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# An Explicit Computation of "bar" Homology Groups of a Non-unital Ring

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## 0. Introduction

Let $R$ be the ring of integers in a number field $F$, $\Lambda$ an $R$-order in a semi-simple $F$-algebra $\Sigma$, and $\Gamma$ a maximal $R$-order in $\Sigma$ containing $\Lambda$. Then there exists $s \in \mathbb{Z}$, $s > 0$, such that $s\Gamma \subseteq \Lambda$, and so $s\Gamma$ is a 2-sided ideal in both $\Lambda$ and $\Gamma$. Put $\underline{q} = s\Gamma$. Then we have a Cartesian square

$$\begin{array}{ccc} \Lambda & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ \Lambda/\underline{q} & \longrightarrow & \Gamma/\underline{q} \end{array} .$$

If the relative $K$-groups $K_n(\Lambda, \underline{q})$ and $K_n(\Gamma, \underline{q})$ coincide (see below), then one can get for all $n \in \mathbb{Z}$ the long exact Mayer-Vietoris sequence

$$\cdots \to K_{n+1}(\Gamma/\underline{q}) \to K_n(\Lambda) \to K_n(\Lambda/\underline{q}) \oplus K_n(\Gamma) \to K_n(\Gamma/\underline{q}) \to K_{n-1}(\Lambda) \to \cdots .$$

This paper was inspired by a desire to understand the relative groups $K_n(s\Gamma) := K_n(\widetilde{s\Gamma}, s\Gamma)$ (see below) where $\widetilde{s\Gamma}$ is the ring obtained from $s\Gamma$ by adjoining a unit to $s\Gamma$. Since the

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additive group of $s\Gamma$ is free as a $\mathbb{Z}$-module, we are led to compute explicitly $\mathrm{Tor}_n^{\widetilde{sA}}(\mathbb{Z},\mathbb{Z})$ and hence the so-called bar homology groups $HB_n(sA)$ (see Theorem 1) in the general setting of $A$ being a ring with identity such that the additive group of the ideal $sA$ of $A$ is a free $\mathbb{Z}$-module. We now explain the mathematical context of our result.

If $\Lambda$ is a ring with identity, let $K_n(\Lambda)$ be the Quillen $K$-groups $\pi_n(BGL(\Lambda)^+)$ (cf. [1]). If $I$ is an 2-sided ideal of $\Lambda$, the relative $K$-group $K_n(\Lambda, I)$ are defined for all $n \geq 1$ as the homotopy groups $\pi_n(F(\Lambda, I))$ of the homotopy fibre $F(\Lambda, I)$ of the morphism $BGL(\Lambda)^+ \to B\overline{GL}(\Lambda/I)^+$ where $\overline{GL}(\Lambda/I)$ is the image of $GL(\Lambda)$ under the canonical map $GL(\Lambda) \to GL(\Lambda/I)$. The fibration $F(\Lambda, I) \to BGL(\Lambda)^+ \to B\overline{GL}(\Lambda/I)^+$ then yields a long exact sequence

$$\cdots \to K_n(\Lambda, I) \to K_n(\Lambda) \to K_n(\Lambda/I) \to K_{n-1}(\Lambda, I) \to K_{n-1}(\Lambda) \to K_{n-1}(\Lambda/I) \to \cdots .$$

If $B$ is any ring without unit, and $\tilde{B}$ is the ring with unit obtained by formally adjoining a unit to $B$, i.e., $\tilde{B} =$ the set of all $(b, s) \in B \times \mathbb{Z}$ with multiplication defined by $(b, s)(b', s') = (bb' + sb' + s'b, ss')$. Define $K_n(B)$ as $K_n(\widetilde{B}, B)$. If $\Lambda$ is an arbitrary ring with identity containing $B$ as 2-sided ideal, then $B$ is said to satisfy excision for $K_n$ if the canonical map

$$K_n(B) := K_n(\widetilde{B}, B) \to K_n(\Lambda, B)$$

is an isomorphism for any ring $\Lambda$ containing $B$.

