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An Analogue of the Krein-Milman Theorem for Star-Shaped Sets

Dedicated to Professor Bernd Silbermann on the occasion of his 60th birthday

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Abstract. Motivated by typical questions from computational geometry (visibility and art gallery problems) and combinatorial geometry (illumination problems) we present an analogue of the Krein-Milman theorem for the class of star-shaped sets. If $S \subseteq \mathbb{R}^n$ is compact and star-shaped, we consider a fixed, nonempty, compact, and convex subset K of the convex kernel $K_0 = \operatorname{ck}(S)$ of S, for instance $K = K_0$ itself. A point $q_0 \in S \setminus K$ will be called an extreme point of S modulo K, if for all $p \in S \setminus (K \cup \{q_0\})$ the convex closure of $K \cup \{p\}$ does not contain q_0. We study a closure operator $\sigma : \mathcal{P}(\mathbb{R}^n \setminus K) \longrightarrow \mathcal{P}(\mathbb{R}^n \setminus K)$ induced by visibility problems and prove that $\sigma(S_0) = S \setminus K$, where S_0 denotes the set of extreme points of S modulo K.

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1. Introduction

The motivation for this paper is twofold. On the one hand, a compact, star-shaped set S might be interpreted as an art gallery in the spirit of [13], where one can ask for minimally sufficient subsets of the boundary of S to control the whole set S in the sense of suitable visibility notions, see also [9]. From this point of view, our results are strongly related to the watchman route problem from computational geometry, see Section 7 of [20]. On the other hand, the problem of illuminating a convex body K from outside is well known in combinatorial geometry, cf. Chapters VI and VII of [1], and [9]. Identifying K with the convex kernel $\mathrm{ck}(S)$ of S, one can ask for optimal configurations of light sources (restricted to the boundary of S) to illuminate the whole boundary of $\mathrm{ck}(S)$. Having these two viewpoints in mind, we were able to find an analogue of the Krein-Milman theorem for star-shaped sets.

Convex sets play an important role in many branches of mathematics and its applications, in particular in geometry, integration theory, and mathematical optimization. Star-shaped sets are more general; e.g., they are also important in integration theory. A special field of research, in which convex and star-shaped sets are studied in common, is that of visibility problems. As introduced in [3], for a nonempty set $S \subset \mathbb{R}^n$ the convex kernel $\mathrm{ck}(S)$ consists, by definition, of all those points $x \in S$ such that for every $z \in S$ the line segment $\overline{zx}$ is contained in S. By definition, S is star-shaped if $\mathrm{ck}(S) \neq \emptyset$ and, by the way, S is convex if and only if $\mathrm{ck}(S) = S$. Star-shaped sets have been examined in connection with visibility problems, in particular in several papers by F. A. Toranzos. Consider for example a compact set $S \subset \mathbb{R}^n$ such that the interior $\mathrm{int}\, S$ of S is connected and S equals the closure of $\mathrm{int}\, S$. An element $z \in S$ *sees* x *via* S if the line segment $\overline{zx}$ is contained in S. In the literature, the set $\mathrm{st}(x, S)$ is, by definition, the set of all $z \in S$ which see x via S. An element $x \in S$ is called a *peak* of S if there exists some neighbourhood U of x such that for all $x' \in S \cap U$ one has $\mathrm{st}(x', S) \subseteq \mathrm{st}(x, S)$. Then it is proved in [19] that $\mathrm{ck}(S)$ equals the set of peaks of S. For other characterizations of $\mathrm{ck}(S)$ see also [6], [2], [17], and [4]; related characterization theorems were given by [15], [16], [2], [5], [8], [11], and [18].

