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The Riemann Surface of a Uniform Dessin

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1. Introduction

In this work we explore the relationship between dessins and Riemann surfaces. From the work of Belyi and Grothendieck [1], [8], Jones/Singerman [10, 11], or Malgoire/Voisin [16], it follows that underlying a dessin $\mathcal{H}$ on a surface X, there is a canonical complex structure on X. This makes X into a Riemann surface $R(\mathcal{H})$. In general, it is difficult to study the correspondence $\mathcal{H} \to R(\mathcal{H})$, but some progress can be made if $\mathcal{H}$ is a uniform dessin, which means that the vertex valencies are all equal, as are the edge valencies and the face valencies. In particular, we shall be interested in the cases where the function $\mathcal{H} \to R(\mathcal{H})$ is injective, which then means that there is a unique dessin underlying the Riemann surface X.

An outline of the paper is as follows. In §2 we recall the basic facts about triangle groups that we use. In §3 we state Belyi's Theorem and give Wolfart's altenative criterion in terms of triangle groups [30, 31]. We also introduce Belyi surfaces. In §4, we introduce dessins and maps and in §5 we explain how we associate a Riemann surface to a dessin. In §6, we discuss Belyi surfaces and dessins with 'nice' properties, such as smooth Belyi surfaces, platonic surfaces and regular dessins. We classify the smooth Belyi surfaces of genus $g \leq 2$. In §7 we do the same for platonic and quasiplatonic surfaces. In §8 we begin our study of the injectivity of the map $\mathcal{H} \to R(\mathcal{H})$. In §9 we discuss arithmetic triangle groups and introduce the important theorem of Margulis. In §10 we give our main theorems and in §11 we give examples where distinct maps and dessins may lie on the same Riemann surface. In §12, we compare the automorphism group of a dessin with the automorphism group of the underlying Riemann surface.

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2. Triangle groups

These are very well-known examples of plane discontinuous groups, but as they play a crucial role in our theory we outline some facts concerning them. Consider a triangle T with angles $\pi/l, \pi/m, \pi/n$.

Let

$$\mathcal{U} = \begin{cases} \Sigma, & \text{if } \frac{1}{l} + \frac{1}{m} + \frac{1}{n} > 1 \\ \mathbf{C}, & \text{if } \frac{1}{l} + \frac{1}{m} + \frac{1}{n} = 1 \\ \mathbf{H}, & \text{if } \frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1 \end{cases}$$

where Σ, $\mathbf{C}$, $\mathbf{H}$ are the sphere, complex plane and hyperbolic plane respectively.

Let Γ^* denote the group generated by reflections in the sides of T, and let Γ denote the subgroup of index 2 in Γ^* consisting of those transformations of Γ^* that are conformal, that is preserve orientation and angles. Then, in the third case, Γ is a Fuchsian group of the first kind of signature $(0; [l, m, n])$ and presentation $< x, y, z | x^l = y^m = z^n = xyz = 1 >$. If l,m,n are finite the elements x, y, z are elliptic elements of periods l, m, n respectively. If $\mathcal{U} = \mathbf{H}$ it is also useful to allow any of l, m, n to be ∞ in which case the corresponding element is parabolic. In this case the triangle T has a vertex on $\partial\mathbf{H}$, with a vertex having a zero angle at this vertex. The presentation of the group Γ is adapted in the following way: an equation of the form $u^\infty = 1$ is regarded as vacuous. The best-known example is the modular group $\mathrm{PSL}(2, \mathbf{Z})$, which has two elliptic generators, x, y, of orders 2 and 3 whose product is the parabolic elment, $z \to z + 1$. Thus $\mathrm{PSL}(2, \mathbf{Z})$ has signature $(0; [2, 3, \infty])$ and presentation $\langle x, y | x^2 = y^3 = 1\rangle \cong C_2 * C_3$.

2.1. Rigidity and maximality

An important property of triangle groups that distinguishes them from other Fuchsian groups is their rigidity. This means that given two triangle groups with the same signature then they are conjugate in $\mathrm{PSL}(2, \mathbf{R})$. Thus we may refer to the triangle group $\Gamma(l, m, n)$, to mean a triangle group with signature $(0; [l, m, n])$, and this is determined uniquely up to conjugacy in $\mathrm{PSL}(2, \mathbf{R})$. Also if Γ_1 is a triangle group and $\Gamma_1 < \Gamma_2$, then Γ_2 is a triangle group. If no such Γ_2 exists then Γ_1 is called a *maximal triangle group*. In [19] a complete list of non-maximal Fuchsian groups was given. These will all have some consequence for the geometry of dessins, but there are 2 families of inclusions that will be of particularly important for this paper. These are as follows:

1. $\Gamma(2, n, n) < \Gamma(2, 4, n)$ with index 2,
2. $\Gamma(2, n, 2n) < \Gamma(2, 3, 2n)$ with index 3.

3. Belyi's Theorem

Let $f : X \to Y$ be a non-constant holomorphic mapping between compact (connected) Riemann surfaces. Then thee exists a positive integer n and a finite set of points $C(f) \subset Y$ such that $|f^{-1}(y)| = n$, for all $y \in Y \setminus C(f)$ and $1 \leq |f^{-1}(y)| < n$, for all $y \in C(f)$. The elements of $C(f)$ are called the *critical* values of f and the function f is *branched* or *ramified* above the points in $C(f)$ and *unramified* above the other points of Y. If $C(f) = \emptyset$ then f is an unramified cover. Now let $T(x, y, z) \in \mathbf{C}[x, y, z]$ be an irreducible homogeneous polynomial with complex coefficients.. Then the algebraic curve

$$C_T = \{[x, y, z] \in P^2(\mathbf{C}) \text{ such that } T(x, y, z) = 0\}$$

can be normalized to give a compact Riemann surface X_T. Conversely, given a compact Riemann surface X, there exists an irreducible homogeneous polynomial $T(x, y, z) \in \mathbf{C}(x, y, z)$ such that X_T, the normalization of C_T is isomorphic to X, ([7]). We say that a Riemann surface is defined over a field $F \subseteq \mathbf{C}$ if $X \cong X_T$ for some polynomial $T(x, y, z) \in F[x, y, z]$. (We shall also use the corresponding affine curve.)

Definition. *Let X be a compact Riemann surface. Then a non-constant holomorphic mapping $\beta : X \to \Sigma$ is called a Belyi function if $C(\beta) \subseteq \{0, 1, \infty\}$.*

Theorem 3.1. (Belyi) *A compact Riemann surface is defined over the field of algebraic numbers $\overline{Q}$ if and only if there exists a Belyi function $\beta : X \to \Sigma$.*

Belyi [1] proved the 'only if' part of this theorem. The 'if' part is a consequence of Weil's rigidity theorem [29]. For other proofs of Weil's Theorem see [31, 12], and for a general discussion [11].