In [3] A. A. Suslin proves that a ring $B$ satisfies excision for $K_n$-theory for $n \leq r$ if and only if

$$\mathrm{Tor}_1^{\tilde{B}}(\mathbb{Z},\mathbb{Z}) = \ldots = \mathrm{Tor}_r^{\tilde{B}}(\mathbb{Z},\mathbb{Z}) = 0.$$

It thus becomes important to compute $\mathrm{Tor}_n^{\tilde{B}}(\mathbb{Z},\mathbb{Z})$.

Now, let $B_*(\Lambda)$ be the complex

$$B_*(\Lambda): \quad \cdots \to \Lambda^{\otimes^n} \xrightarrow{d_n} \Lambda^{\otimes^{n-1}} \cdots \to \Lambda^{\otimes^2} \xrightarrow{d_1} \Lambda$$

where the differentials $d_n$ are defined by

$$d_n(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n-1} (-1)^{i-1}(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n).$$

Let $HB_n(\Lambda)$ be the $n$-th homology group of $B_*(\Lambda)$ (cf. [2] P. 12). In [3] A. A. Suslin also proves that for any ring $\Lambda$ (maybe without identity) and for any $n > 0$, there is the canonical homomorphism $\mathrm{Tor}_n^{\tilde{\Lambda}}(\mathbb{Z},\mathbb{Z}) \to HB_n(\Lambda)$, which is an isomorphism for all $n$ if the additive group of the ring $\Lambda$ is torsion-free. This explains the motivation for our study in this paper.

## 1. Main Result

Let $A$ be a ring with identity. Let $s \in \mathbb{Z}$, $s > 0$. Then $sA$ is an ideal of $A$. From now on, we assume that the additive group of $sA$ is a free $\mathbb{Z}$-module with basis $sx_i, i \in I$, where $I$ is a totally ordered index set with the smallest element 1. We can assume that there is a $\lambda \in \mathbb{Z}$ such that $\lambda s x_1 = s$. This is because there exist $a_1, a_2, \ldots, a_m \in \mathbb{Z}$ such that

$s = a_1sx_1 + \cdots + a_msx_m$ and for the vector $(a_1, a_2, \ldots, a_m)$ there is an invertible $m \times m$ matrix $g \in GL_m(\mathbb{Z})$ such that $(a_1, a_2, \ldots, a_m)g = (\lambda, 0, \ldots, 0)$ for some $\lambda \in \mathbb{Z}$. Thus $g^{-1}(sx_1, 0, \ldots, 0)^t$, $g^{-1}(0, sx_2, \ldots, 0)^t$, $\ldots$, $g^{-1}(0, 0, \ldots, sx_m)^t$ as well as $sx_i, i \in I, i > m$, constitute the required basis. We want to calculate the groups $\mathrm{Tor}_n^{\widetilde{sA}}(\mathbb{Z}, \mathbb{Z})$ for such a ring $A$ and any $n$.

For any positive integer $n$, denote the cartesian product of $n$ copies of $I$ by $I^n$. We define a partition of $I^n$ as $I^n = I_1^n \bigcup I_2^n \bigcup I_3^n$ by induction as follows:

$$\begin{gathered}
I_1^1 = \emptyset, \qquad I_2^1 = I, \qquad I_3^1 = \emptyset; \\
I_1^2 = \{(1, \alpha_2) \in I^2 | \alpha_2 \in I_2^1 \textstyle\bigcup I_3^1\}, \qquad I_2^2 = \emptyset, \qquad I_3^2 = I^2 \setminus (I_1^2 \textstyle\bigcup I_2^2); \\
\vdots \\
I_1^n = \{(\alpha_1, \alpha_2, \ldots, \alpha_{n-2}, 1, \alpha_n) \in I^n | (\alpha_1, \alpha_2, \ldots, \alpha_{n-2}, \alpha_n) \in I_2^{n-1} \textstyle\bigcup I_3^{n-1}\}, \\
I_2^n = \{(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n) \in I^n | (\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \in I_1^{n-1}\}, \\
I_3^n = I^n \setminus (I_1^n \textstyle\bigcup I_2^n).
\end{gathered}$$

One could easily check that $I_1^n, I_2^n, I_3^n$ are pairwise disjoint.