In the present paper we start from some slightly modified visibility problem. We are interested to analyse the points $z \in S \setminus \mathrm{ck}(S)$ which "see many points in front of $\mathrm{ck}(S)$". For a compact and star-shaped set S and a nonempty compact and convex subset K of $K_0 := \mathrm{ck}(S)$, we study the operator $\sigma = \sigma_K : \mathcal{P}(\mathbb{R}^n \setminus K) \longrightarrow \mathcal{P}(\mathbb{R}^n \setminus K)$ such that for $A \subseteq \mathbb{R}^n \setminus K$ the set $\sigma(A)$ consists of A as well as all those points $x \in \mathbb{R}^n \setminus (A \cup K)$ such that there exists some $z \in A$ with $\overline{zx} \cap K = \emptyset$, but the ray with initial point z passing through x meets K. Thus, if $A \subseteq S \setminus K$, then $\sigma(A) \setminus A$ consists of those points of $S \setminus (A \cup K)$ which lie on some line segment $\overline{zx}$ with $z \in A$, $x \in K$. This means in particular that z sees x via S.

A point $q_0 \in S \setminus K$ will be called an *extreme point of* S *modulo* K, if $q_0 \notin \mathrm{conv}\,(K \cup \{p\})$ holds for all $p \in S \setminus (K \cup \{q_0\})$ or, equivalently (cf. Lemma 2.4), if $q_0 \notin \sigma(\{p\})$ holds for all such p.

The main result of our paper (see Theorem 2.7 below) states that the set S_0 of extreme points of S modulo K satisfies

$$K \cup \sigma(S_0) = S\,.$$

This just constitutes an analogue of the so-called Krein-Milman Theorem (cf. [7], but for Minkowski's earlier formulation [12, § 12]; a wider discussion is given in [14, § 1.4]) which states that *every compact, convex subset of* $\mathbb{R}^n$ *is the convex hull of its extreme points.*

It is easily seen that for convex $K \subseteq \mathbb{R}^n$ and $A \subseteq \mathbb{R}^n \setminus K$ the set $K \cup \sigma_K(A)$ is always star-shaped, cf. Proposition 2.3. Thus it turns out that a compact set $S \subseteq \mathbb{R}^n$ is star-shaped if and only if there exists some nonempty compact and convex subset K of $\mathbb{R}^n$ as well as some $A \subseteq S \setminus K$ with $S = K \cup \sigma_K(A)$, see Theorem 2.8. It should be noticed that in [10] we have already proved the following similar characterization of convex sets: If K is compact and $\mathbb{R}^n \setminus K$ is connected, then K is convex if and only if $\sigma_K = \sigma$ is a closure operator.

The fact that σ_K is a closure operator for convex K is used repeatedly in the present paper; therefore we recall the short proof of this part of our previous characterization to make the paper self-contained, see Theorem 2.1.

2. Results and proofs

In what follows, assume $n \geq 1$. For two points $a, b \in \mathbb{R}^n$ with $a \neq b$ let

$$\overline{ab} := \{a + \lambda \cdot (b - a) \mid 0 \leq \lambda \leq 1\} \tag{2.1}$$

denote the closed *line segment* between a and b, while

$$s(a, b) := \{a + \lambda \cdot (b - a) \mid \lambda \geq 0\} \tag{2.2}$$

means the *ray* with initial point a passing through b.

For $K \subseteq \mathbb{R}^n$ and $E := \mathbb{R}^n \setminus K$, define the operator $\sigma_K : \mathcal{P}(E) \longrightarrow \mathcal{P}(E)$ by

$$\sigma_K(A) := A \cup \left\{ b \in E \setminus A \;\middle|\; \begin{array}{l} \text{there exists some } a \in A \text{ with} \\ \overline{ab} \cap K = \emptyset, \text{ but} s(a, b) \cap K \neq \emptyset \end{array} \right\}. \tag{2.3}$$

Thus $\sigma_K(A) \setminus A$ consists of those points of $E \setminus A$ which "may be seen from A against K". We have the following basic result, cf. also [10].

