $E = iv_1$ as our base point in $EIII$. Then $J_E = ad\,(E)$ in V. It is well known and not hard to see, that with this definition we have a Hermitian positive simple Jordan triple system associated to our space $EIII$.

The roots $\{\eta, \beta_{11}\}$ form a strongly orthogonal pair. We take $e_1 = U_{-\eta}$ and $e_2 = U_{-\beta_{11}}$ and then

$$L\,(e_j, e_j)\,e_j = 2e_j, \qquad j = 1, 2, \tag{4}$$

because $\eta\,(H_\eta) = \beta_{11}\,(H_{\beta_{11}}) = 2$ (see the Appendix for notation and a proof of (4)). Hence they are tripotents and also $L\,(e_1, e_2) = L\,(e_2, e_1) = 0$, since $[e_1, e_2] = 0$ and $[e_1, [e_2, E]] = 0$. Furthermore

$$V_2\,(e_j) = Re_j + iRe_j = Ce_j, \qquad j = 1, 2$$

and so they are primitive tripotents [4, p. 515, VI.2.5].

Now we obtain the simultaneous Pierce decomposition for $\{e_1, e_2\}$ and to that end we consider first the Pierce decomposition for each e_j, $j = 1, 2$.

$$\begin{aligned}
V_0\,(e_1) &= \sum_j CU_{\beta_j} \qquad j = 11, 13, 14, 15, 16. \\
V_1\,(e_1) &= \sum_l CU_{\beta_l} \qquad l = 2, \ldots, 10, 12.
\end{aligned}$$

$$\begin{aligned} V_0(e_2) &= \sum_j CU_{\beta_j} \qquad j = 1, 7, 9, 10, 12. \\ V_1(e_2) &= \sum_l CU_{\beta_l} \qquad l = 2, \ldots, 8, 13, \ldots, 16. \end{aligned}$$

Then (using the obvious notation which indicates only the positive roots involved)

$$\begin{aligned} V_{00} &= V_0(e_1) \cap V_0(e_2) = \{0\}, \\ V_{01} &= V_0(e_2) \cap V_1(e_1) = \{7, 9, 10, 12\}, \\ V_{02} &= V_0(e_1) \cap V_1(e_2) = \{13, 14, 15, 16\}, \\ V_{12} &= V_1(e_1) \cap V_1(e_2) = \{2, 3, 4, 5, 6, 8\}, \\ V_{11} &= Ce_1, \qquad V_{22} = Ce_2. \end{aligned}$$

Notice that the dimensions $a = 6$ and $b = 4$ agree with [4, p. 526].

Then we have the simultaneous Pierce decomposition for $\{e_1, e_2\}$.

$$V = V_{11} \oplus V_{22} \oplus V_{01} \oplus V_{02} \oplus V_{12}.$$

Now if we take

$$x = \sqrt{2}e_1 + \frac{1}{\sqrt{2}}e_2, \tag{5}$$

then we have a regular element in the maximal flat subspace $S = Re_1 \oplus Re_2$ contained in V. We take now the connected compact group $K = Spin(10).U(1) \subset (Aut(V))_0$. Let $\mathfrak{k}$ be the Lie algebra of K.

For the canonical embedding into $\mathfrak{e}_6$ as adjoint orbit of the point $E = iv_1$ we have

$$\begin{aligned} T_E(EIII)^{\perp} &= \mathfrak{k} \\ T_E(EIII) &= V. \end{aligned}$$

On the other hand, as we saw in Subsection 2.1, to construct such an space in V, it is not required to take the whole group $(Aut(V))_0$ but a subgroup satisfying the three conditions.

Let M be the K-orbit of the point x defined at (5) and let K_x be the isotropy subgroup. We clearly have

$$\begin{aligned} T_x(M)^{\perp} &= RU_{-\eta} \oplus RU_{-\beta_{11}} = Re_1 \oplus Re_2 = S, \\ T_x(M) &= RU_{\eta} \oplus RU_{\beta_{11}} \oplus V_{01} \oplus V_{02} \oplus V_{12}. \end{aligned}$$

We want to see that M is an extrinsic SCR space in V.

To that end we study the effect of $L(x,x)$ on $V_{\alpha\beta}$ (it is necessary to put $\lambda_0 = 0$ and use formula (3)).

$$\begin{aligned} L(x,x)y &= 2y \qquad y \in V_{01}, \\ L(x,x)y &= \tfrac{1}{2}y \qquad y \in V_{02}, \\ L(x,x)y &= \tfrac{5}{2}y \qquad y \in V_{12}, \\ L(x,x)y &= 4y \qquad y \in V_{11}, \\ L(x,x)y &= y \qquad y \in V_{22}. \end{aligned} \tag{6}$$

Set $\mu = L(x, x)$ and $\rho = \exp(2\pi i\mu)$. Since $i\mu \in Int(V) = \mathfrak{k}$ (see Theorem 6 in Subsection 5.5) we clearly have $\rho \in K$. We have to study the eigenspaces and projectors of ρ. We have

$$\mu = P_{22} + \frac{1}{2}P_{02} + 2P_{01} + \frac{5}{2}P_{12} + 4P_{11}, \tag{7}$$

and then

$$\rho = \exp(2\pi i\mu) = P_{22} - P_{02} + P_{01} - P_{12} + P_{11}. \tag{8}$$

This means that ρ acts as the identity on $V_{11} \oplus V_{22} \oplus V_{01}$ and as $(-id)$ on $V_{02} \oplus V_{12}$. Since $\rho \in K$ it leaves M invariant; then $\rho \in K_x$ because it fixes x. We define

$$s_x = (\rho|\, M)$$

and want to see that M is a minimal symmetric CR-manifold with symmetry s_x at x.

