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On Pseudosymmetric Para-Kählerian Manifolds

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Abstract. In the present paper, we consider para-Kählerian manifolds satisfying various curvature conditions of the pseudosymmetric type. Let (M, J, g) be a para-Kählerian manifold. We prove the following theorems: The Ricci-pseudosymmetry of (M, J, g) reduces to the Ricci-semisymmetry. The pseudosymmetry as well as the Bochner-pseudosymmetry and the paraholomorphic projective-pseudosymmetry of the manifold (M, J, g) always reduces to the semisymmetry in dimensions > 4. The paraholomorphic projective-pseudosymmetry reduces to the pseudosymmetry in dimension 4. Moreover, we establish new examples of para-Kählerian manifolds being Ricci-semisymmetric (in dimensions $\geqslant 6$) as well as pseudosymmetric (in dimension 4) or Bochner-pseudosymmetric (in dimension 4). We have given examples of semisymmetric para-Kählerian manifolds in [7] and [8].

1. Preliminaries

By a para-Kählerian manifold we mean a triple (M, J, g), where M is a connected differentiable manifold of dimension $n = 2m$, J is a $(1,1)$-tensor field and g is a pseudo-Riemannian metric on M satisfying the conditions

$$J^2 = I, \qquad g(JX, JY) = -g(X, Y), \qquad \nabla J = 0,$$

for any $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is Lie algebra of vector fields on M, ∇ is the Levi-Civita connection of g and I is the identity tensor field.

Let (M,J,g) be a para-Kählerian manifold. By $R(X,Y)$ we denote its curvature operator, $R(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}$. The Riemann-Christoffel curvature tensor R, the Ricci curvature tensor S and the scalar curvature r are defined by

$$\begin{gathered} R(X,Y,Z,W)=g(R(X,Y)Z,W), \\ S(X,Y)=\operatorname{Tr}\{\,Z\mapsto R(Z,X)Y\,\}, \\ r=\operatorname{Tr}_g S. \end{gathered}$$

Let $\widetilde{S}$ be the Ricci operator given by $S(X,Y)=g(\widetilde{S}X,Y)$. For these tensor fields, the following identities are satisfied

$$\begin{gathered} R(JX,JY)=-R(X,Y), \quad R(JX,Y)=-R(X,JY), \\ S(JX,Y)=-S(JY,X), \quad S(JX,JY)=-S(X,Y), \\ \operatorname{Tr}\{\,Z\mapsto R(X,Y)JZ\,\}=-2S(X,JY), \\ \operatorname{Tr}\{\,Z\mapsto R(JZ,X)Y\,\}=S(X,JY). \end{gathered} \tag{1}$$

Next, for a symmetric $(0,2)$-tensor field A on M and $X,Y\in\mathfrak{X}(M)$, we define the endomorphism $X\wedge_A Y$ of $\mathfrak{X}(M)$ by

$$(X\wedge_A Y)Z=A(Y,Z)X-A(X,Z)Y, \quad Z\in\mathfrak{X}(M).$$

In the case when $A=g$, we shall write $\wedge$ instead of $\wedge_g$. The Bochner curvature tensor B is defined by [1], [8]

$$\begin{aligned} B(X,Y)=R(X,Y)&-\frac{1}{n+4}(X\wedge(\widetilde{S}Y)+(\widetilde{S}X)\wedge Y-(JX)\wedge(\widetilde{S}JY) \\ &-(\widetilde{S}JX)\wedge(JY)+2g(JX,Y)\widetilde{S}J+2g(\widetilde{S}JX,Y)J) \\ &+\frac{r}{(n+4)(n+2)}(X\wedge Y-(JX)\wedge(JY)+2g(JX,Y)J). \end{aligned}$$