A useful alternative criterion, based on Belyi's Theorem, for a compact Riemann surface to be defined over $\overline{Q}$ is due to Wolfart [30, 31].

Theorem 3.2. *A compact Riemann surface X is defined over the field of algebraic numbers $\overline{Q}$ if and only if $X \cong \mathcal{U}/\Lambda$ where Λ is a finite index subgroup of a cocompact triangle group.*

Thus a compact Riemann surface X is defined over $\overline{Q}$ if and only if there exists a Belyi function $\beta : X \to \Sigma$ or equivalently when $X \cong \mathcal{U}/\Lambda$ where Λ is a finite index subgroup of a cocompact triangle group. We call such a surface a *Belyi surface*. It is important to note that the subgroup Λ need not be a surface group, (i.e. torsion-free). If it is, then we call X a *smooth Belyi surface.*

4. Dessins d'enfants

Roughly speaking a "dessin d'enfant" is a particular type of drawing on a compact orientable surface X that gives us a picture of a Belyi function. Basically, we want it to give a Schreier coset graph of a permutation representation of a triangle group that we can draw on X. Historically, the first examples considered were (oriented) *maps*. This theory goes back to Heffter[9] (or even Hamilton) and a modern version is described in some detail in [10]. We remind the reader of some definitions from this latter paper. We consider locally finite embeddings of a class of graphs, called *allowed graphs* in surfaces. The edges of the graphs are homeomorphic to the closed unit interval [0,1] or to a circle S^1. Those homeomorphic to S^1 will contain one vertex, but those homeomorphic to the interval will contain either two verices (at the images of 0 and 1), or just one vertex (at the image of 0). In the latter case we call the edge a free edge. Local finiteness means that a vertex only lies in finitely many edges, the *valency* of the vertex. To form a map we require embeddings of graphs into orientable surfaces with the property that the complement of the embedded graph is a union of two-cells (the faces of the map.) A *dart* of the map is an arrow pointing along an edge e towards a vertex v on e. Thus free edges only contain one dart, while other edges contain two darts. If Ω_d denotes the set of darts of a map, then on Ω_d we have an involution x which just interchanges the darts of a non-free edge and fixes the darts of every free edge. Thus the edges of a map are in bijective correspondence with the cycles of x on Ω_d, the two-cyles corresponding to non-free edges and the one-cycles correspondng to free edges. We also have a permutation y of Ω_d, which follows the darts aound Ω_d in an anticlockwise direction, (using the orientation of X.) We then find that the permutation $y^{-1}x$ follows the edges around a face, again following an anticlockwise orientation. As xy and $y^{-1}x$ have the same order, we find that

$$x^2 = y^m = (xy)^n = 1,$$

so that if $G = gp < x, y >$, then G is an image of the triangle group with signature $\Gamma(2, m, n)$, where m is the l.c.m. of the vertex valencies and n is the l.c.m. of the face sizes. As we do not want '2' to have a special role here, we need to extend our definition. (Also see §5 for remarks on the ordering of the periods.)

In 1975, Robert Cori [3], introduced a theory of planar hypermaps and in 1988, Corn and Singerman [5], pointed out that planarity was not a necessary assumption, and considered hypermaps in any orientable surface, also see [4]. A *hypermap* or *dessin d'enfant* is basically an embedding of a hypergraph in an orientable surface. There are many ways of drawing hypermaps and already in 1973, Walsh [27] had pointed out that there is a natural correspondence between hypermaps and bipartite maps. This leads us to our definition.

Definition. *A dessin $\mathcal{H}$ on a surface X is an embedding of a bipartite graph $\mathcal{B}$ into X such that the components of $X \setminus \mathcal{B}$ are simply connected.*

(Of course, we are identifying $\mathcal{B}$ with its embedding.) Now as $\mathcal{B}$ is bipartite, it has two kinds of vertices, which we may call black vertices and white vertices. Also, every edge of $\mathcal{B}$ can only join vertices of different colours. We suppose that the edges of the dessin are numbered from the set $\Omega = \{1, 2, 3 \cdots\}$. Each edge joins a black vertex to a white vertex, and incident with every black vertex, we have some of these edges. Using the anticlockwise orientation of the surface gives us a cyclic permutation of these edges. Thus if we have b black vertices, we have a permutation r_0 that is a product of b disjoint cycles. Similarly, if we have w white vertices then we get a permutation $r_1$consisting of w disjoint cycles, again using the anticlockwise orientation. We then find that the permutation $r_2 := (r_0 r_1)^{-1}$ describes the edges going around a face, each cycle of length u corresponding to a $2u$-gonal face. We illustrate the situation with the following example. Let

$r_0 = (1248)(365)(7)$

$r_1 = (1)(23)(4567)(8)$

$r_2 = (18473)(25)(6)$

Then we have the following diagram.

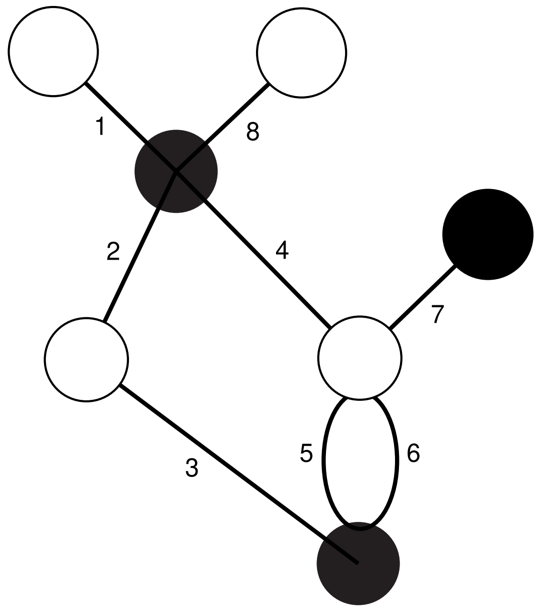

Figure 1

Definition. *A dessin $\mathcal{H}$ has type (l, m, n) if l is the least common multiple of the vertex valencies, m is the least common multlple of the edge valencies and n is half the least common multiple of the face valencies of $\mathcal{H}$. Alternatively, l, m, n are the least common multiples of the cycle lengths of the permutations r_0, r_1, r_2. For example in Fig.* 1 *we have a dessin of type* $(12, 4, 10)$.

A reader approaching the theory through the Grothendieck route as explained in [17] will meet some special dessins such as clean dessins or pre-clean dessins. Those following [10] will meet maps and maps without free edges. We clarify the two approaches by the following dictionary.