We first have to decide which is the subspace $H_x(M) \subset T_x(M)$ that we want to use. From the nature of ρ it is clear that we have to take

$$H_x(M) = V_{02} \oplus V_{12} = \{2, 3, 4, 5, 6, 8, 13, 14, 15, 16\}. \tag{9}$$

We need to find the corresponding subspace $\mathfrak{h}$ in the algebra $\mathfrak{k} = Riv_1 \oplus \mathfrak{so}(5) \subset \mathfrak{e}_6$ such that $\mathfrak{h}.x = H_x(M)$.

The way in which the subalgebra of type $\mathfrak{d}_5$ is contained in $\mathfrak{e}_6$ is as the roots which do not contain α_1, There are 20 positive roots which are

$$\begin{array}{ll}
\delta_1 = \alpha_3 & \delta_{11} = \alpha_2 \\
\delta_2 = \alpha_3 + \alpha_4 & \delta_{12} = \alpha_4 + \alpha_5 \\
\delta_3 = \alpha_2 + \alpha_3 + \alpha_4 & \delta_{13} = \alpha_2 + \alpha_4 + \alpha_5 \\
\delta_4 = \alpha_4 & \delta_{14} = \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 \\
\delta_5 = \alpha_2 + \alpha_4 & \delta_{15} = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 \\
\delta_6 = \alpha_3 + \alpha_4 + \alpha_5 & \delta_{16} = \alpha_5 \\
\delta_7 = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 & \delta_{17} = \alpha_5 + \alpha_6 \\
\delta_8 = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 & \delta_{18} = \alpha_4 + \alpha_5 + \alpha_6 \\
\delta_9 = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 & \delta_{19} = \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 \\
\delta_9 = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 & \delta_{20} = \alpha_6.
\end{array}$$

By (9), the formulae in Subsection 5.3 of the Appendix and the identities

$$\begin{array}{ll}
\beta_2 = \eta - \delta_{11} & \beta_8 = \beta_{11} + \delta_{11} \\
\beta_3 = \eta - \delta_5 & \beta_6 = \beta_{11} + \delta_5 \\
\beta_4 = \eta - \delta_3 & \beta_5 = \beta_{11} + \delta_3 \\
\beta_5 = \eta - \delta_{13} & \beta_4 = \beta_{11} + \delta_{13} \\
\beta_6 = \eta - \delta_7 & \beta_3 = \beta_{11} + \delta_7 \\
\beta_8 = \eta - \delta_{15} & \beta_2 = \beta_{11} + \delta_{15} \\
\beta_{14} = \eta - \delta_{17} & \beta_{13} = \beta_{11} - \delta_{20} \\
\beta_{16} = \beta_{11} - \delta_{14} & \beta_{15} = \beta_{11} - \delta_{18}
\end{array}$$

we see that the subspace $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{e}_6$ is

$$\mathfrak{h} = \{\delta_3, \delta_5, \delta_7, \delta_{11}, \delta_{13}, \delta_{14}, \delta_{15}, \delta_{17}, \delta_{18}, \delta_{20}\}. \tag{10}$$

Now we compute the isotropy subalgebra $\mathfrak{k}_x$ of $\mathfrak{k}$ at the point x. The roots are those δ such that $\pm\delta$ cannot be added to η and β_{11}. They are $\{\delta_1, \delta_2, \delta_4, \delta_6, \delta_{12}, \delta_{16}\}$ which is clearly the root part of an algebra of type $\mathfrak{a}_3$ generated by $\{\alpha_3, \alpha_4, \alpha_5\}$. Let us get now the Abelian part of the isotropy.

We have

$$\begin{aligned}
&\left[\sum_{j=1}^{6} a_j i v_j, \lambda_1 e_1 + \lambda_2 e_2\right] \\
= \quad & -\lambda_1 \left(\sum_{j=1}^{6} a_j\right) U_\eta - \lambda_2 \left(a_1 + \frac{1}{2}a_3 + \frac{1}{3}a_4 + \frac{1}{2}a_5 + a_6\right) U_{\beta_{11}}.
\end{aligned}$$

Hence the Abelian part of the isotropy is the subspace determined by the two coefficients equal to zero which in turn yields

$$\begin{aligned}
a_1 &= -\frac{1}{2}a_3 - \frac{1}{3}a_4 - \frac{1}{2}a_5 - a_6, \\
a_2 &= -\frac{1}{2}a_3 - \frac{2}{3}a_4 - \frac{1}{2}a_5.
\end{aligned}$$

Replacing these expressions in the original sum, we obtain the form of the mentioned subspace which we shall denote by $\mathfrak{A}$.

$$\begin{aligned}
&a_3\left(iv_3 - \frac{1}{2}iv_2 - \frac{1}{2}iv_1\right) + a_4\left(iv_4 - \frac{2}{3}iv_2 - \frac{1}{3}iv_1\right) + \\
&a_5\left(iv_5 - \frac{1}{2}iv_2 - \frac{1}{2}iv_1\right) + a_6\left(iv_6 - iv_1\right),
\end{aligned} \tag{11}$$

where $a_j \in R$, $j = 3, 4, 5, 6$ are arbitrary. Then

$$\mathfrak{k}_x = \sum_j RU_{\delta_j} \oplus RU_{-\delta_j} \oplus \mathfrak{A} \qquad j = 1, 2, 4, 6, 12, 16.$$

Here we have to check that $\mathfrak{h}$, $\langle *, * \rangle$ and K_x satisfy conditions (i), (ii) and (iii) of Subsection 4.1. Since $K \subset (Aut\,(V))_0$ it leaves $\langle *, * \rangle$ invariant and we are taking in M the induced metric from V.