Recall that the Bochner curvature $(0,4)$-tensor, $B(X,Y,Z,W)=g(B(X,Y)Z,W)$, has the same algebraic properties as the usual curvature tensor. Moreover, for this tensor, we have

$$\begin{gathered} B(JX,JY)=-B(X,Y), \\ \operatorname{Tr}\{\,Z\mapsto B(Z,X)Y\,\}=0, \quad \operatorname{Tr}\{\,Z\mapsto B(JZ,X)Y\,\}=0. \end{gathered} \tag{2}$$

The paraholomorphic projective curvature tensor P of (M,J,g) is defined in the following manner [9], [10], [7]

$$P(X,Y)=R(X,Y)-\frac{1}{n+2}(X\wedge_S Y-(JX)\wedge_S(JY)+2g(\widetilde{S}JX,Y)J).$$

Notice that this tensor has the following properties

$$\begin{gathered} P(X,Y)=-P(Y,X), \quad \operatorname{Tr}\{\,Z\mapsto P(Z,X)Y\,\}=0, \\ \textstyle\sum_i \varepsilon_i P(X,e_i,e_i,W)=\dfrac{1}{n+2}(nS(X,W)-rg(X,W)). \end{gathered} \tag{3}$$

In the above and in the sequel, $(e_1,e_2,\dots,e_n)$ is an orthonormal frame and ε_i is the indicator of e_i, $\varepsilon_i=g(e_i,e_i)=\pm1$.

2. Main result

For a $(0,k)$-tensor ($k \geqslant 1$) field T on a pseudo-Riemannian manifold (M,g), we define a $(0,k+2)$-tensor field $R \cdot T$ by the condition

$$(R \cdot T)(U,V,X_1,\dots,X_k) = -\sum_{s=1}^{k} T(X_1,\dots,R(U,V)X_s,\dots,X_k). \tag{4}$$

A pseudo-Riemannian manifold (M,g) is called: semisymmetric if $R \cdot R = 0$; Ricci-semisymmetric if $R \cdot S = 0$ (see [2], [6], [11]).

To formulate the notions of various pseudosymmetry type curvature conditions, we define also a $(0,k+2)$-tensor ($k \geqslant 1$) field $Q(g,T)$

$$Q(g,T)(U,V,X_1,\dots,X_k) = -\sum_{s=1}^{k} T(X_1,\dots,(U \wedge V)X_s,\dots,X_k). \tag{5}$$

A pseudo-Riemannian manifold (M,g) is said to be Ricci-pseudosymmetric [6] if there exists a function $L_S : M \to \mathbb{R}$ such that

$$R \cdot S = L_S Q(g,S).$$

Clearly, every Ricci-semisymmetric manifold is also Ricci-pseudosymmetric. The converse is not true in general [6]. However, we shall prove that the Ricci-pseudosymmetry reduces to the Ricci-semisymmetry in the class of para-Kählerian metrics.

Theorem 1. *Every Ricci-pseudosymmetric para-Kählerian manifold is Ricci-semisymmetric.*

Proof. Assume that a para-Kählerian manifold (M,J,g) satisfies the condition

$$(R \cdot S)(U,V,X,Y) = L_S Q(g,S)(U,V,X,Y). \tag{6}$$

Note that in virtue of (1) and (4), we have

$$(R \cdot S)(JU,JV,X,Y) = -(R \cdot S)(U,V,X,Y).$$

Thus by (6), we have

$$L_S Q(g,S)(U,V,X,Y) = -L_S Q(g,S)(JU,JV,X,Y).$$

Suppose that L_S is non-zero at a certain point $p \in M$. Then the above equality gives

$$Q(g,S)(U,V,X,Y) = -Q(g,S)(JU,JV,X,Y),$$

or in view of (5)

$$\begin{aligned}
&S(U,Y)g(V,X) - S(V,Y)g(U,X) + S(U,X)g(V,Y)\\
&-S(V,X)g(U,Y) = -S(Y,JU)g(X,JV) + S(Y,JV)g(X,JU)\\
&-S(X,JU)g(Y,JV) + S(X,JV)g(Y,JU).
\end{aligned}$$

This, by contraction with respect to V, X and applying of (1), we find

$$S(Y,U) = \frac{r}{n} g(Y,U),$$

that is, the manifold is Einstein. This gives $R \cdot S = 0$, which completes the proof. □

Now, we give examples of Ricci-semisymmetric para-Kählerian manifolds.