Theorem 2.1. *Assume that $K \subseteq \mathbb{R}^n$ is convex, and put $E = \mathbb{R}^n \setminus K$. Then $\sigma_K : \mathcal{P}(E) \longrightarrow \mathcal{P}(E)$ is a closure operator; that means:*

(H0) *For all $A \subseteq E$ one has $A \subseteq \sigma_K(A)$.*

(H1) *For $A \subseteq B \subseteq E$ one has $\sigma_K(A) \subseteq \sigma_K(B)$.*

(H2) *For all $A \subseteq E$ one has $\sigma_K(\sigma_K(A)) = \sigma_K(A)$.*

Proof. (H0) is trivial. Regarding (H1) we see that if $A \subseteq B \subseteq E$ and $b \in \sigma_K(A) \setminus B$, then there exists some $a \in A$ with $\overline{ab} \cap K = \emptyset$ but $s(a, b) \cap K \neq \emptyset$. Since also $a \in B$, we conclude that $b \in \sigma_K(B)$, and (H1) follows.

To verify (H2), assume $A \subseteq E$ and $e_1 \in \sigma_K(\sigma_K(A))$. We have to show that $e_1 \in \sigma_K(A)$. If $e_1 \notin \sigma_K(A)$, there exists some $e_2 \in \sigma_K(A)$ with $\overline{e_2e_1} \cap K = \emptyset$ but $s(e_2, e_1) \cap K \neq \emptyset$. $e_1 \notin \sigma_K(A)$ implies $e_2 \notin A$, see Fig. 1. Therefore we have some $e_3 \in A$ with $\overline{e_3e_2} \cap K = \emptyset$ but $s(e_3, e_2) \cap K \neq \emptyset$. Choose $x_1 \in s(e_2, e_1) \cap K$ and $x_2 \in s(e_3, e_2) \cap K$. Then we get $e_1 \in \overline{e_2x_1}$ and $e_2 \in \overline{e_3x_2}$ and thus also

$$e_1 \in \text{ conv } \{e_2, x_1\} \subseteq \text{ conv } \{e_3, x_1, x_2\}.$$

Hence there exists some $x_3 \in \overline{x_1x_2}$ with $e_1 \in \overline{e_3x_3}$. In particular, we have $x_3 \in s(e_3, e_1)$. Since K is *convex*, we have $x_3 \in K$ and thus also $s(e_3, e_1) \cap K \neq \emptyset$. Moreover, one has $\overline{e_3e_1} \cap K = \emptyset$, because otherwise there would exist some $x \in K$ with $e_1 \in \overline{xx_3}$. However, this is not possible, because K is convex and $e_1 \notin K$. Altogether, we get $e_1 \in \sigma_K(\{e_3\}) \subseteq \sigma_K(A)$, in contradiction to our hypothesis $e_1 \notin \sigma_K(A)$. $\square$

Remark. In [10, Theorem 2.9] we proved also the following converse of Theorem 2.1: Assume that $K \subseteq \mathbb{R}^n$ is compact and that $E = \mathbb{R}^n \setminus K$ is connected. If, in addition, the operator $\sigma_K : \mathcal{P}(E) \longrightarrow \mathcal{P}(E)$ is a closure operator, then K is convex.

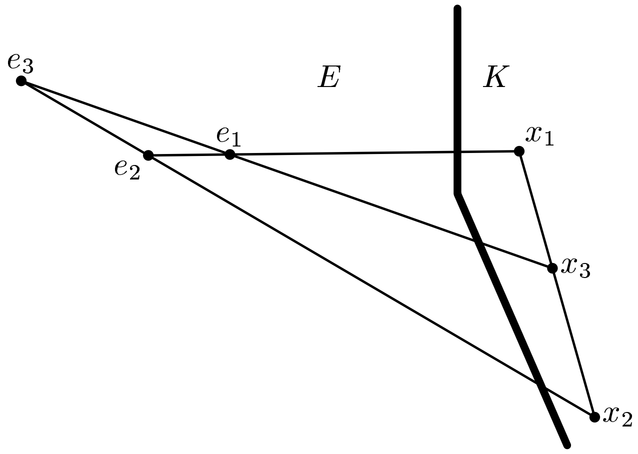

Figure 1

For a subset $K \subseteq \mathbb{R}^n$ and $A \subseteq E = \mathbb{R}^n \setminus K$ we put

$$\tau_K(A) := K \cup \sigma_K(A)\,. \tag{2.4}$$

We have the following simple

Lemma 2.2. *Assume K is a convex subset of $\mathbb{R}^n$ with $K \neq \emptyset$. Then for all $a \in E = \mathbb{R}^n \setminus K$ we have*

$$\text{conv}\ (K \cup \{a\}) = K \cup \sigma_K(\{a\}) = \tau_K(\{a\})\,. \tag{2.5}$$