(i) If $Z \in \mathfrak{h}$, $iZ = J_E(Z) \in \mathfrak{h}$, $\langle iZ, iZ \rangle = i(-i)\langle Z, Z \rangle$.

(ii) By definition $s_x = (\rho|\,M)$ and $\rho \in K$ since $i\mu \in Int\,(V)$. Furthermore it fixes x then $s_x \in K_x$.

(iii) If $g \in K_x$ we have to see that $Ad\,(g)\,\mathfrak{h} \subset \mathfrak{h}$ and $Ad\,(g)\,iZ = iAd\,(g)\,Z$.

The last identity holds in V since $K \subset Aut\,(V)$ and automorphisms are C-linear by definition.

To show that $Ad(g)\mathfrak{h} \subset \mathfrak{h}\ \forall g \in K_x$ is tantamount to showing that $[\mathfrak{k}_x, \mathfrak{h}] \subset \mathfrak{h}$ in the algebra $\mathfrak{k}$.

Since the Abelian part $\mathfrak{A}$ of $\mathfrak{k}_x$ leaves $\mathfrak{h}$ invariant, we need to verify that the root part also does. By considering (10) we see that if $\gamma = \pm\delta_j$ $(j = 1, 2, 4, 6, 12, 16)$ and $\varepsilon = \pm\delta_s$ $(s = 3, 5, 7, 11, 13, 14, 15, 17, 18, 20)$ then either $(\gamma + \varepsilon) = 0$ or $(\gamma + \varepsilon) = \pm\delta_h$ $(h = 3, 5, 7, 11, 13, 14, 15, 17, 18, 20)$. Hence

$$[\mathfrak{k}_x, \mathfrak{h}] \subset \mathfrak{h}.$$

Now according to Subsection 4.1 we have to see that the subalgebra $\mathfrak{b}$ of $\mathfrak{k}$ generated by the subspace $\mathfrak{h} \subset \mathfrak{k}$ plus the isotropy subalgebra $\mathfrak{k}_x$ equals the whole algebra $\mathfrak{k}$.

We first analyze which new roots of $\mathfrak{k}$ there appear when we generate the subalgebra $\mathfrak{b}$. Since

$$\begin{aligned} \delta_3 - \delta_5 &= \alpha_3 \\ \delta_5 - \delta_{11} &= \alpha_4 \\ \delta_{17} - \delta_{20} &= \alpha_5 \end{aligned}$$

and $\mathfrak{h}$ already contains $\delta_{11} = \alpha_2$ and $\delta_{20} = \alpha_6$ we see that

$$\left\{U_{\pm\alpha_j} : j = 2, \ldots, 6\right\} \in \mathfrak{b}.$$

Since

$$\left[U_{\alpha_j}, U_{-\alpha_j}\right] = iH_{\alpha_j},$$

we clearly have

$$\left\{iH_{\alpha_j} : j = 2, \ldots, 6\right\} \in \mathfrak{b}. \tag{12}$$

Now we compute the abelian subalgebra that we obtain with the subspace (11) and the subspace generated by (12). That is

$$\mathfrak{U} = \{v : a_3, \ldots, a_6, b_2, \ldots, b_6 \in R\}$$

where

$$\begin{aligned} v \;=\; & a_3\left(iv_3 - \frac{1}{2}iv_2 - \frac{1}{2}iv_1\right) + a_4\left(iv_4 - \frac{2}{3}iv_2 - \frac{1}{3}iv_1\right) + \\ & a_5\left(iv_5 - \frac{1}{2}iv_2 - \frac{1}{2}iv_1\right) + a_6\left(iv_6 - iv_1\right) + \sum_{j=2}^{6} b_j iH_{\alpha_j}. \end{aligned}$$

We need to prove that the subalgebra $\mathfrak{U}$ coincides with the Cartan subalgebra in $\mathfrak{k}$ which is none other than the Cartan subalgebra of $\mathfrak{e}_6$ i.e.

$$\left\{\sum_{j=1}^{6} a_j iv_j : a_j \in R,\ \forall j\right\}.$$

This is easy to see, by checking that the vectors iv_s are in $\mathfrak{U}$ for $s = 1, \ldots, 6$ and therefore

$$\mathfrak{k} = \mathfrak{k}_x + \mathfrak{b}.$$

Then our manifold M is a minimal symmetric CR-manifold with symmetry s_x at x.