Example 1. Let (x_i) be the Cartesian coordinates in $\mathbb{R}^6$ and $\partial_i = \partial/\partial x^i$. Define a pseudo-Riemannian metric g by

$$[g(\partial_i, \partial_j)] = \begin{bmatrix} x_6 + x_3^2 & 0 & 1 & 0 & 0 & 0 \\ 0 & x_5 + x_4^2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and a $(1,1)$-tensor field J by

$$\begin{aligned} &J\partial_1 = -\partial_1 + (x_6 + x_3^2)\partial_3, \quad J\partial_2 = \partial_2 - (x_5 + x_4^2)\partial_4, \\ &J\partial_3 = \partial_3, \quad J\partial_4 = -\partial_4, \quad J\partial_5 = -\partial_5, \quad J\partial_6 = \partial_6. \end{aligned}$$

It is a straightforward verification that (J, g) is a para-Kählerian structure on $\mathbb{R}^6$ wich is Ricci-semisymmetric and non-semisymmetric (e.g., the component $(R \cdot R)_{131212} = -1/2 \neq 0$). To get Ricci-semisymmetric non-semisymmetric manifolds in dimensions $n = 6 + 2p$, $p \geqslant 1$, it is sufficient to take the product of the para-Kählerian manifold $(\mathbb{R}^6, J, g)$ and the standard para-Kählerian flat space $(\mathbb{R}^{2p}, J_0, g_0)$. □

A pseudo-Riemannian manifold (M, g) is said to be pseudosymmetric [6] if there exists a function $L_R : M \to \mathbb{R}$ such that

$$R \cdot R = L_R Q(g, R). \tag{7}$$

Clearly, every semisymmetric manifold is also pseudosymmetric. The converse is not true in general [6].

Theorem 2. *Let (M, J, g) be a pseudosymmetric para-Kählerian manifold.*
(a) *If $\dim M = 4$, then (M, J, g) is Ricci flat.*
(b) *If $\dim M > 4$, then (M, J, g) is semisymmetric.*

Proof. Assume that the condition (7) is satisfied everywhere on M. Now, in the same way as in the proof of Theorem 1, we have

$$L_R Q(g,R)(U,V,X,Y,Z,W) = -L_R Q(g,R)(JU,JV,X,Y,Z,W). \tag{8}$$

Suppose that the function L_R is non-zero at a point $p \in M$. Therefore, (8) takes the form

$$Q(g,R)(U,V,X,Y,Z,W) = -Q(g,R)(JU,JV,X,Y,Z,W).$$

Contracting, the last relation with respect to X, V, we obtain

$$\sum_i \epsilon_i Q(g,R)(U,e_i,e_i,Y,Z,W) = -\sum_i \epsilon_i Q(g,R)(JU,Je_i,e_i,Y,Z,W)$$

which, with the help (5), can be rewritten in the following form

$$\begin{aligned}
&nR(U,Y,Z,W) - R(U,Y,Z,W) + R(Y,U,Z,W) + R(Z,Y,U,W)\\
&+ R(W,Y,Z,U) + g(U,Z)S(Y,W) - g(U,W)S(Y,Z) =\\
&+ R(U,Y,Z,W) - R(Y,U,Z,W) - R(JU,W,Y,JZ) + R(JU,Z,Y,JW)\\
&+ 2g(Y,JU)S(Z,JW) - g(Z,JU)S(W,JY) + g(W,JU)S(Z,JY).
\end{aligned}$$

Hence, using (1) and the first Bianchi identity, we get

$$\begin{aligned}
&(n-4)R(U,Y,Z,W) - 2g(Y,JU)S(Z,JW) + g(Z,JU)S(W,JY)\\
&\quad -g(W,JU)S(Z,JY) + g(U,Z)S(Y,W) - g(U,W)S(Y,Z) = 0.
\end{aligned} \tag{9}$$

(a) Let $n = 4$. Substituting JY instead of Y in (9), contracting the obtained relation with respect to Y, U and using (1), we find $S = 0$.