Dessin	Hypermap
Pre-clean dessin	Map
Clean dessin	Map without free edges
Black vertex	hypervertex
White vertex	hyperedge
Face	hyperface

Thus a dessin is map if and only if $m \leq 2$. In this case, every white vertex has degree one or two. We omit these white vertices, so that if the degree of the white vertex is two, it becomes the centre of an edge with two black vertices; if the degree is one it becomes the free end point of a free edge. Notice that the set Ω_d of darts becomes the underlying set Ω of the dessin $\mathcal{H}$.

5. The Riemann surface $R(\mathcal{H})$

It is clear from the above section that the group generated by r_0, r_1, r_2 is an image of a triangle group. Indeed, if the permutations r_0, r_1, r_2 have orders l, m, n, respectively, then we could use the triangle group Γ with periods l, m, n. However, we could replace any of these periods l, m, n with multiples of these periods, or even with ∞ and the procedure that we now describe does not substantially alter. So let Γ denote the triangle group $\Gamma(l, m, n)$ and let G denote the group generated by r_0, r_1, r_2 as in the previous section. We then have an epimorphism $\theta : \Gamma(l, m, n) \to G$. Let G_α denote the stabilizer of $\alpha \in \Omega$ and $H = \theta^{-1}(G_\alpha)$. Then H is a subgroup of $\Gamma(l, m, n)$ of index $|\Omega|$, and so H is a cocompact Fuchsian group, which has been called the hypermap subgroup of $\mathcal{H}$ or the fundamental group of $\mathcal{H}$ in $\Gamma(l, m, n)$. Note that even though the triple (l, m, n) when applied to hypermaps imposes a specific ordering on (l, m, n), (for example l is the l.c.m. of the vertex valencies) two triangle groups in which the periods are permuted are isomorphic so for many purposes we do not worry about the order of the periods. Note that H is uniquely determined up to conjugacy in $\Gamma(l, m, n)$ and so we can define

$$R(\mathcal{H}) = \mathcal{U}/H, \tag{1}$$

as the Riemann surface underlying the dessin $\mathcal{H}$. It follows from standard arguments that if we replace either of l, m, n by finite multiples of l, m, n then we obtain the same complex structure. Also if we replace any of these finite periods by infinite periods, then the standard way of defining the complex structure at parabolic fixed points will still lead to the same complex structure on $\mathcal{U}/H$. Thus equation (1) can be used to define the Riemann surface of a dessin $\mathcal{H}$.

Now $H \subset \Gamma(l, m, n)$ and so we can see the Belyi function β on $\mathcal{U}/H$ as the natural projection $\mathcal{U}/H \to \mathcal{U}/\Gamma$. Conversely, every Belyi function has this form [11], pp.569–571. Thus the complex structure on the surface underlying $\mathcal{H}$ is obtained by lifting the complex structure on Σ using the meromorphic function β.

This line of thought also shows that the dessin $\mathcal{H}$ is obtained by lifting the trivial dessin $\mathcal{H}_0$ on Σ, consisting of a unique black vertex at 0, a unique white white vertex at 1 and a unique edge that follows the shortest geodesic that joins 0 and 1. Thus given a dessin on a surface X, we can regard each black vertex as having a label 0, each white vertex as having a label 1 and each face as having a label ∞ at its centre. These are precisely the points that map under the Belyi function to $0,1,\infty$ in Σ.

6. Dessins and Belyi surfaces with nice properties

Recall that a Belyi surface is any Riemann surface of the form $\mathcal{U}/\Lambda$ where Λ is any subgroup of a cocompact triangle triangle group. By Belyi's theorem these are precisely the Riemann surfaces that arise out of algebraic curves defined over $\overline{Q}$ and hence for each genus there are an uncountable number of examples. We now specialise to looking at the most well-behaved dessins and Belyi surfaces.

We say that a dessin $\mathcal{H}$ of type (l, m, n) is *uniform* if the permutation r_0 is a product of l cycles, r_1 is a product of m cycles and r_2 is a product of n cycles. Alternatively, every vertex of $\mathcal{H}$ has valency l, every edge has valency m and every face has valency $2n$. In this case the fundamental group of $\mathcal{H}$ in $\Gamma(l, m, n)$ is torsion-free an so the Riemann surface $R(\mathcal{H})$ is a smooth Belyi surface as defined at the end of Section 3. Conversely, every smooth Belyi surface is the underlying Riemann surface of a uniform dessin.

An *automorphism* of a dessin $\mathcal{H}$ is a bijection of Ω that commutes with r_0 and r_1 (and hence with r_2). These bijections form a group Aut$\mathcal{H}$ that is the centralizer of H in the symmetric group on Ω, and is isomorphic to $N_\Gamma(H)/H$, [10]. As the automorphism group of the Riemann surface is $N_\mathcal{G}(H)/H$, where $\mathcal{G}$ is the automorphism group of $\mathcal{U}$, (this is the group of Möbius transformations $PSL(2,\mathbf{R})$ in the usual case when H is hyperbolic), we see that every automorphism of the dessin is an automorphism of the underlying Riemann surface. A dessin is called *regular* if its automorphism group is transitive on the set Ω. A Riemann surface that underlies a regular dessin is called *quasiplatonic*; if the dessin is also clean then the Riemann surface is called *platonic.* (In the quasiplatonic case the automorphism group is generated by elements of orders l, m, whose product has order n; the platonic case occurs when some of l, m, n is equal to 2.). Alternatively, the automorphism group of a quasiplatonic dessin is a *smooth* homomorphic image of the triangle group $\Gamma(l, m, n)$, where a smooth homomorphism is one that preserves the orders of elements of finite order – these are often called surface-kernel homomorphisms, as their kernels are surface groups. The automorphism group of a platonic surface is a smooth image of $\Gamma(2, m, n)$.