**Lemma 1.** *For any $\alpha_n$ and $\alpha_n'$ in $I$ both elements $(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n)$ and $(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n')$ in $I^n$ are in the same partition of $I^n$.*

*Proof.* When $n = 1$, it is obviously true. Suppose it is true for $n-1$. If $(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n) \in I_1^n$, then $\alpha_{n-1} = 1$ and $(\alpha_1, \alpha_2, \ldots, \alpha_{n-2}, \alpha_n) \in I_2^{n-1} \bigcup I_3^{n-1}$. By the induction assumption, one has $(\alpha_1, \alpha_2, \ldots, \alpha_{n-2}, \alpha_n') \in I_2^{n-1} \bigcup I_3^{n-1}$ and therefore $(\alpha_1, \alpha_2, \ldots, \alpha_{n-2}, \alpha_{n-1}, \alpha_n') \in I_1^n$. If $(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n) \in I_2^n$, then $(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \in I_1^{n-1}$, thus $(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n') \in I_2^n$ by the definition of $I_2^n$. If $(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n) \in I_3^n$, then $(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n) \notin I_1^n \bigcup I_2^n$. By the results we have proved above one gets $(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n') \notin I_1^n \bigcup I_2^n$, so $(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n') \in I_3^n$. $\square$

To simplify notations for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in I^n$ and $i \leq n (n \geq 2)$ we denote $(\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1} \ldots, \alpha_n)$ by $\alpha[\hat{i}]$ and $(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, 1)$ by $\alpha(1)$. By Lemma 1 both $\alpha$ and $\alpha(1)$ are in the same partition of $I^n$. For any element $\alpha \in I^n$ pick a symbol $e_\alpha$ and make a right free $\widehat{sA}$-module with basis $e_\alpha$, $\alpha \in I^n$, which enable us to calculate $\mathrm{Tor}_n^{\widetilde{sA}}(\mathbb{Z}, \mathbb{Z})$.

**Lemma 2.** *There is a free chain complex of the $\widetilde{sA}$-module $\mathbb{Z}$,*

$$\cdots \xrightarrow{d_{n+1}} \oplus_{\alpha \in I^n} e_\alpha \widetilde{sA} \xrightarrow{d_n} \oplus_{\alpha \in I^{n-1}} e_\alpha \widetilde{sA} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} \oplus_{\alpha \in I} e_\alpha \widetilde{sA} \xrightarrow{d_1} \widetilde{sA} \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

*where $\epsilon$ is the augmentation map defined by $\epsilon(x, m) = m$ and $d_1$ is defined by $d_1(e_\alpha) = (sx_\alpha, 0)$ for any $\alpha \in I$ and when $n \geq 2$ $d_n$ is defined by*

$$d_n(e_\alpha) = \begin{cases} e_{\alpha[\widehat{n-1}]}(-s, s), & \text{if } \alpha \in I_1^n, \\ e_{\alpha[\hat{n}]}(sx_{\alpha_n}, 0), & \text{if } \alpha \in I_2^n, \\ e_{\alpha[\hat{n}]}(sx_{\alpha_n}, 0) - e_{\alpha[\hat{n}](1)}(\lambda sx_{\alpha_{n-1}} x_{\alpha_n}, 0), & \text{if } \alpha \in I_3^n. \end{cases}$$

*Proof.* It is easy to see that $\epsilon d_1 = 0$. For $n \geq 2$ and $\alpha \in I^n$ set $y_\alpha = d_{n-1}d_n(e_\alpha)$. It is sufficient to prove that $y_\alpha = 0$.