(b) Let $n > 4$. Contracting (9) with respect to Y, Z, we find

$$S(U,W) = \frac{r}{n} g(U,W).$$

This implies $R \cdot S = 0$. Using this fact in (9), we obtain $R \cdot R = 0$. $\square$

Examples of semisymmetric para-Kählerian manifolds can be found in [7] and [8]. Below, we give an example of a 4-dimensional pseudosymmetric para-Kählerian manifold which is non-semisymmetric.

Example 2. Let U be the open subset of $\mathbb{R}^4$ consisting of points at which $x_1 > 0$. Define a pseudo-Riemannian metric g by

$$[g(\partial_i, \partial_j)] = \begin{bmatrix} -2x_1 & 0 & 0 & 0 \\ 0 & 2x_1 & 0 & 0 \\ 0 & 0 & 2x_1^{-1} & -x_2x_1^{-1} \\ 0 & 0 & -x_2x_1^{-1} & 2x_2^2x_1^{-1} - 2x_1 \end{bmatrix}$$

and a $(1,1)$-tensor field J by

$$\begin{aligned}
&J\partial_1 = \partial_2, \quad J\partial_2 = \partial_1, \quad J\partial_3 = -x_2x_1^{-1}\partial_3 - (2x_1)^{-1}\partial_4,\\
&J\partial_4 = \left(2x_2^2x_1^{-1} - 2x_1\right)\partial_3 + x_2x_1^{-1}\partial_4.
\end{aligned}$$

One verifies that (J, g) is para-Kählerian structure on U. Moreover, it can be checked that the structure is non-semisymmetric and pseudosymmetric with $L_R = (2x_1^3)^{-1}$. □

Now, we consider a para-Kählerian manifold, whose Bochner curvature tensor fulfills the condition

$$R \cdot B = L_B Q(g, B),$$

where L_B is a function on M. Such a manifold will be called Bochner-pseudosymmetric. In the special case when $R \cdot B = 0$, the manifold is said to be Bochner-semisymmetric [8].

Theorem 3. *Every Bochner-pseudosymmetric para-Kählerian manifold of dimension $n > 4$ is Bochner-semisymmetric.*

Proof. Let (M, J, g) be a para-Kählerian manifold which is Bochner-pseudosymmetric. In the same manner as in the proof of Theorem 1, we find

$$L_B Q(g, B)(U, V, X, Y, Z, W) = -L_B Q(g, B)(JU, JV, X, Y, Z, W).$$

Let L_B be non-zero at $p \in M$. Then we have

$$Q(g, B)(U, V, X, Y, Z, W) = -Q(g, B)(JU, JV, X, Y, Z, W),$$

or in view of (5) with $T = B$

$$\begin{aligned}
&B(U, Y, Z, W)g(V, X) - B(V, Y, Z, W)g(U, X) + B(X, U, Z, W)g(V, Y)\\
&- B(X, V, Z, W)g(U, Y) + B(X, Y, U, W)g(V, Z) - B(X, Y, V, W)g(U, Z)\\
&+ B(X, Y, Z, U)g(V, W) - B(X, Y, Z, V)g(U, W)\\
&= B(JV, Y, Z, W)g(X, JU) - B(JU, Y, Z, W)g(X, JV)\\
&+ B(X, JV, Z, W)g(Y, JU) - B(X, JU, Z, W)g(Y, JV) + B(X, Y, JV, W)g(Z, JU)\\
&- B(X, Y, JU, W)g(Z, JV) + B(X, Y, Z, JV)g(W, JU) - B(X, Y, Z, JU)g(W, JV).
\end{aligned}$$

Contracting the last identity with respect to X, V and next using (2) and the first Bianchi identity for B, we find

$$(n - 4)B(U, Y, Z, W) = 0.$$

This gives immediately $B = 0$, which completes the proof. □

Remark 1. In paper [8], we have shown that for a para-Kählerian manifold, the Bochner semisymmetry always implies the semisymmetry at points where the Bochner tensor does not vanish.