6.1. Smooth Belyi surfaces

A smooth Belyi surface of type (l, m, n) is any surface of the form $\mathcal{U}/H$, where H is a torsion-free subgroup of finite index in $\Gamma(l, m, n)$. The only Riemann surface of genus 0 is the Riemann sphere and as the platonic solids are uniform dessins, this is a smooth Belyi surface. So there is precisely one smooth Belyi surface of genus 0. Now suppose that we have a smooth Belyi surface of genus 1. We then see that (up to a permutation of l, m, n) that $(l, m, n) = (2, 4, 4), (3, 3, 3)$ or $(2, 3, 6)$. However as every Euclidean group isomorphic to $\Gamma(3, 3, 3)$ lies as a subgroup of index 2 in a group isomorphic to $\Gamma(2, 3, 6)$ [5], any genus 1 surface group that represents a smooth Belyi surface must lie in $\Gamma(2, 4, 4,)$ or in $\Gamma(2, 3, 6)$. This genus 1 surface group is a lattice and as usual we may assume that this is similar to the lattice Λ_τ where τ is a complex number with positive imaginary part. According to Lemma 1 of [23] $\Gamma(2, 4, 4)$ contains a lattice similar to Λ_τ if and only if $\tau \in \mathbf{Q}(i)$; $\Gamma(2, 3, 6)$ contains a lattice similar to Λ_τ if and only if $\tau \in \mathbf{Q}(\rho)$, where $\rho = \frac{1}{2}(-1 + \sqrt{-3})$. If F denotes the modular figure $\{z \in \mathbf{H}$ such that $|z| > 1, -\frac{1}{2} < Re(z) < \frac{1}{2}\}$ then F contains a countable infinity of such points $\tau \in \mathbf{Q}(i)$ or $\tau \in \mathbf{Q}(\rho)$ so that there are a countable infinity of smooth Belyi surfaces of genus 1.

6.2. Smooth Belyi surfaces of genus $g > 1$

We now need to consider genus g surface groups inside a hyperbolic triangle group Γ. If H_1 and H_2 are two such surface groups then they are conjugate in Γ if and only if the corresponding dessins $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic. This shows that the number of dessins on a surface of genus $g > 1$ of type (l, m, n) is finite. However, it is possible for H_1 and H_2 to be non-conjugate in Γ but to be conjugate in PSL$(2, \mathbf{R})$. In this case the two Belyi surfaces are conformally equivalent wheras the underlying dessins are not isomorphic so that the number of smooth Belyi surfaces is not greater than the number of isomorphism classes of dessins. We now present a table listing all uniform genus 2 dessins. In this table the first column gives the types of the dessin, the second column the number of edges of the embedded bipartite graph and the third column gives the number N of hypermaps of the given type. As we have seen, if we have a uniform dessin $\mathcal{H}$ of type (l, m, n) with the fundamental group of $\mathcal{H}$ in $\Gamma(l, m, n)$ equal to H, then H is isomorphic to a genus 2 surface group and so the Riemann-Hurwitz formula gives us the list of possible triples (l, m, n). To find the number of non-isomorphic dessins in each case we used Conder's small index subgroup programme [2], to find the number of conjugacy classes of subgroups of $\Gamma(l, m, n)$.

Type	index	N
(5,5,5)	5	4
(3,6,6)	6	4
(2,8,8)	8	4
(4,4,4)	8	6
(3,3,9)	9	4
(2,5,10)	10	7
(2,4,12)	12	6
(2,6,6)	12	13
(3,3,6)	12	8
(3,4,4)	12	10
(3,3,5)	15	9
(2,4,8)	16	19
(2,3,18)	18	9
(2,5,5)	20	21
(2,3,12)	24	25
(2,4,6)	24	40
(3,3,4)	24	28
(2,3,10)	30	20
(2,3,9)	36	37
(2,4,5)	40	75
(2,3,8)	48	77
(2,3,7)	84	155

The isomorphism classes of genus 2 uniform dessins

7. Platonic and quasiplatonic surfaces

Many of the Riemann surfaces that have a large number of automorphisms are platonic. Indeed, by the Riemann-Hurwitz formula, any Riemann surface of genus $g \geq 2$ with more than $24(g-1)$ automorphims is platonic, so that for example, all Hurwitz surfaces, (those with the maximal number $84(g-1)$ of automorphisms) are platonic. Also every Wiman surface (admitting maximal order cylic groups of order $4g+2$) are platonic as these are images of $\Gamma(2, 4, 4g+2)$. Other examples that occur are the Accola-Maclachlan surfaces, Bring's curve and every surface that comes from compactifying the quotient space of a principal congruence subgroup of the modular group.

Theorem 7.1. *All quasiplatonic surfaces of genus $g \leq 2$ are platonic. There exists a quasiplatonic surface of genus 3 that is not platonic.*

Proof. For $g = 0$ we only have one Riemann surface namely the Riemann sphere. If there is a quasiplatonic surface of type (l, m, n) on the sphere then we must have $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} > 1$, and hence one of l, m, n must equal 2 and so the dessin is platonic. The platonic solids show that the Riemann sphere is platonic. For $g = 1$ a quasiplatonic dessin must have type which is either of the form $(2, 3, 6)$, $(2, 4, 4)$ or $(3, 3, 3)$. However, it is shown in [5] that a normal subgroup of finite index in $\Gamma(3, 3, 3)$ (necessarily of genus 1) is also normal in $\Gamma(2, 3, 6)$, and this shows that a quasiplatonic dessin of genus 1 is necessarily platonic.

The genus 2 regular hypermaps were found in [5] and [6]. They correspond to regular hypermaps of types (5,5,5), (6,3,6), (4,4,4), (3,4,4), and (3,3,4) with automorphism groups of orders 5,6,8,12,24, respectively. This gives 5 quasiplatonic surfaces in genus 2. However, there are only 3 platonic surfaces of genus 2 and these have automorphism groups of orders 10, 24, 48. They correspond to regular maps of types (5,10), (4,6) and (3,8), and we can show that each of the above quasiplatonic surfaces are included in this set of 3 platonic surfaces. For example the quasiplatonic surface of type (5,5,5) is the same as the platonic surface of type (5, 10).

However in genus 3, we can find a quasiplatonic but not platonic surface as follows. There is a smooth homomorphism $\theta : \Gamma(3, 9, 9) \to C_9$ with kernel K say. By the Riemann-Hurwitz formula K has genus 3 and is the fundamental group of a regular dessin $\mathcal{K}$ of type (3,9,9). If $\mathcal{U}/K$ carried a regular dessin, then K is also normal in a triangle group $\Gamma(2, m, n)$. By considering the normalizer of K in $\mathrm{PSL}(2, \mathbf{R})$, (which is a Fuchsian group), we find that as it is triangle group that contains both $\Gamma(3, 9, 9)$ and $\Gamma(2, m, n)$ it must be a triangle group with a period 2, and so by the list of inclusion relationships in [19], it must be $\Gamma(2, 6, 9)$. But there is no smooth homomorphism from $\Gamma(2, 6, 9)$ onto a group of order 18, and so this is not possible. Thus $\mathcal{K}$ carries a regular dessin but no regular clean dessin.