If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in I_1^n$, then by the definition of $I_1^n$ we have $\alpha_{n-1} = 1$ and $\alpha[\widehat{n-1}] \in I_2^{n-1} \bigcup I_3^{n-1}$. Thus

$$y_\alpha = d_{n-1}(e_{\alpha[\widehat{n-1}]}(-s, s)).$$

Note that $(\alpha[\widehat{n-1}])[\widehat{n-1}] = \alpha[\hat{n}][\widehat{n-1}]$. When $\alpha[\widehat{n-1}] \in I_2^{n-1}$, then

$$y_\alpha = e_{\alpha[\hat{n}][\widehat{n-1}]}(sx_{\alpha_n}, 0)(-s, s) = 0.$$

When $\alpha[\widehat{n-1}] \in I_3^{n-1}$, then

$$y_\alpha = [e_{\alpha[\hat{n}][\widehat{n-1}]}(sx_{\alpha_n}, 0) - e_{\alpha[\hat{n}][\widehat{n-1}](1)}(\lambda sx_{\alpha_{n-2}}x_{\alpha_n}, 0)](-s, s) = 0.$$

If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in I_2^n$, then $\alpha[\hat{n}] = (\alpha_1, \alpha_2, \dots, \alpha_{n-1}) \in I_1^{n-1}$, so $I_1^{n-1} \neq \emptyset$ and this implies that $n \geq 3$ since $I_1^1 = \emptyset$, thus

$$y_\alpha = d_{n-1}(e_{\alpha[\hat{n}]}(sx_{\alpha_n}, 0)) = e_{\alpha[\hat{n}][\widehat{n-2}]}(-s, s)(sx_{\alpha_n}, 0) = 0.$$

Suppose now that $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \alpha_n) \in I_3^n$, then $\alpha[\hat{n}] \notin I_1^{n-1}$, so $\alpha[\hat{n}] \in I_2^{n-1}$ or $\alpha[\hat{n}] \in I_3^{n-1}$. Thus

$$y_\alpha = d_{n-1}(e_{\alpha[\hat{n}]}(sx_{\alpha_n}, 0) - e_{\alpha[\hat{n}](1)}(\lambda sx_{\alpha_{n-1}}x_{\alpha_n}, 0)).$$

When $\alpha[\hat{n}] \in I_2^{n-1}$, by Lemma 1 $\alpha[\hat{n}](1) \in I_2^{n-1}$, too. Thus

$$y_\alpha = e_{\alpha[\hat{n}][\widehat{n-1}]}(sx_{\alpha_{n-1}}, 0)(sx_{\alpha_n}, 0) - e_{\alpha[\hat{n}][\widehat{n-1}]}(sx_1, 0)(\lambda sx_{\alpha_{n-1}}x_{\alpha_n}, 0) = 0$$

since $\lambda sx_1 = s$.

When $\alpha[\hat{n}] = (\alpha_1, \alpha_2, \dots, \alpha_{n-1}) \in I_3^{n-1}$, by Lemma 1 $\alpha[\hat{n}](1) \in I_3^{n-1}$, too. In this case

$$\begin{aligned} y_\alpha = &[e_{\alpha[\hat{n}][\widehat{n-1}]}(sx_{\alpha_{n-1}}, 0) - e_{\alpha[\hat{n}][\widehat{n-1}](1)}(\lambda sx_{\alpha_{n-2}}x_{\alpha_{n-1}}, 0)](sx_{\alpha_n}, 0) \\ &-[e_{\alpha[\hat{n}][\widehat{n-1}]}(sx_1, 0) - e_{\alpha[\hat{n}][\widehat{n-1}](1)}(\lambda sx_{\alpha_{n-2}}x_1, 0)](\lambda sx_{\alpha_{n-1}}x_{\alpha_n}, 0) = 0 \end{aligned}$$

since $\lambda sx_1 = s$. Thus we have finished the proof of Lemma 2. $\square$

**Lemma 3.** *The chain complex in Lemma 2 is acyclic, and so, one gets a free resolution of the* $\widehat{sA}$*-module* $\mathbb{Z}$.