Proof. Since K is convex, we obtain

$$\begin{aligned} \text{conv}\ (K \cup \{a\}) &= K \cup \{a\} \cup \{\lambda \cdot a + (1-\lambda) \cdot x | x \in K, 0 < \lambda < 1\} \\ &= K \cup \{a\} \cup \{f \in E\backslash\{a\} | f \in \overline{ax} \text{ for some } x \in K\} \\ &= K \cup \{a\} \cup \{f \in E\backslash\{a\} | s(a,f) \cap K \neq \emptyset \text{ and } \overline{af} \cap K = \emptyset\} \\ &= K \cup \sigma_K(\{a\})\,. \end{aligned}$$

$\square$

Now we turn over to star-shaped sets. We have the following

Proposition 2.3. *Assume K is a convex subset of $\mathbb{R}^n$ and $K \neq \emptyset$. Then for every subset $A \subseteq E = \mathbb{R}^n\backslash K$, the set $\tau_K(A) = K \cup \sigma_K(A)$ is star-shaped; more precisely, every point $x \in K$ is a star-centre of $\tau_K(A)$.*

Proof. Without loss of generality, assume that $A \neq \emptyset$. Suppose $x \in K$ and $y \in K \cup \sigma_K(A)$. Then we have $y \in K \cup \sigma_K(\{a\})$ for some $a \in A$; thus Lemma 2.2 implies

$$\overline{xy} \subseteq K \cup \sigma_K(\{a\}) \subseteq K \cup \sigma_K(A)$$

as claimed. □

In the rest of this paper, we want to prove also some converse of Proposition 2.3.

In what follows, assume that $S \subseteq \mathbb{R}^n$ is compact and star-shaped. Moreover, let

$$K_0 := \text{ ck}(S) := \{x \in S | \overline{xy} \subseteq S \text{ for all } y \in S\} \tag{2.6}$$

denote the *convex kernel* of S. Assume that $K \subseteq K_0$ is compact and convex with $K \neq \emptyset$, and put

$$E := \mathbb{R}^n \backslash K\,,\ d := \dim(\text{aff } (K))\,, \tag{2.7}$$

where aff means the affine closure. Furthermore, vol_m will denote the m-dimensional volume for $m \in \mathbb{N}$. Finally, for $p \in S \backslash K$ put

$$d(p) := \dim(\text{aff } (K \cup \{p\}))\,, \tag{2.8 a}$$

$$v(p) := \text{vol}_{d(p)}(\text{conv } (K \cup \{p\}))\,, \tag{2.8 b}$$

$$w(p) := \sup\{v(q) | q \in S\backslash K, p \in \sigma_K(\{q\})\}\,, \tag{2.8 c}$$

$$D(p) := w(p) - v(p)\,. \tag{2.8 d}$$

Clearly, one has $d(p) \in \{d, d+1\}$ and $v(p) \leq w(p)$ for all $p \in S\backslash K$. Note that $p \in \sigma_K(\{q\})$ implies $d(p) = d(q)$ whenever $p, q \in S \setminus K$. $D(p)$ will be called the *defect* of p. We have

Lemma 2.4. *For $q_0 \in S\backslash K$, the following statements are equivalent*:

(i) *For all $x \in K$ and all $p \in S\backslash(K \cup \{q_0\})$ one has $q_0 \notin \overline{xp}$.*

(ii) *For all $p \in S\backslash(K \cup \{q_0\})$ one has $q_0 \notin$ conv $(K \cup \{p\})$.*

(iii) *For all $p \in S\backslash(K \cup \{q_0\})$ one has $q_0 \notin \sigma_K(\{p\})$.*

(iv) $D(q_0) = 0$.