4.3. Example in $MIII\,(2,3)$

We take as V the tangent space, at some point, to the compact irreducible Hermitian symmetric space $MIII(2,3) = SU(5)/S\,(U(2) \times U(3))$. Once more, to introduce the structure of Jordan triple system in V, we use the canonical isometric Euclidean embedding of $MIII$ in $\mathfrak{su}\,(5)$, the compact Lie algebra of $SU(5)$, and define $V = T_E(MIII)$ with

$$L(x,y)z = A_{\alpha(x,y)}z + R(x,y)z \qquad \forall x,\ y,\ z \in V.$$

Then we have a Hermitian positive simple Jordan triple system associated to our space $MIII$.

The Lie algebra $\mathfrak{su}\,(5)$ is of type A_4 so we have simple roots $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. The complex dimension of V is 6 and it is generated by the root vectors of those roots of A_4 which contain the root α_2. The rank of V is clearly 2 and we have our basic point $E = iv_2$.

The root $\eta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ is the maximal one and the pair

$$\begin{aligned} \gamma_1 &= \eta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\ \gamma_2 &= \alpha_2 + \alpha_3, \end{aligned}$$

is a strongly orthogonal pair of roots in V so we may consider the following pair of orthogonal primitive tripotents

$$e_1 = U_{-\eta}, \qquad e_2 = U_{-\gamma_2}.$$

We have the Pierce subspaces for each one of them namely

$$\begin{aligned} V_0\,(e_1) &= \{\gamma_2, \alpha_2\}, \\ V_0\,(e_2) &= \{\gamma_1, \alpha_1 + \alpha_2\}, \\ V_1\,(e_1) &= \{\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4\}, \\ V_1\,(e_2) &= \{\alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4\}, \\ V_2\,(e_1) &= Ce_1, \qquad V_2\,(e_2) = Ce_2, \end{aligned}$$

and the simultaneous Pierce decomposition for the frame $\{e_1, e_2\}$.

$$\begin{aligned} V_{00} &= V_0\,(e_1) \cap V_0\,(e_2) = \{0\}, \\ V_{01} &= V_1\,(e_1) \cap V_0\,(e_2) = \{\alpha_1 + \alpha_2\}, \\ V_{02} &= V_0\,(e_1) \cap V_1\,(e_2) = \{\alpha_2\} \\ V_{12} &= V_1\,(e_1) \cap V_1\,(e_2) = \{\alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4\}, \\ V_{11} &= Ce_1, \qquad V_{22} = Ce_2. \end{aligned}$$

The dimensions $a = 2$ and $b = 1$ agree with [4, p. 525].

Let us take now, as in the previous example, $x = \sqrt{2}e_1 + \frac{1}{\sqrt{2}}e_2$ and obtain the same eigenvalues and eigenspaces of $L\,(x,x)$ (see (6)).

Set $\mu = L\,(x,x)$ and $\rho = \exp\,(2\pi i\mu)$. Since $i\mu \in Int\,(V) = \mathfrak{k}$ we clearly have $\rho \in K$. We see that we have "formally" the same expressions for μ and ρ as before (see (7) and (8)).

This means that ρ acts as the identity on $V_{11} \oplus V_{22} \oplus V_{01}$ and as $(-id)$ on $V_{02} \oplus V_{12}$. On the other hand ρ leaves M invariant and fixes x so $\rho \in K_x$. We define, as before,

$$s_x = (\rho| M)$$

and have to see that M is a minimal symmetric CR-manifold with symmetry s_x at x.

To that end we take

$$H_x(M) = V_{02} \oplus V_{12} = \{\alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4\}.$$

We have to find the corresponding subspace in the algebra $\mathfrak{k} = Riv_2 \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(3) = A_1 \oplus A_2 \oplus Riv_2$. The way in which this subalgebra is contained in $\mathfrak{su}(5)$ is as the roots which do not contain α_2 and $\sum_{j=1}^{4} Riv_j$.

Since we have

$$\begin{aligned} -\eta + \alpha_1 &= -(\alpha_2 + \alpha_3 + \alpha_4), \\ -\eta + \alpha_4 &= -(\alpha_1 + \alpha_2 + \alpha_3), \\ -\gamma_2 + \alpha_3 &= -\alpha_2, \end{aligned}$$

it is clear that the subspace of $\mathfrak{k}$ which satisfies $\mathfrak{h}.x = H_x(M)$ is

$$\mathfrak{h} = \{\alpha_1, \alpha_3, \alpha_4\}. \tag{13}$$

Now we compute the isotropy subalgebra. Since $\dim_R M = 10$ and $\dim_R \mathfrak{k} = 12$ we see that the isotropy subalgebra $\mathfrak{k}_x$ is Abelian and has dimension 2 which yields

$$\mathfrak{k}_x = \{a_3(iv_3 - iv_2) + a_4(iv_4 - iv_1) : a_3, a_4 \in R\}.$$

It is now clear from (13) that in the subalgebra $\mathfrak{b}$ of $\mathfrak{k}$ generated by $\mathfrak{h}$ we have

$$\left\{U_{\pm\alpha_j} : j = 1, 3, 4\right\} \in \mathfrak{b}$$

and hence

$$\left\{iH_{\alpha_j} : j = 1, 3, 4\right\} \in \mathfrak{b}.$$

We have to verify that

$$\mathfrak{k} = \mathfrak{k}_x + \mathfrak{b}$$

and to that end we consider the subspace

$$\mathfrak{U} = \left\{ a_3(iv_3 - iv_2) + a_4(iv_4 - iv_1) + \sum_{j=1,3,4} b_j iH_{\alpha_j} : a_3, a_4, b_1, b_3, b_4 \in R \right\}.$$

It is easy to see that $\mathfrak{U}$ contains the basis vectors $\{iv_1, iv_2, iv_3, iv_4\}$.