*Proof.* Since $d_1(e_\alpha) = (sx_\alpha, 0)$ for any $\alpha \in I$ and $sA$ is generated by $sx_\alpha$, $\alpha \in I$, it follows that $\ker(\epsilon) = \mathrm{Im}(d_1)$. Next we prove that $\ker(d_n) = \mathrm{Im}(d_{n+1})$.

By the definition of $d_n$ one gets

$$e_\alpha(-s, s) \in \ker(d_n), \quad \text{for any } \alpha \in I_2^n \bigcup I_3^n,$$

$$e_\alpha(sx_j, 0) \in \ker(d_n), \quad \text{for any } \alpha \in I_1^n \text{ and any } j \in I,$$

$$e_\alpha(sx_j, 0) - e_{\alpha(1)}(\lambda sx_{\alpha_n}x_j, 0) \in \ker(d_n), \quad \text{for any } \alpha \in I_2^n \bigcup I_3^n \text{ and any } j \in I.$$

Let $B_n$ denote the submodule of $\oplus_{\alpha\in I^n} e_\alpha\widetilde{sA}$ generated by

$$\begin{array}{ll} e_\alpha(-s,s), & \alpha\in I_2^n\bigcup I_3^n,\\ e_\alpha(sx_j,0), & \alpha\in I_1^n,\ j\in I,\\ e_\alpha(sx_j,0)-e_{\alpha(1)}(\lambda s x_{\alpha_n}x_j,0), & \alpha\in I_2^n\bigcup I_3^n,\ j\in I.\end{array}$$

Then $B_n\subseteq\ker(d_n)$ since all of its generators are in $\ker(d_n)$.

We now prove that each generator of $B_n$ is in $\operatorname{Im}(d_{n+1})$. Suppose that

$$\alpha=(\alpha_1,\alpha_2,\dots,\alpha_{n-1},\alpha_n)\in I_2^n\bigcup I_3^n.$$

Then by the definition of $I_1^{n+1}$ we have

$$\alpha'=(\alpha_1,\alpha_2,\dots,\alpha_{n-1},1,\alpha_n)\in I_1^{n+1}.$$

Thus $e_\alpha(-s,s),\alpha\in I_2^n\bigcup I_3^n$, is the image of $e_{\alpha'}$ under $d_{n+1}$. Suppose that

$$\alpha=(\alpha_1,\alpha_2,\dots,\alpha_{n-1},\alpha_n)\in I_1^n,$$

then for any $j\in I$,

$$\beta=(\alpha_1,\alpha_2,\dots,\alpha_{n-1},\alpha_n,j)\in I_2^{n+1},$$

so $e_\alpha(sx_j,0),\alpha\in I_1^n$, is the image of $e_\beta$ under $d_{n+1}$. Suppose that

$$\alpha=(\alpha_1,\alpha_2,\dots,\alpha_{n-1},\alpha_n)\in I_2^n\bigcup I_3^n.$$

When $\alpha_n\neq 1$, then

$$\beta=(\alpha_1,\alpha_2,\dots,\alpha_{n-1},\alpha_n,j)\in I_3^{n+1}$$

and

$$e_\alpha(sx_j,0)-e_{\alpha(1)}(\lambda s x_{\alpha_n}x_j,0),$$

is the image of $e_\beta$ under $d_{n+1}$. When $\alpha_n=1$ then $\alpha=\alpha(1)$ and

$$e_\alpha(sx_j,0)-e_{\alpha(1)}(\lambda s x_1 x_j,0)=0$$

since $\lambda s x_1=s$ and it is the image of 0 under $d_{n+1}$. So $B_n\subseteq\operatorname{Im}(d_{n+1})$. Hence, to finish the proof of Lemma 3 it is sufficient to prove that $\ker(d_n)\subseteq B_n$.