Proof. The equivalence of (i) and (ii) is clear, because K is convex. Moreover, (ii) and (iii) are equivalent by Lemma 2.2.
(iii) ⇒ (iv) follows directly from (2.8 c) and (2.8 d).
(iv) ⇒ (iii): Assume there exists some $p \in S\backslash(K \cup \{q_0\})$ with $q_0 \in \sigma_K(\{p\})$. Then we have $\sigma_K(\{q_0\}) \subseteq \sigma_K(\{p\})$, because σ_K is a closure operator by Theorem 2.1. Moreover, one has $p \notin \sigma_K(\{q_0\})$, because otherwise there would exist $x_1, x_2 \in K$ with $\{p, q_0\} \subseteq \overline{x_1x_2} \subseteq K$. Thus, by Lemma 2.2 the compact sets conv $(K \cup \{q_0\})$ and conv $(K \cup \{p\})$ satisfy conv $(K \cup \{q_0\}) \subsetneq$ conv $(K \cup \{p\})$. However, this means $v(q_0) < v(p) \leq w(q_0)$ and thus $D(q_0) > 0$, in contradiction to (iv). □

Definition 2.5. *A point $q_0 \in S\backslash K$ is called an extreme point of S modulo K, if the four equivalent conditions of Lemma 2.4 are satisfied.*

Let S_0 denote the set of extreme points of S modulo K. We want to show that S_0 is the uniquely determined minimal subset of S with $\tau_K(S_0) = S$. First we prove the following statement (see also [21, Theorem 6.2.17]).

Lemma 2.6. *The maps v_1: aff $(K) \cap S \longrightarrow \mathbb{R}$ and $v_2 : S \longrightarrow \mathbb{R}$ defined by*

$$v_1(p) := \mathrm{vol}_d(\mathrm{conv}\ (K \cup \{p\}))\,, \tag{2.9 a}$$

$$v_2(p) := \mathrm{vol}_{d+1}(\mathrm{conv}\ (K \cup \{p\})) \tag{2.9 b}$$

are continuous.

Proof. For $p \in S$ one has

$$v_2(p) = \frac{1}{d+1} \cdot\ \mathrm{vol}_d(\mathrm{conv}\,(K)) \cdot d(p,\ \mathrm{aff}\,(K))\,, \tag{2.10}$$

where $d(p,\ \mathrm{aff}\,(K))$ means the distance from p to the affine subspace aff (K) of $\mathbb{R}^n$.
(The equality (2.10) holds also in case $\mathrm{aff}\,(K) = \mathbb{R}^n$; then one has $v_2 \equiv 0$.) Thus v_2 is continuous. In case of v_1, it suffices to prove:

> For every $\varepsilon > 0$ there exists some $\delta > 0$ such that for all $p_1, p_2 \in$ aff $(K) \cap S$ with $||p_1 - p_2|| < \delta$ one has $|v_1(p_1) - v_1(p_2)| < \varepsilon$.

Here and in the sequel, $|| \cdot ||$ means the Euclidean norm. Moreover, for $x_0 \in \mathbb{R}^n$ and $r > 0$, we put

$$B(x_0, r) := \{x \in \mathbb{R}^n \mid ||x - x_0|| < r\}\,.$$

Without loss of generality, we may assume that 0 lies in the relative interior of K; that means aff (K) is a linear subspace of $\mathbb{R}^n$, and for some $r_0 > 0$ one has

$$B(0, r_0) \cap\ \mathrm{aff}\,(K) \subseteq K\,. \tag{2.11}$$

Choose some $\xi > 0$ such that for the compact set $S' :=$ aff $(K) \cap S$ we have

$$\mathrm{vol}_d\left((1+\xi)\cdot S'\right) - \mathrm{vol}_d(S') = \big((1+\xi)^d - 1\big) \cdot \mathrm{vol}_d(S') < \varepsilon\,. \tag{2.12}$$