Then we have

$$\mathfrak{k} = \mathfrak{k}_x + \mathfrak{b}.$$

and we need to check that the conditions (i), (ii) and (iii) of Subsection 4.1 are satisfied.

(i) $\langle iZ.x, iZ.x\rangle_R = \langle Z.x, Z.x\rangle$ for all $Z.x \in \mathfrak{h}.x$.

(ii) K_x contains the element s_x.

(iii) Every element $g \in K_x$ leaves the subspace $\mathfrak{h}$ invariant and commutes there with multiplication by i, i.e. $Z \in \mathfrak{h}$ implies $Ad\,(g)\,Z \in \mathfrak{h}$ $\forall g \in K_x$ and $Ad\,(g)\,iZ = iAd\,(g)\,Z$.

That $s_x \in K_x$ is obvious since $s_x \in K$ and it fixes x so we have (ii). (i) is clearly satisfied and that K_x leaves $\mathfrak{h}$ invariant is immediate since $\mathfrak{k}_x$ is Abelian.

Then this manifold M is also a minimal symmetric CR-manifold with symmetry s_x at x.

4.4. The general situation in $\boldsymbol{EIII}$ and $\boldsymbol{MIII\,(2,3)}$

In this section we want to consider which are all the spaces that can be constructed by this method in $EIII$ and $MIII\,(2,3)$. For the sake of briefness we do not include the spaces of Kaup-Zaitsev that are present in these Hermitian positive simple Jordan Triple Systems. We do not claim that these are all the possible extrinsic SCR-spaces in $EIII$ and $MIII\,(2,3)$. These are the ones constructible by this method.

Let us set

$$x = \frac{a_1}{\sqrt{2}}e_1 + \frac{a_2}{\sqrt{2}}e_2 \tag{14}$$

where e_1 and e_2 are the primitive idempotents chosen in each case and a_1, a_2 are integers.

We have always the simultaneous Pierce decomposition for $\{e_1, e_2\}$.

$$V = V_{11} \oplus V_{22} \oplus V_{01} \oplus V_{02} \oplus V_{12}.$$

For our general x from (14) the eigenvalues are

$$\begin{array}{lll} L\,(x,x)\,y & = \frac{a_1^2}{2}y & y \in V_{01}, \\ L\,(x,x)\,y & = \frac{a_2^2}{2}y & y \in V_{02}, \\ L\,(x,x)\,y & = \frac{a_1^2+a_2^2}{2}y & y \in V_{12}, \\ L\,(x,x)\,y & = 2\frac{a_1^2}{2}y & y \in V_{11}, \\ L\,(x,x)\,y & = 2\frac{a_2^2}{2}y & y \in V_{22}. \end{array}$$

The cases $a_1 = 1$, $a_2 = 0$ (and vice versa) and $a_1 = a_2 = 1$ are the spaces of Kaup-Zaitsev [8] and our Proposition 5. Note that $e_1 + e_2$ is a maximal tripotent here [4, p. 503].

We have the following table that indicates which is the space $H_x\,(M)$ to be taken in each case

	a_1	a_2	$H_x\,(M)$
1	even	odd	$V_{02} \oplus V_{12}$
2	odd	odd	$V_{01} \oplus V_{02}$
3	odd	even	$V_{01} \oplus V_{12}$
4	even	even	$\rho = id.$

For the space V associated to $EIII$ the case (1) considered in the table is essentially the one studied in detail in the previous subsection.

In the case (2) we have

$$H_x(M) = V_{01} \oplus V_{02} = \{7, 9, 10, 12, 13, 14, 15, 16\}$$

and then one sees that

$$\mathfrak{h} = \{\delta_8, \delta_9, \delta_{10}, \delta_{14}, \delta_{17}, \delta_{18}, \delta_{19}, \delta_{20}\},$$

and the isotropy subalgebra $\mathfrak{k}_x$ is the same as in case (1).

Since

$$\begin{aligned} \delta_9 - \delta_8 &= \alpha_4 \\ \delta_{10} - \delta_9 &= \alpha_5 \\ \delta_{19} - \delta_{18} &= \alpha_2 \\ \delta_{14} - \delta_{18} &= \alpha_3 \\ \delta_{20} &= \alpha_6, \end{aligned}$$

we immediately see that

$$\mathfrak{k} = \mathfrak{k}_x + \mathfrak{b}$$

and so the spaces of case (2) for $EIII$ are extrinsic SCR-manifolds.

Case (3) of $EIII$ also works since here

$$H_x(M) = V_{01} \oplus V_{12} = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 12, \},$$

and then

$$\mathfrak{h} = \{\delta_3, \delta_5, \delta_7, \delta_8, \delta_9, \delta_{10}, \delta_{11}, \delta_{13}, \delta_{15}, \delta_{19}\}.$$

Here again

$$\begin{aligned} \delta_3 - \delta_5 &= \alpha_3 \\ \delta_7 - \delta_3 &= \alpha_5 \\ \delta_8 - \delta_7 &= \alpha_6 \\ \delta_9 - \delta_8 &= \alpha_4 \\ \delta_{11} &= \alpha_2, \end{aligned}$$

yielding immediately that

$$\mathfrak{k} = \mathfrak{k}_x + \mathfrak{b}.$$

Case (4) does not have to be considered since here $\rho = id$ in V.