8. The function $\mathcal{H} \to R(\mathcal{H})$

In (1) of Section 5 we defined the Riemann surface $R(\mathcal{H})$ that underlies a dessin $\mathcal{H}$. If $\mathcal{H}$ has genus $g > 0$ then R is certainly not surjective into $\mathcal{M}_g$, (moduli space of genus g) because by Belyi's Theorem, this image only contains Riemann surfaces defined over $\overline{Q}$. We now want to find families of dessins on which R is injective. We solved this problem for regular maps in [21]. The image then consists of Platonic surfaces, which for $g > 1$ forms a very small subset of moduli space. We will now restrict the domain to consist of uniform maps of type (m, n), that is we are assuming that $l = 2$. (As explained in §5 we are permuting periods for convenience, so l now corresponds to hyperedge valencies, m to hypervertex valencies and n to hyperface valencies.) In other words, we are posing the following problem: When does a uniform map of type (m, n) on a smooth Belyi surface X, uniquely determine the complex structure on X. Recall that the complex structure is determined as follows: Given the uniform map $\mathcal{H}$, of type (m, n) we have a surface subgroup H of $\Gamma(2, m, n)$, and the complex structure is the natural complex structure on $\mathcal{U}/H$.

8.1. Non-maximal maps

A uniform map $\mathcal{M}$ (i.e. uniform clean dessin) of type (m,n) is called maximal if the triangle group $\Gamma(2,m,n)$ is maximal. Otherwise we call the map non-maximal. (See §2.1.) From the inclusion relationships amongst triange groups listed in §2.1 we find the two general families of inclusion relationships, namely

- $\Gamma(2,n,n) < \Gamma(2,4,n)$ with index 2,
- $\Gamma(2,n,2n) < \Gamma(2,3,2n)$ with index 3.

Besides these, the only other inclusion relationships amongst triangle groups with a period equal to 2 are

- $\Gamma(2,8,8) < \Gamma(2,3,8)$ with index 6,
- $\Gamma(2,7,7) < \Gamma(2,3,7)$ with index 9.

Thus we have the following result.

Theorem 8.1. *A uniform map of type (m,n) is non-maximal if an only if $m = n$, $m = 2n$, or $2m = n$.*

8.2. Geometry of the inclusions

We first deal with the case $m = n$. We know that $\Gamma(2,n,n) < \Gamma(2,4,n)$. If $M < \Gamma(2,n,n)$ is a surface subgroup then in the usual way [10] we can construct a uniform map of type (n,n) on the Riemann surface $\mathcal{U}/M$ and this Riemann surface also has a uniform map of type $(4,n)$ defined on it. By constructing the fundamental region for $\Gamma(2,n,n)$ as a union of two copies of the fundamental region for $\Gamma(2,4,n)$, we see that the map of type $(4,n)$ is defined as follows. Start with a map $\mathcal{M}$ of type (n,n). Two edges are adjacent if they intersect in a vertex. We define a *medial* to be a geodesic joining the midpoints of two adjacent edges. The medial map of $\mathcal{M}$, denoted by $\mathrm{Med}(\mathcal{M})$, has as vertex set the mid-points of the edges of $\mathcal{M}$, the edge set of $\mathrm{Med}(\mathcal{M})$ are the medials of $\mathcal{M}$ and their are two kinds of face centres of $\mathrm{Med}\mathcal{M}$; firstly those at the vertices of $\mathcal{M}$ and secondly, those at the face-centres of $\mathcal{M}$. It is easy to see that the vertices of $\mathrm{Med}(\mathcal{M})$ have valency 4, and the face sizes are equal to n, so that $\mathrm{Med}(\mathcal{M})$ has type $(4,n)$ as expected.

Now we deal with the inclusion $\Gamma(2,n,2n) < \Gamma(2,3,2n)$. We will show that on the Riemann surface underlying a uniform map $\mathcal{M}_1$ of type $(2n,n)$, there is a uniform map $\mathrm{trunc}(\mathcal{M}_1)$ of type $(3,2n)$. This map is called the *truncation* of $\mathcal{M}_1$. In a similar way as in the construction of the medial map above this construction comes from expressing a fundamental region for $\Gamma(2,n,2n)$ as a union of three copies of a fundamental region for $\Gamma(2,3,2n)$. We start by trisecting each edge of $\mathcal{M}_1$. These points of trisection will be vertices v_1, v_2 of our new uniform map $\mathrm{trunc}(\mathcal{M}_1)$. Let V be the vertex of $\mathcal{M}_1$, such that Vv_1v_2 lie on an edge E of $\mathcal{M}_1$ and such that v_1 lies between V and v_2. The edge E bounds two faces F_1, F_2 of $\mathcal{M}_1$, each of which have V as a vertex. We can now construct two edges E_1, E_2 of $\mathcal{M}_1$, each passing through V and such that E_1 and E bound the face F_1, and E_2 and E bound the face F_2. Thus through V we have constructed the three edges E, E_1 and E_2. As above, we now trisect E_1 and E_2, and on these edges we choose the closest points (say u and w) on E_1 and E_2 respectively. The vertices of $\mathrm{trunc}\mathcal{M}_1$ consist of these points of trisection, (like v_1, v_2, u

and w above) and the edges of trunc$\mathcal{M}_1$ either lie along the edges of $\mathcal{M}_1$, (such as v_1v_2), or are the edges joining the closest vertices on E, E_1 and E_2, such as v_1u or v_1w above. These edges bound the faces of trunc$\mathcal{M}_1$. So each face consists of the middle thirds of the n edges of the faces of $\mathcal{M}_1$, together with n edges that join the closest vertices to a vertex of $\mathcal{M}_1$, such as v_1u and v_1w above. Thus trunc$\mathcal{M}_1$ has type $(3, 2n)$.

9. Arithmetic dessins

Let $\mathcal{H}$ denote a dessin of type (l, m, n). Then $\mathcal{H}$ is called *of arithmetic type* if $\Gamma(l, m, n)$ is an arithmetic Fuchsian group. We call such dessins *arithmetic*, otherwise we have a *non-arithmetic* dessin. From the results of Takeuchi [24] we may read off the possible types of arithmetic dessins. We will restrict ourselves to arithmetic clean dessins (arithmetic maps) of type (m, n) and we may assume that $m \leq n$. We also assume that $\frac{1}{m} + \frac{1}{n} < \frac{1}{2}$. We then find that the arithmetic maps are of the types

1. $(3, n)$, $7 \leq n \leq 12$, or $n = 14, 16, 18, 24, 30$,
2. $(4, n)$, $5 \leq n \leq 8$, or $n = 10, 12, 18$,
3. $(5, 5), (5, 6), (5, 8).(5, 10), (5, 20), (5, 30)$,
4. $(6, 6), (6, 8), (6, 12)$,
5. $(7, 7), (7, 14)$,
6. $(8, 8), (8, 16)$,
7. $(9, 18)$,
8. $(10, 10)$,
9. $(12, 12), (12, 24)$,
10. $(15, 30)$,
11. $(18, 18)$.