Any element $x\in\oplus_{\alpha\in I^n}e_\alpha\widetilde{sA}$ can be expressed as a sum

$$x=\sum_{\alpha\in I_1^n}e_\alpha(u_\alpha,k_\alpha)+\sum_{\alpha\in I_2^n\bigcup I_3^n}e_\alpha(u_\alpha,k_\alpha)$$

where $k_\alpha\in\mathbb{Z}$ and $u_\alpha\in sA$. Since the additive group of $sA$ is free with basis $sx_j,\ j\in I$ it follows that $\sum_{\alpha\in I_1^n}e_\alpha(u_\alpha,0)\in B_n$ by definition of $B_n$. For any $\alpha\in I_2^n\bigcup I_3^n$ let $k_\alpha=sh_\alpha+l_\alpha$ where $0\le l_\alpha<s$. Then

$$\sum_{\alpha\in I_2^n\bigcup I_3^n}e_\alpha(u_\alpha,k_\alpha)=\sum_{\alpha\in I_2^n\bigcup I_3^n}e_\alpha(u_\alpha-sh_\alpha,0)+\sum_{\alpha\in I_2^n\bigcup I_3^n}e_\alpha(0,l_\alpha)+\sum_{\alpha\in I_2^n\bigcup I_3^n}e_\alpha(-sh_\alpha,sh_\alpha)$$

where $\sum_{\alpha\in I_2^n\bigcup I_3^n} e_\alpha(-sh_\alpha, sh_\alpha) \in B_n$. Since $e_\alpha(sx_j,0) - e_{\alpha(1)}(\lambda sx_{\alpha_n}x_j,0) \in B_n$ for any $\alpha \in I_2^n\bigcup I_3^n$, the sum $\sum_{\alpha\in I_2^n\bigcup I_3^n} e_\alpha(u_\alpha - sh_\alpha, 0)$ can be expressed as

$$\sum_{\alpha\in I_2^n\bigcup I_3^n} e_\alpha(u_\alpha - sh_\alpha, 0) = \sum_{\substack{\alpha\in I_2^n\bigcup I_3^n\\ \alpha_n=1}} e_\alpha(u'_\alpha, 0) + b$$

where $u'_\alpha \in sA$ and $b \in B_n$. Thus $x \in \oplus_{\alpha\in I^n} e_\alpha\widetilde{sA}$ it can be expressed as

$$x = \sum_{\alpha\in I_1^n} e_\alpha(0,k_\alpha) + \sum_{\alpha\in I_2^n\bigcup I_3^n} e_\alpha(0,l_\alpha) + \sum_{\substack{\alpha\in I_2^n\bigcup I_3^n\\ \alpha_n=1}} e_\alpha(u'_\alpha,0) + b'$$

where $k_\alpha \in \mathbb{Z}$ for $\alpha \in I_1^n$, $0 \le l_\alpha < s$ for $\alpha \in I_2^n\bigcup I_3^n$, $u'_\alpha \in sA$ for $\alpha \in I_2^n\bigcup I_3^n$ with $\alpha_n = 1$ and $b' \in B_n$. Thus, if $x \in \ker(d_n)$, then

$$d_n(x) = \sum_{\alpha\in I_1^n} d_n(e_\alpha)(0,k_\alpha) + \sum_{\alpha\in I_2^n\bigcup I_3^n} d_n(e_\alpha)(0,l_\alpha) + \sum_{\substack{\alpha\in I_2^n\bigcup I_3^n\\ \alpha_n=1}} d_n(e_\alpha)(u'_\alpha,0) = 0.$$

Let

$$y = \sum_{\alpha\in I_1^n} d_n(e_\alpha)(0,k_\alpha),$$

$$z_2 = \sum_{\alpha\in I_2^n} d_n(e_\alpha)(0,l_\alpha) + \sum_{\substack{\alpha\in I_2^n\\ \alpha_n=1}} d_n(e_\alpha)(u'_\alpha,0),$$

and

$$z_3 = \sum_{\alpha\in I_3^n} d_n(e_\alpha)(0,l_\alpha) + \sum_{\substack{\alpha\in I_3^n\\ \alpha_n=1}} d_n(e_\alpha)(u'_\alpha,0).$$