For the space V associated to $MIII\,(2,3)$ again the case (1) is essentially the one studied in detail in the previous section.

In the case (2) we have

$$H_x(M) = V_{01} \oplus V_{02} = \{\alpha_1 + \alpha_2, \alpha_2\}$$

and then one sees that

$$\mathfrak{h} = \{\alpha_3 + \alpha_4, \alpha_3\}.$$

Clearly the isotropy subalgebra $\mathfrak{k}_x$ is the same as in case (1) and since in the subalgebra $\mathfrak{b}$, generated by $\mathfrak{h}$, the root α_1 is missing and it is in $\mathfrak{k}$ we see that

$$\mathfrak{k} \supsetneq \mathfrak{k}_x + \mathfrak{b}$$

and so the construction does not work in this case.

In case (3) we have

$$H_x(M) = V_{01} \oplus V_{12} = \{\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4\}$$

and then

$$\mathfrak{h} = \{\alpha_1, \alpha_4, \alpha_3 + \alpha_4, \alpha_3\}.$$

Here we get the roots α_1, α_3 and α_4 and then we see that

$$\mathfrak{k} = \mathfrak{k}_x + \mathfrak{b}.$$

This completes the analysis of the different cases.

5. Appendix

5.1. Notation on compact Hermitian symmetric spaces

We recall the definition of an irreducible Hermitian symmetric space in order to fix some notation used in the paper.

Let $\mathfrak{g}_C$ be a complex simple Lie algebra and let $\mathfrak{h}_C \subset \mathfrak{g}_C$ be a Cartan subalgebra. Let $\Delta = \Delta(\mathfrak{g}_C, \mathfrak{h}_C)$ be the root system of $\mathfrak{g}_C$ relative to $\mathfrak{h}_C$ and $\pi = \{\alpha_1, \ldots, \alpha_\ell\} \subset \Delta$ be a system of simple roots of Δ. Let us denote by Δ^+ the set of positive roots and by η the maximal root of Δ. Let us assume that there is some $\alpha_j \in \pi$ such that the coefficient $m(\alpha_j)$ of α_j in η is $m(\alpha_j) = 1$ (this is the necessary and sufficient condition for the existence of the Hermitian symmetric space M associated to the Lie algebra $\mathfrak{g}_C$). To distinguish this particular simple root we denote it by α_o.

Let us consider now the following subsets of Δ.

$$\Phi^r = \{\alpha \in \Delta : c(\alpha, \alpha_o) = 0\}$$
$$\Phi^u = \{\alpha \in \Delta : c(\alpha, \alpha_o) = 1\}$$

where $c(\alpha, \alpha_o)$ indicates the coefficient of the simple root α_o in α. Then $\Delta = \Phi^r \cup \Phi^u \cup (-\Phi^u)$. The subalgebra

$$\mathfrak{p}_C = \mathfrak{p}_C^r \oplus \mathfrak{p}_C^u = \left[\mathfrak{h}_C \oplus \sum_{\alpha \in \Phi^r} \mathfrak{g}_{C\alpha}\right] \oplus \sum_{\alpha \in \Phi^u} \mathfrak{g}_{C\alpha}$$

is called the *parabolic* subalgebra defined by $\mathfrak{h}_C$, π, and α_o.

Let G_C be the complex simply connected Lie group whose Lie algebra is $\mathfrak{g}_C$ and let P_C be the analytic subgroup of G_C corresponding to $\mathfrak{p}_C$. It is closed in G_C because it is the normalizer of its own Lie algebra. Then $M = G_C/P_C$ is a complex manifold which is a Hermitian symmetric space. All irreducible Hermitian symmetric spaces can be constructed in this way.

Let $\{H_\alpha : \alpha \in \pi\} \cup \{X_\alpha : \alpha \in \Delta\}$ be the canonical basis of $\mathfrak{g}_C$ (as in [6, p. 176]). Then the subalgebra

$$\mathfrak{g}_u = \sum_{\alpha \in \pi} RiH_\alpha + \sum_{\alpha \in \Delta^+} R(X_\alpha - X_{-\alpha}) + Ri(X_\alpha + X_{-\alpha}),$$

$(i = \sqrt{-1})$ is a compact real form of $\mathfrak{g}_C$.

Let $G_u \subset G_C$ be the analytic subgroup of G_C corresponding to $\mathfrak{g}_u$. G_u is compact and it is well known that if M is compact G_u acts transitively on M, so $M = G_u/(G_u \cap P_C)$.

It is usual to define, associated to the roots in $\pi = \{\alpha_1, \ldots, \alpha_\ell\}$, the vectors $\{v_1, \ldots, v_\ell\}$ in $\sum_{j=1}^{\ell} RH_{\alpha_j}$ by the rule

$$\alpha_j(v_k) = \delta_{j,k}/m(\alpha_j) \qquad 1 \leq j, k \leq \ell. \tag{15}$$

Let v_o denote the vector associated to our chosen root α_o.