The assertion of Theorem 3 does not hold in dimension 4; see the following example.

Example 3. Let h be a function on $\mathbb{R}$ such that $h \neq 0$ and $h' \neq 0$ at any point. On $\mathbb{R}^4$, define a pseudo-Riemannian metric g by

$$[g(\partial_i, \partial_j)] = \begin{bmatrix} -h'(x_1)/2 & 0 & 0 & 0 \\ 0 & h(x_1) & 0 & 0 \\ 0 & 0 & h'(x_1)/2 & -x_2 h'(x_1) \\ 0 & 0 & -x_2 h'(x_1) & -h(x_1) + 2x_2^2 h'(x_1) \end{bmatrix}$$

and a $(1,1)$-tensor field J by

$$J\partial_1 = \partial_3, \quad J\partial_2 = 2x_2\partial_3 + \partial_4, \quad J\partial_3 = \partial_1, \quad J\partial_4 = \partial_2 - 2x_2\partial_2.$$

Then (J, g) is a para-Kählerian structure which is non-pseudosymmetric and Bochner pseudosymmetric with

$$L_B = \frac{h(x_1)h''(x_1) - h'^2(x_1)}{h^2(x_1)h'(x_1)}.$$ □

A para-Kählerian manifold (M, J, g) will be called paraholomorphic projective-pseudosymmetric if there exists a function $L_P : M \to \mathbb{R}$ such that

$$R \cdot P = L_P Q(g, P).$$

Theorem 4. *Let (M, J, g) be a paraholomorphic projective-pseudosymmetric para-Kählerian manifold.*
(a) *If $\dim M = 4$, then (M, J, g) is Ricci flat and pseudosymmetric.*
(b) *If $\dim M > 4$, then (M, J, g) is semisymmetric.*

Proof. If $R \cdot P = 0$ at a certain point of M, then $R \cdot R = 0$ at this point (it was really shown in the paper [7], Theorem 1, since this is a pointwise property). In the sequel, we assume that $R \cdot P \neq 0$ at a point of M. Let G be the contracted tensor P,

$$G(X, W) = \sum_i \epsilon_i P(X, e_i, e_i, W).$$

Thus, by (3), we have

$$G(X, W) = \frac{1}{n+2}(nS(X, W) - rg(X, W)). \tag{10}$$

Since (M, J, g) is paraholomorphic projective-pseudosymmetric, the following formula is fulfilled

$$(R \cdot P)(U, V, X, Y, Z, W) = L_P Q(g, P)(U, V, X, Y, Z, W). \tag{11}$$

Contracting (11) with respect to Y, Z and using (4) and (5), we obtain

$$(R \cdot G)(U, V, X, W) = L_P Q(g, P)(U, V, X, W).$$

Hence, using (10) and (4), we get

$$(R \cdot S)(U, V, X, W) = L_P Q(g, S)(U, V, X, W).$$

This by Theorem 1 implies $R \cdot S = 0$. Note that L_P is non-zero at p. Then $Q(g, S) = 0$ at this point. Therefore, in virtue of (3) and (11), we find $R \cdot R = L_P Q(g, R)$. Thus, (M, J, g) is pseudosymmetric. To finish the proof it is sufficient to use Theorem 2. □

Final remarks. 1. The notion of the para-Kählerian manifold used in the presented paper is different from that applied in papers [6], [5], where the structure tensor J is an almost complex structure and the metric g is positive definite.

2. The local components of geometric objects (that is, the Levi-Civita connection, the Riemann, Ricci, Bochner and paraholomorphic projective curvature tensors and the scalar curvature) in our examples were calculated with the help of *Mathematica* programs.

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