Since $d_n(e_\alpha) = e_{\alpha\widehat{[n-1]}}(-s,s)$ if $\alpha \in I_1^n$, it follows that

$$y = \sum_{\alpha\in I_1^n} e_{\alpha\widehat{[n-1]}}(-sk_\alpha, sk_\alpha).$$

Since $d_n(e_\alpha) \in \sum_{\alpha\in I^{n-1}} e_\alpha(sA,0)$ if $\alpha \in I_2^n\bigcup I_3^n$, it follows that

$$z_2 + z_3 \in \sum_{\alpha\in I^{n-1}} e_\alpha(sA,0).$$

From $y + z_2 + z_3 = 0$ it follows that $k_\alpha = 0$. Hence $y = 0$ and $z_2 + z_3 = 0$. If $\alpha \in I_2^n$ then $\alpha[\hat{n}] \in I_1^{n-1}$ and $d_n(e_\alpha) = e_{\alpha[\hat{n}]}(sx_{\alpha_n},0)$ thus $z_2 \in \sum_{\alpha\in I_1^{n-1}} e_\alpha(sA,0)$. If $\alpha \in I_3^n$ then $\alpha[\hat{n}] \notin I_1^{n-1}$, by Lemma 1 $\alpha[\hat{n}](1) \notin I_1^{n-1}$ also. By the definition we have $d_n(e_\alpha) = e_{\alpha[\hat{n}]}(sx_{\alpha_n},0) - e_{\alpha[\hat{n}](1)}(\lambda sx_{\alpha_{n-1}}x_{\alpha_n},0)$ thus $z_3 \in \sum_{\alpha\notin I_1^{n-1}} e_\alpha(sA,0)$. From $z_2 + z_3 = 0$ it follows that $z_2 = 0$ and $z_3 = 0$. From $z_2 = 0$ it follows that

$$\sum_{\alpha\in I_2^n} e_{\alpha[\hat{n}]}(l_\alpha sx_{\alpha_n},0) + \sum_{\substack{\alpha\in I_2^n\\ \alpha_n=1}} e_{\alpha[\hat{n}]}(sx_1u'_\alpha,0) = 0.$$

Since $sx_1u'_\alpha \in s^2A$ and $sA$ is a free $\mathbb{Z}$-module with basis $sx_i, i \in I$ and $0 \leq l_\alpha < s$ it follows that $l_\alpha = 0, \alpha \in I_2^n$. Hence

$$\sum_{\substack{\alpha\in I_2^n\\ \alpha_n=1}} e_{\alpha[\hat{n}]}(sx_1u'_\alpha, 0) = 0.$$

However there is an injection from set $\{\alpha \in I_2^n | \alpha_n = 1\}$ to $\{\alpha[\hat{n}] | \alpha \in I_2^n, \alpha_n = 1\}$. Thus $sx_1u'_\alpha = 0$ and $su'_\alpha = 0$ since $\lambda sx_1 = 1$. Furthermore $u'_\alpha = 0$ since $u'_\alpha \in sA$ and $sA$ is a free $\mathbb{Z}$-module. Similarly one proves that $l_\alpha = 0$ for any $\alpha \in I_3^n$ and $u'_\alpha = 0$ for any $\alpha \in I_3^n$ with $\alpha_n = 1$. Thus, $\ker(d_n) \subseteq B_n$. So $\ker(d_n) = \mathrm{Im}(d_{n+1})$. $\square$