The set $\{iv_1, \ldots, iv_\ell\}$ is clearly a basis for $\mathfrak{h}_u$ and obviously $\mathfrak{s} = iv_oR$. Let $\mathfrak{k} = \mathfrak{g}_u \cap \mathfrak{p}_C^r$, and $\mathfrak{m} \subset \mathfrak{g}_u$ denote the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form $B = B_C\,|\mathfrak{g}_u \times \mathfrak{g}_u$.

It is clear that

$$\begin{aligned} \mathfrak{k} &= \textstyle\sum_{\alpha\in\pi} RiH_\alpha + \sum_{\alpha\in\Delta\cap\Phi^r} R(X_\alpha - X_{-\alpha}) + Ri(X_\alpha + X_{-\alpha}) \\ \mathfrak{m} &= \textstyle\sum_{\alpha\in\Phi^u} R(X_\alpha - X_{-\alpha}) + Ri(X_\alpha + X_{-\alpha}). \end{aligned} \tag{16}$$

We define, for α in Δ^+,

$$\begin{aligned} U_\alpha &= \tfrac{1}{\sqrt{2}}(X_\alpha - X_{-\alpha}) \\ U_{-\alpha} &= \tfrac{i}{\sqrt{2}}(X_\alpha + X_{-\alpha}). \end{aligned} \tag{17}$$

5.2. The canonical Euclidean embedding

The orbit of $E = iv_o$ by the adjoint action of G_u on $\mathfrak{g}_u$ is a homogeneous space of type $G_u/(G_u \cap P_C)$ and we may consider it an embedding of M into $\mathfrak{g}_u$. In fact we have $f : M \to \mathfrak{g}_u$ defined as $f(g\,(G_u \cap P_C)) = Ad(g)E$. If we take in $\mathfrak{g}_u$ the scalar product defined by $\langle X, Y\rangle = -B\,(X, Y)$ then it becomes a Euclidean space. We take on M the induced metric and so f becomes an isometry. However, since M is irreducible, any other invariant metric on M is a positive constant multiple of $\langle , \rangle$. Clearly, the metric on M is determined by our choice of the point E. Since $T_E\,(M) = [\mathfrak{g}_u, E] = [\mathfrak{m}, E]$, by taking λE $(\lambda \in R)$ instead of E to generate the embedding, the induced metric is multiplied by λ^2 and so by taking an adequate factor of E, we may get all possible invariant metrics on M. Let us notice that $[\mathfrak{m}, E] = \mathfrak{m}$ (then $T_E\,(M)^\perp = \mathfrak{k}$) but $ad(E)$ does not act trivially on $\mathfrak{m}$. In fact $ad(E) = J_E$ is the almost complex structure at E.

By [6, 6.1(ii), p.382], our compact connected irreducible Hermitian symmetric space M can be written as $M = G/K$ where G is a compact connected centerless simple Lie group and K has nondiscrete center $Z\,(K)$ and is a maximal connected proper subgroup of G. Also by [6, 6.2, p. 382], $Z\,(K)$ is analytically isomorphic to S^1. Here $G = G_u/Z\,(G_u)$ whose Lie algebra is also $\mathfrak{g}_u$ and K is the analytic subgroup of G corresponding to the subalgebra $\mathfrak{k}$.

5.3. Necessary formulae

We include here some formulae which are used in the text. See for instance [3, p. 223]. Let ε, $\rho \in \Delta^+$, $\varepsilon \neq \rho$. Then

$$
\begin{aligned}
[U_\varepsilon, U_\rho] &= \frac{1}{\sqrt{2}}\left\{N_{\varepsilon,\rho}U_{\varepsilon+\rho} + sg\,(\rho-\varepsilon)\,N_{\varepsilon,-\rho}U_{|\varepsilon-\rho|}\right\},\\
[U_\varepsilon, U_{-\rho}] &= \frac{1}{\sqrt{2}}\left\{N_{\varepsilon,\rho}U_{-(\varepsilon+\rho)} + N_{\varepsilon,-\rho}U_{-|\varepsilon-\rho|}\right\},\\
[U_{-\varepsilon}, U_{-\rho}] &= \frac{1}{\sqrt{2}}\left\{N_{-\varepsilon,-\rho}U_{\varepsilon+\rho} + sg\,(\varepsilon-\rho)\,N_{-\varepsilon,\rho}U_{|\varepsilon-\rho|}\right\},\\
[U_\varepsilon, U_{-\varepsilon}] &= iH_\varepsilon,\\
[U_{\pm\varepsilon}, H] &= \pm\varepsilon(iH)U_{\mp\varepsilon}.
\end{aligned}
$$

where the terms are zero if $\varepsilon \pm \rho$ is not a root, $N_{\alpha,\beta}$ is as in [6, p. 176, 5.5] and

$$
|\varepsilon-\rho| = \begin{cases} \varepsilon-\rho & if\ \ (\varepsilon-\rho)\in\Delta^+ \\ \rho-\varepsilon & if\ \ (\rho-\varepsilon)\in\Delta^+ \end{cases},
$$

$$
sg\,(\varepsilon-\rho) = \begin{cases} 1 & if\ \ (\varepsilon-\rho)\in\Delta^+ \\ -1 & if\ \ (\rho-\varepsilon)\in\Delta^+ \end{cases},
$$

5.4. Auxiliary computations

Set $Y = U_{-\gamma}$. We want to compute $\alpha_E\,(Y,Y) = [Y,[Y,E]]$.