Finally, put $\delta := \xi \cdot r_0$. Then for $p_1, p_2 \in S'$ with $||p_1 - p_2|| < \delta$ we have

$$\frac{1}{1+\xi} \cdot p_2 \ = \ \frac{1}{1+\xi} \cdot p_1 + \frac{\xi}{1+\xi} \cdot \left(\frac{1}{\xi} \cdot (p_2 - p_1)\right)$$

as well as

$$||\frac{1}{\xi} \cdot (p_2 - p_1)|| < \frac{\delta}{\xi} = r_0\,,$$

and thus $\frac{1}{1+\xi} \cdot p_2 \in \mathrm{conv}\,(K \cup \{p_1\})$ by (2.11). (See also Figure 2 in case $d = 2$.) Therefore, we get $p_2 \in (1+\xi) \cdot \mathrm{conv}\,(K \cup \{p_1\})$ and thus

$$\begin{aligned} v_1(p_2) - v_1(p_1) \ &\leq \ \mathrm{vol}_d(\mathrm{conv}\ (K \cup \{p_1, p_2\})) - \ \mathrm{vol}_d(\mathrm{conv}\ (K \cup \{p_1\})) \\ &\leq \ \mathrm{vol}_d\left((1+\xi)\cdot\ \mathrm{conv}\ (K \cup \{p_1\})\right) - \ \mathrm{vol}_d(\mathrm{conv}\ (K \cup \{p_1\})) \\ &\leq \ \big((1+\xi)^d - 1\big) \cdot\ \mathrm{vol}_d(S') \\ &< \ \varepsilon \end{aligned}$$

by (2.12). By exchanging the roles of p_1 and p_2, we get also $v_1(p_1) - v_1(p_2) < \varepsilon$ as claimed. $\square$

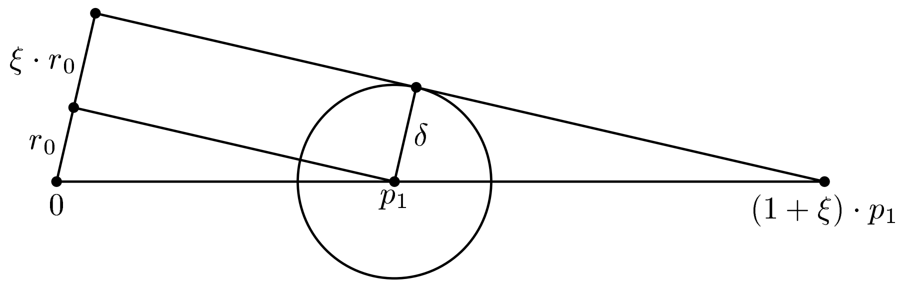

Figure 2

Now we are able to prove the announced analogue of the well-known Krein-Milman Theorem.

Theorem 2.7. *The set S_0 of extreme points of S modulo K satisfies*

$$\tau_K(S_0) = K \cup \sigma_K(S_0) = S\,. \tag{2.13}$$

If, moreover, $S' \subseteq S \backslash K$ satisfies $\tau_K(S') = S$, then one has $S_0 \subseteq S'$. In other words, $S' = S_0$ is the uniquely determined minimal subset of $S \backslash K$ satisfying $\tau_K(S') = S$.

Proof. Since $K \subseteq$ ck (S), we have $\overline{xp} \subseteq S$ for all $x \in K$ and all $p \in S$ and thus $\tau_K(S_0) \subseteq S$. If, moreover, $S' \subseteq S \backslash K$ satisfies $S_0 \backslash S' \neq \emptyset$, then, by the definitions of S_0 and σ_K, one has $q_0 \notin \sigma_K(S')$ for all $q_0 \in S_0 \backslash S'$ and thus $\tau_K(S') \neq S$. It remains to prove that $S \backslash (K \cup S_0) \subseteq \sigma_K(S_0)$. Assume $p \in S \backslash (K \cup S_0)$ and, according to (2.8 c), choose some sequence $(q_m)_{m \in \mathbb{N}}$ in $S \backslash K$ with $p \in \sigma_K(\{q_m\})$ for all $m \in \mathbb{N}$ and $\lim\limits_{m \to \infty} v(q_m) = w(p)$. Then there exists also some sequence $(x_m)_{m \in \mathbb{N}}$ in K as well as some sequence $(\lambda_m)_{m \in \mathbb{N}}$ in $[0,1]$ with

$$p = \lambda_m \cdot q_m + (1 - \lambda_m) \cdot x_m \ \text{ for all } \ m \in \mathbb{N}. \tag{2.14}$$