In a similar way, we say that a triangle group $\Gamma(l, m, n)$ has maximal type, if $\Gamma(l, m, n)$ is a maximal Fuchsian group. An interesting consequence of the results of Takeuchi and the list of smooth genus 2 maps given at the end of §6, is that every smooth dessin of genus 2 is arithmetic. Previously, this was only known for regular dessins of genus 2, ([13]).

9.1. Commensurability

Two Fuchsian groups Γ_1 and Γ_2 are said to be *commensurable* if $\Gamma_1 \cap \Gamma_2$ has finite index in both of them.

Definition. *Let Γ be a finite covolume Fuchsian group. Then we define*

$$\mathrm{Comm}(\Gamma) = \{t \in \mathrm{PGL}(2, \mathbf{R}) \textit{ such that } t\Gamma t^{-1} \textit{ is commensurable with } \Gamma\}.$$

Comm(Γ) is called the *commensurator* of Γ. Its importance is because of the following result due to Margulis [32].

Theorem 9.1. *Let Γ be a finite covolume Fuchsian group. Then Γ is a subgroup of finite index in* Comm(Γ) *if and only if* Γ *is non-arithmetic.*

We also define $\mathrm{Comm}^+(\Gamma)$ to be the subgroup of Comm(Γ) consisting of conformal transformations. We now suppose that Γ is a non-arithmetic cocompact Fuchsian triangle group with one elliptic period equal to 2, say $\Gamma = \Gamma(2,m,n)$. By Theorem 9.1, $\mathrm{Comm}^+(\Gamma)$ is also a cocompact Fuchsian triangle group that contains Γ. If we assume that $m \leq n$ then one of the following holds:

1. $\Gamma = \Gamma(2,m,n)$ is maximal and then $\mathrm{Comm}^+(\Gamma) = \Gamma$,
2. $m = n$ and then $\mathrm{Comm}^+(\Gamma) = \Gamma(2,4,n)$,
3. $2m = n$ and then $\mathrm{Comm}^+(\Gamma) = \Gamma(2,3,n)$.

10. Injectivity of R

We now examine some cases where the function R applied to maps is injective. In other words, we ask whether we may have nonisomorphic uniform maps on the same Riemann surface. As a map and its dual lie on the same Riemann surface we shall always identify a map with its dual map.

Theorem 10.1. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two non-arithmetic maps of maximal type. Then $R(\mathcal{M}_1)$ and $R(\mathcal{M}_2)$ are conformally equivalent if and only if $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic maps.*

Proof. Recall how we pass from a map $\mathcal{M}$ of type (m,n) to the Riemann surface $R(\mathcal{M})$. Given $\mathcal{M}$ we have three permutations X, Y, Z of the darts of $\mathcal{M}$ with

$$X^2 = Y^m = Z^n = XYZ = 1.$$

Let $\Gamma(2,m,n)$ be the triangle group with presentation

$$\langle x, y, z | x^2 = y^m = z^n = xyz = 1\rangle.$$

Now, in $\Gamma(2,m,n)$ the orders of the finite cyclic subgroups are 2, m and n, so that $\Gamma(2,m,n) \cong \Gamma(2,m',n')$ if and only if $m = m'$ and $n = n'$ or $m = n'$ and $n = m'$. Now surface subgroups of $\Gamma(2,m,n)$ correspond to the dual maps of maps obtained from surface subgroups of $\Gamma(2,n,m)$. As we are identifying a map with its dual we see from [10] that two surface subgroups M_1 and M_2 give isomorphic maps of type (m,n) if and only there exists $g \in \mathrm{PSL}(2,\mathbf{R})$ with $M_2 = gM_1g^{-1}$. Thus $R(\mathcal{M}_1) = \mathcal{U}/M_1 \cong R(\mathcal{M}_2) = \mathcal{U}/M_2$ are isomorphic. Conversely, if $\mathcal{U}/M_1$ and $\mathcal{U}/M_2$ are isomorphic Riemann surfaces where M_1 and M_2 are surface subgroups of triangle groups Γ_1 and Γ_2, then there exists $t \in \mathrm{PSL}(2,\mathbf{R})$ such that $M_1 = tM_2t^{-1}$ and so $M_1 \leq \Gamma_1 \cap t\Gamma_2t^{-1}$. Now Γ_1 and $t\Gamma_2t^{-1}$ are maximal, non-arithmetic and commensurable and so by Margulis's Theorem

$$\Gamma_1 = \mathrm{Comm}^+\Gamma_1 = \mathrm{Comm}^+(t\Gamma_2t^{-1}) = t\Gamma_2t^{-1},$$

and hence $\Gamma_1 = t\Gamma_2 t^{-1}$. Thus the map $\mathcal{M}_2$ associated to the inclusion $M_2 < \Gamma_2$ is isomorphic to the map $\mathcal{M}_1$ associated to the inclusion $M_1 < \Gamma_1$.

We now explore some ways in which we may have non-isomorphic maps $\mathcal{M}_1$, $\mathcal{M}_2$ that lie on conformally equivalent Riemann surfaces. By Theorem 10.1 this *cannot* occur if $\mathcal{M}_1$, and $\mathcal{M}_2$ are non-arithmetic of maximal type. Thus the conformal equivalence of $R(\mathcal{M}_1)$ and $R(\mathcal{M}_2)$ implies that

(i) at least one of $\mathcal{M}_1$, $\mathcal{M}_2$ are of non-maximal type, or

(ii) if $\mathcal{M}_1$, $\mathcal{M}_2$ are of maximal type then one of $\mathcal{M}_1$, $\mathcal{M}_2$ is arithmetic.

We show that both of $\mathcal{M}_1$, $\mathcal{M}_2$ are arithmetic. Suppose that $\mathcal{M}_i$ is of type (m_i, n_i) for $i = 1, 2$ and $\mathcal{M}_i = \mathcal{U}/M_i$, where M_i are surface groups. Then as $R(\mathcal{M}_1)$ and $R(\mathcal{M}_2)$ are conformally equivalent M_1 is conjugate to M_2 in $\mathrm{PSL}(2, \mathbf{R})$. After conjugation, we may suppose that $M_1 = M_2$. Thus $\Gamma(2, m_1, n_1)$ and $\Gamma(2, m_2, n_2)$ are commensurable and hence both of $\mathcal{M}_1$, $\mathcal{M}_2$ are arithmetic.