$$
[Y,E] = [U_{-\gamma}, iv_1] = -\gamma\,(-v_1)\,U_\gamma = U_\gamma
$$

since $\gamma\,(v_1) = 1\ \forall\gamma\in\Phi^u$. Then

$$
\alpha_E\,(Y,Y) = [U_{-\gamma}, U_\gamma] = -\,[U_\gamma, U_{-\gamma}] = -iH_\gamma.
$$

Set

$$
X = \sum_{\beta_j\in\Phi^u} a_j U_{\beta_j} + a_{-j}U_{-\beta_j}
$$

then

$$
ad\,(\alpha_E\,(Y,Y))\,X = \sum_{\beta_j\in\Phi^u} a_j[-iH_\gamma, U_{\beta_j}] + a_{-j}[-iH_\gamma, U_{-\beta_j}].
$$

Then using the above formulae

$$
ad\,(\alpha_E\,(Y,Y))\,X = \sum_{\beta_j\in\Phi^u} -a_j\beta_j\,(H_\gamma)\,U_{-\beta_j} + a_{-j}\beta_j\,(H_\gamma)\,U_{\beta_j}.
$$

Furthermore

$$
-\left[ad\,(E)\right]^{-1} : \begin{cases} U_\gamma = & U_{-\gamma} \\ U_{-\gamma} = & -U_\gamma; \end{cases}
$$

and hence

$$
\begin{aligned}
L\,(Y,Y)\,X &= A_{\alpha(Y,Y)}X = -\left[ad\,(E)\right]^{-1} ad\,(\alpha_E\,(Y,Y))\,X = \\
&= \sum_{\beta_j\in\Phi^u} a_j\beta_j\,(H_\gamma)\,U_{\beta_j} + a_{-j}\beta_j\,(H_\gamma)\,U_{-\beta_j}.
\end{aligned}
$$

This shows that

$$L(Y,Y)Y = 2Y$$

and since $\beta_j(H_\eta) = 2$ if and only if $\beta_j = \eta$ and $\beta_j(H_{\beta_{11}}) = 2$ if and only if $\beta_j = \beta_{11}$ then we have

$$V_2(e_j) = Re_j + iRe_j = Ce_j.$$

5.5. A useful fact

We need the following fact which must be well known but we have not seen remarked in the literature.

Theorem 6. *Let N be a compact connected irreducible Hermitian symmetric space. We may write it as $N = G/K$ where G is a compact, connected centerless simple Lie group and K has nondiscrete center and is a maximal connected proper subgroup* [6, p. 382]. *Let $V = T_E(N)$ be the Hermitian positive simple Jordan triple system associated to N and let $\mathfrak{k}$ denote the Lie algebra of K. Then*

$$\mathfrak{k} = Int(V).$$

Proof. As we indicated above the subalgebra $Int(V) = Span_R\{iL(x,x) : x \in V\}$ of $\mathfrak{D}$ contains the set $\{(L(x,y) - L(y,x)) : x, y \in V\}$ ([4, p. 518]). On the other hand

$$\begin{aligned}(L(x,y) - L(y,x)) &= A_{\alpha(x,y)} + R(x,y) - \left(A_{\alpha(y,x)} + R(y,x)\right) \\ &= 2R(x,y) \qquad \forall x,\ y \in V\end{aligned}$$

and by [6, p. 243, (4.1), (iii)] the algebra $\mathfrak{k}$ is generated by $\{R(x,y) : x,\ y, \in V\}$. Then we clearly have that

$$\mathfrak{k} \subset Int(V).$$

In order to prove the other inclusion we only need to show that $\dim \mathfrak{k} \geq \dim Int(V)$.

We use the notation from the Appendix concerning the canonical embedding of the compact Hermitian symmetric space N. It is well known that this embedding of N is extrinsic symmetric in the sense of Ferus [5]. It is also well known that this property of the embedding implies that its second fundamental form is *onto* which means that the set $\{\alpha(x,x) : x \in V\}$ generates $T_E(N)^\perp$. For the particular case of the canonical embedding of our space N, it is easy to see that if $E = [K]$ in G/K then $T_E(N)^\perp = \mathfrak{k}$ (the Lie algebra of K).

Let us take now in V a set of elements $\{z_1, \ldots, z_t\}$ such that the corresponding set

$$\{iL(z_j, z_j) : j = 1, \ldots, t\}$$

is a basis of $Int(V)$. Then, for any collection $\{b_1, \ldots, b_t\}$, of real numbers such that $\sum_{j=1}^t b_j iL(z_j, z_j) = 0$ we have $b_j = 0$ for $j = 1, \ldots, t$.

Now consider the set

$$\{\alpha(z_j, z_j) : j = 1, \ldots, t\} \subset T_E(N)^\perp.$$

It is linearly independent in $T_E(N)^{\perp}$ because, if for some real coefficients $\{b_1, \ldots, b_t\}$,

$$\sum_{j=1}^{t} b_j \alpha(z_j, z_j) = 0$$

then

$$0 = A_{\left(\sum_{j=1}^{t} b_j \alpha(z_j, z_j)\right)} = \sum_{j=1}^{t} b_j A_{\alpha(z_j, z_j)} = \sum_{j=1}^{t} b_j L(z_j, z_j).$$

which yields $b_j = 0$ for $j = 1, \ldots, t$. Then $\dim \mathfrak{k} \geq \dim Int(V)$ and the proof is complete. □

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