Since S and K are compact, we may assume that $(q_m)_{m \in \mathbb{N}}$ and $(x_m)_{m \in \mathbb{N}}$ converge to some $q \in S$ and some $x \in K$, respectively. Since $d(p) = d(q_m)$ holds for all $m \in \mathbb{N}$, Lemma 2.6 implies

$$\mathrm{vol}_{d(p)}\,(\mathrm{conv}\ (K \cup \{q\})) = \lim_{m \to \infty} v(q_m) = w(p) > v(p)$$

and thus $q \in S \backslash (K \cup \{p\})$ and $v(q) = w(p)$. Since also $p \neq x$, it follows that $(\lambda_m)_{m \in \mathbb{N}}$ converges, too. So one has $\lambda := \lim\limits_{m \to \infty} \lambda_m \in (0,1)$. Moreover, (2.14) implies

$$p = \lambda \cdot q + (1 - \lambda) \cdot x \tag{2.15}$$

and thus $p \in \sigma_K(\{q\})$. Since σ_K is a closure operator, any $q' \in S \backslash K$ satisfying $q \in \sigma_K(\{q'\})$ also fulfils $p \in \sigma_K(\{q'\})$. Therefore, (2.8 c) yields

$$w(q) \leq w(p) = v(q) \leq w(q)\,.$$

This means $D(q) = 0$. Thus we get $q \in S_0$ and $p \in \sigma_K(S_0)$ as claimed. $\square$

By summarizing Proposition 2.3 and Theorem 2.7, we get

Theorem 2.8. *Assume K is a compact and convex subset of $\mathbb{R}^n$ with $K \neq 0$. Then for a compact set $S \subseteq \mathbb{R}^n$ with $K \subseteq S$, the following statements are equivalent:*

(i) *S is a star-shaped set with $K \subseteq ck(S)$.*

(ii) *Some subset $A \subseteq S \backslash K$ satisfies $S = K \cup \sigma_K(A) = \tau_K(A)$.*

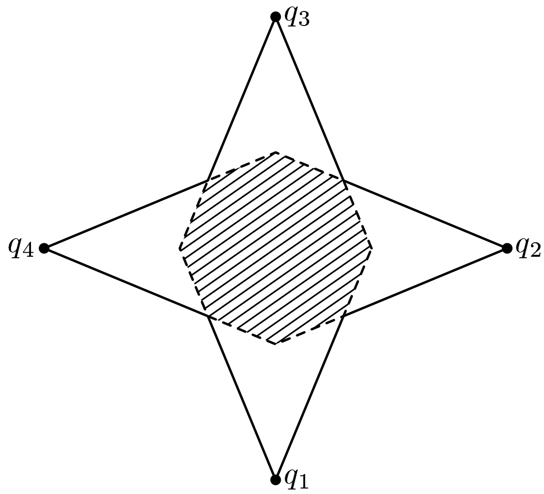

Figure 3

The final figures present two star-shaped sets. In Figure 3, the convex kernel K_0 is marked, and q_1, q_2, q_3, q_4 are the extreme points modulo K_0. In Figure 4, the convex kernel consists only of $\{x\}$, and the union of the line segments $\overline{a_1a_2}$ and $\overline{a_3a_4}$ is the set of extreme points modulo $\{x\}$.

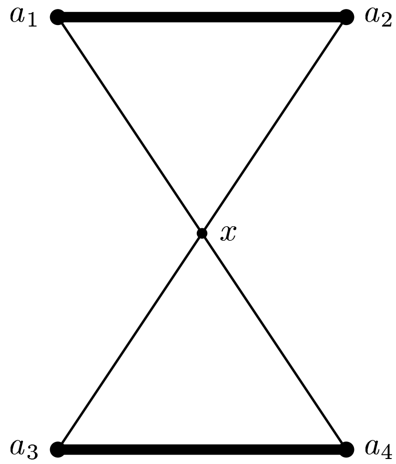

Figure 4

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