Theorem 10.2. *Suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ are maximal maps of arithmetic but of distinct types, and that $R(\mathcal{M}_1)$ is conformally equivalent to $R(\mathcal{M}_2)$. Then there are four cases (up to duality):*

(i) *$\mathcal{M}_1$ and $\mathcal{M}_2$ have two of the types* $(3, 7)$, $(3, 14)$, $(4, 7)$*;*

(ii) *$\mathcal{M}_1$ and $\mathcal{M}_2$ have types* $(3, 8)$, $(6, 8)$*;*

(iii) *$\mathcal{M}_1$ and $\mathcal{M}_2$ have types* $(4, 5)$, $(4, 10)$*;*

(iv) *$\mathcal{M}_1$ and $\mathcal{M}_2$ have types* $(3, 9)$, $(3, 18)$.

Proof. By the above discussion we may assume that both $\mathcal{M}_1$ and $\mathcal{M}_2$ are arithmetic. If $\mathcal{M}_1$ has type (3,7) and if $R(\mathcal{M}_1) = R(\mathcal{M}_2)$ then, if $R(\mathcal{M}_2)$ is of maximal type, then from Table 5 of [15] we find that $(m, n) = (3, 14)$ or $(4, 7)$. The results in (ii), (iii) and (iv) follow similarly from [25, 15].

11. Examples

(a) We first give an example of maps of type (3,7) and of type (7,14) on Klein's Riemann surface of genus 3. This is uniformized by a normal torsion-free subgroup M of genus 3 inside $\Gamma(2, 3, 7)$ and the automorphism group of the surface is PSL(2,7) of order 168. This contains a cyclic subgroup of order 7 corresponding to the subgroup $\Gamma(7, 7, 7)$ of $\Gamma(2, 3, 7)$ containing M as a normal subgroup of genus 7. Now $\Gamma(7, 7, 7) < \Gamma(2, 7, 14)$, which as shown in [22] corresponds to an edge-transitive but not regular one faced map on Klein's surface. This gives uniform maps of types (3,7) and (7,14) on Klein's surface. (As shown in [22] the Klein surface is the only platonic surface containing a one faced edge transitive but not regular map. It also contains the truncation of the map of type (7,14) which has type (3,14)). The Klein surface has equation

$$y^7 = x(x-1)^2.$$

(b) To get an example of maps of type (3,7) and (4,7) on the same Riemann surface, we consider the Riemann surface of genus 7 admitting PSL(2,8) of order 504 as automorphism group. This was discussed in detail by Macbeath[14]. This is the Hurwitz group of second largest order. This contains a map of type (3,7). Now PSL(2,8) contains an affine subgroup of order 56 whose inverse image in $\Gamma(2,3,7)$ is $\Gamma(2,7,7)$ corresponding to a map $\mathcal{M}$ of type (7,7). Its medial map Med$\mathcal{M}$ is a maximal map of type (4,7). The equation of the Macbeath surface is given in [14]. Also see example (d) in §11.1.

(c) We discuss examples of maps of type (4,5) and (4,10) on the Riemann surface B that corresponds to Bring's curve. This curve is given by the complete intersection

$$\sum_{i=1}^{5} x_i = 0 \qquad \sum_{i=1}^{5} x_i^2 = 0 \qquad \sum_{i=1}^{5} x_i^3 = 0 \tag{1}$$

in four-dimensional projective space $\mathbb{P}^4$ and B is known to have genus 4 and from (1) we see that it admits S_5 as automorphism group. (See [28], for a nice account of this surface). Another way of constructing this surface is by considering a smooth homomorphism $\phi : \Gamma(2,4,5) \to S_5$. If K is the kernel of ϕ then $\mathcal{U}/K$ is a Riemann surface of genus 4 admitting S_5 as its automorphism group and then we can show that $\mathcal{U}/K \cong B$. Thus B is the underlying Riemann surface of the regular map of type (4,5) of genus 4, which is actually the first regular map described by Coxeter and Moser in their famous account of regular maps in [4], Chapter 8, who point out that it goes back to Kepler's stellation of the regular dodecahedron, (see [28] for further details about this). Now S_5 has a subgroup $H =$Aff(1,5) (the one-dimensional affine group over the field with 5 elements, which has order 20. It is generated by two elements $\alpha : t \mapsto 2t$, $\beta : t \mapsto 3t + 3$ of order 4 whose procuct is $t \mapsto t + 1$ which has order 5. By [18] it follows that $\phi^{-1}(H)$ is the triangle group $\Gamma(4,4,5)$. This is a subgroup of index 2 in $\Gamma(2,4,10)$ and so $K < \Gamma(2,4,10)$ (but not normally). Thus B also admits a uniform map of type (4,10). As Weber points out in [28], this was observed by Threlfall, [26].

(d) All our examples lie on well-known Riemann surfaces with a large automorphism group. (Klein's surface of genus 3, Bring's surface of genus 4 and Macbeath's surface of genus 7.) In all these cases one of the pair of uniform maps is regular. We now discuss an example of maps of type (3,9) and (3,18) on the same Riemann surface. As there is no well-known Riemann surface of low genus admitting a regular map of type (3,9), or (3,18) we find a Riemann surface of genus 2 on which there are uniform (but not regular) maps of types (3,9) and (3,18). Consider the arithmetic groups $\Gamma_1 = \Gamma(3,2,9)$ and $\Gamma_2 = \Gamma(3,2,18)$. By [19], Γ_1 contains a subgroup isomorphic to $\Gamma(3,3,9)$ with index 4 and Γ_2 contains a subgroup of index 2 also isomorphic to $\Gamma(3,3,9)$. As any triangle groups with the same signature are conjugate in PSL(2,$\mathbf{R}$), we may assume that $\Gamma_1 \cap \Gamma_2 = \Gamma(3,3,9)$. Now $\Gamma(3,3,9)$ contains $\Gamma(0;3,3,3,3)$ with index 3 and this group contains a surface group Λ as a normal subgroup of index 3. In this way, we obtain a Riemann surface X of genus 2 admitting an automorphism group isomorphic to C_3 as a group of automorphisms. This Riemann surface X admits a uniform map of type (3,9) and a uniform map of type (3,18). To obtain further information about X, we show that X admits two distinct hypermaps of type (3,9,3). We consider the following group isomorphic to $\Gamma(3,9,3)$

$$\langle y_1, y_2, y_0 | y_1^3 = y_2^9 = y_0^3 = y_0 y_1 y_2 = 1\rangle$$

and the following two permutation representations ϕ_1, ϕ_2 of $\Gamma(3,9,3)$,

$$\phi_1(y_1) = (1,3,5)(2,6,7)(4,8,9)$$

$$\phi_1(y_2) = (1,7,3,4,5,8,2,9,6)$$

$$\phi_1(y_0) = (1,2,4)(3,6,8)(5,9,7).$$

We note that the automorphism group of this dessin is generated by

$$(1,4,2)(5,9,7)(6,3,8)$$

and thus this automorphism group is isomorphic to C_3. Similarly, we can get another dessin from the permutation representation ϕ_2 where

$$\phi_2(y_1) = (1,2,4)(3,6,8)(5,9,7)$$

$$\phi_2(y_2) = (1,9,3,4,7,8,2,5,6)$$

$$\phi_2(y_0) = (1,3,5)(2,6,7)(4,8,9)$$

and this also has an automorphism group generated by

$$(1,4,2)(5,9,7)(6,3,8).$$

The two hypermaps we have constructed are not isomorphic but one is obtained from the other by interchanging hypervertices and hyperedges, that is black and white vertices in the bipartite model introduced in Section 4. In particular they both lie on the same Riemann surface X which is the one above containing the maps of type (3,9) and (3,18). By the low index subgroup program we can show that there are two dessins of type (3,9,3) admitting C_3 as an automorphism group so these are the two we have found, and they lie on a unique Riemann surface. We will identify this surface X as the Riemann surface of the equation

$$y^3 = (x-1)^2(x-\rho)(x-\rho^2)$$

where $\rho = e^{2\pi i/3}$. This surface is a three sheeted branched cover of the Riemann sphere Σ with 4 branch points of order 2 over $1, \rho, \rho^2, \infty$ and so by the Riemann-Hurwitz formula, X has genus 2. Now $\beta : X \mapsto \Sigma$, defined by $\beta(x,y) = x^3$ is a Belyi function of degree 9. The points 0 and 1 have 3 inverse images and the point ∞ has one inverse image, and we can then show that the above dessin of type (3,3,9) (obtained from the (3,9,3,) dessin) is the Riemann surface of this dessin.

Note. Theorems 10.1 and 10.2 solve the problem of the injectivity of R when R is restricted to maps of maximal type.

11.1. Maps of non-maximal type

If $\mathcal{M}$ has type (m,n) with $\frac{1}{m}+\frac{1}{n}<\frac{1}{2}$ and $m \leq n$ and $\mathcal{M}$ is not of maximal type then either

(a) $m=n$ and on the same Riemann surface we can construct the medial map of type $(4,n)$ which is of maximal type if $n \neq 8$ (see §9.1)

(b) $n=2m$ and on the same Riemann surface we have the truncation of $\mathcal{M}$ of type $(3,2m)$ which is of maximal type.

(c) (the case $n=8$ in (a)). This comes from the chain of inclusions $\Gamma(2,8,8)<\Gamma(2,4,8)<\Gamma(2,3,8)$. If M is any surface subgroup of $\Gamma(2,8,8)$ this gives a uniform map of type (8,8) on $\mathcal{U}/M$, its medial map of type (4,8) and the truncation of the medial map of type (3,8). For example, M may be the normal subgroup of $\Gamma(2,3,8)$ of genus 2 corresponding to the Riemann surface of the hyperelliptic equation $y^2=x^5-x$, or the normal subgroup of genus 3 correspoding to the Riemann surface of the Fermat quartic.

(d) This comes from the inclusion $\Gamma(2,7,7)<\Gamma(2,3,7)$ with index 9. For example let M denote the surface subgroup corresponding to the Macbeath curve of genus 7 ([14]). The automorphism group of the Macbeath curve is PSL(2,8) of order 504. It is the second largest group for which the Hurwitz bound is attained; its genus is 7. $\Gamma(2,7,7)$ corresponds to the affine subgroup of order 56, and hence to a map with 28 edges, 8 vertices and 8 faces. This is Edmond's map of genus 7, (an embedding of the complete graph K_8) the smallest chiral map of genus $g>1$. (Also see example (b) in §11.)

12. Automorphisms of maps and Riemann surfaces

If $\mathcal{M}$ is a map then every automorphism of $\mathcal{M}$ extends to an automorphism of $R(\mathcal{M})$. For if $\mathcal{M}$ corresponds to a subgroup $M<\Gamma=\Gamma(2,m,n)$, then $\mathrm{Aut}\mathcal{M}=\Gamma/M<N/M=\mathrm{Aut}R(\mathcal{M})$, where N is the normalizer of M in $\mathrm{PSL}(2,\mathbf{R})$. We are interested in a converse of this result.

Theorem 12.1. *Let $\mathcal{M}$ be a uniform non-arithmetic map of maximal type. Then every conformal automorphism of $R(\mathcal{M})$ can be realized by an automorphism of $\mathcal{M}$.*

Proof. If M is the fundamental group of $\mathcal{M}$ in the triangle group Γ then $\mathrm{Aut}\mathcal{M} \cong \mathcal{N}_{-}(\mathcal{M})/\mathcal{M}$ and $\mathrm{Aut}R(\mathcal{M}) \cong \mathcal{N}_{\mathcal{G}}(\mathcal{M})/\mathcal{M}$, (see §6). We now show that $N_\Gamma(M)=N_{\mathcal{G}}(M)$. As $\Gamma \leq \mathcal{G}$, $N_\Gamma(M) \leq N_{\mathcal{G}}(M)$. Also $N_{\mathcal{G}}(M) \leq \mathrm{Comm}^+(\Gamma)$. As Γ is non-arithmetic, $\mathrm{Comm}^+(\Gamma)$ is discrete by Theorem 9.1. As Γ is maximal, $\mathrm{Comm}^+(\Gamma)=\Gamma$. Thus $N_{\mathcal{G}}(M) \leq \Gamma$ and so $N_\Gamma(M)=N_{\mathcal{G}}(M)$ as desired.

12.1. Hyperelliptic and reflexible maps

Definitions. *A map $\mathcal{M}$ is said to be hyperelliptic if it admits an automorphism group H of order 2 such that $\mathcal{M}/H$ has genus 0. A Riemann surface X is said to be reflexible if it admits an anticonformal involution T with fixed points.*

The idea of a Riemann surface being hyperelliptic and the idea of a map being reflexible are well-known. In [20] it is shown that a regular map $\mathcal{M}$ is hyperelliptic if and only if $R(\mathcal{M})$ is hyperelliptic. This is related to part (i) of the following result whose proof follows easily to that of Theorem 12.1.

Theorem 12.2. *Let $\mathcal{M}$ be a non-arithmetic uniform map of maximal type. Then*

(i) *$\mathcal{M}$ is hyperelliptic if and only if $R(\mathcal{M})$ is hyperelliptic,*

(ii) *$\mathcal{M}$ is reflexible if and only if $R(\mathcal{M})$ is reflexible.*

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