**Theorem 1.** *Let $s \in \mathbb{Z}$ and $s > 0$. Assume that $A$ is a ring with identity and the additive group of $sA$ is a free $\mathbb{Z}$-module with basis $sx_i, i \in I$, where $I$ is a totally ordered set. Then*

$$\mathrm{Tor}_n^{\widetilde{sA}}(\mathbb{Z}, \mathbb{Z}) = (\mathbb{Z}/s\mathbb{Z})^{|I_2^n \bigcup I_3^n|}.$$

*Hence, $HB_n(sA) = (\mathbb{Z}/s\mathbb{Z})^{|I_2^n \bigcup I_3^n|}$.*

*Proof.* By tensoring the exact sequence in Lemma 3 with $\mathbb{Z}$ we get a complex

$$\cdots \xrightarrow{d_{n+1}\otimes 1} \oplus_{\alpha\in I^n} e_\alpha \widetilde{sA} \otimes_{\widetilde{sA}} \mathbb{Z} \xrightarrow{d_n \otimes 1} \oplus_{\alpha\in I^{n-1}} e_\alpha \widetilde{sA} \otimes_{\widetilde{sA}} \mathbb{Z} \xrightarrow{d_{n-1}\otimes 1} \cdots$$
$$\xrightarrow{d_2\otimes 1} \oplus_{\alpha\in I} e_\alpha \widetilde{sA} \otimes_{\widetilde{sA}} \mathbb{Z} \xrightarrow{d_1 \otimes 1} \widetilde{sA} \otimes_{\widetilde{sA}} \mathbb{Z}$$

We have

$$\oplus_{\alpha\in I^n} e_\alpha \widetilde{sA} \otimes_{\widetilde{sA}} \mathbb{Z} = \oplus_{\alpha\in I^n} e_\alpha \otimes_{\mathbb{Z}} \mathbb{Z}.$$

It is easy to see that if $\alpha \in I_2^n \bigcup I_3^n$, then $(d_n \otimes 1)(e_\alpha \otimes 1) = 0$, thus

$$\oplus_{\alpha\in I_2^n \bigcup I_3^n} e_\alpha \otimes_{\mathbb{Z}} \mathbb{Z} \subseteq \ker(d_n \otimes 1).$$

There is a bijection between $I_1^n$ and $I_2^{n-1} \bigcup I_3^{n-1}$ defined by $\alpha \mapsto \alpha[\widehat{n-1}]$, so we have that if

$$(d_n \otimes 1)(\sum_{\alpha\in I_1^n} e_\alpha \otimes k_\alpha) = \sum_{\alpha[\widehat{n-1}]\in I_2^{n-1} \bigcup I_3^{n-1}} e_{\alpha[\widehat{n-1}]} \otimes sk_\alpha = 0,$$

then $k_\alpha = 0$. So

$$\ker(d_n \otimes 1) = \oplus_{\alpha\in I_2^n \bigcup I_3^n} e_\alpha \otimes_{\mathbb{Z}} \mathbb{Z}.$$

Since $(d_{n+1} \otimes 1)(\oplus_{\alpha\in I_2^{n+1} \bigcup I_3^{n+1}} e_\alpha \widetilde{sA} \otimes_{\widetilde{sA}} \mathbb{Z}) = 0$ it follows that

$$\mathrm{Im}(d_{n+1} \otimes 1) = (d_{n+1} \otimes 1)(\oplus_{\alpha\in I_1^{n+1}} e_\alpha \otimes_{\mathbb{Z}} \mathbb{Z})$$

$$= \oplus_{\alpha\in I_1^{n+1}} e_{\alpha[\hat{n}]} \otimes s\mathbb{Z} = \oplus_{\alpha\in I_2^n \bigcup I_3^n} e_\alpha \otimes s\mathbb{Z}.$$

Hence

$$\mathrm{Tor}_n^{\widetilde{sA}}(\mathbb{Z}, \mathbb{Z}) = \ker(d_n \otimes 1)/\mathrm{Im}(d_{n+1} \otimes 1) = (\mathbb{Z}/s\mathbb{Z})^{|I_2^n \bigcup I_3^n|}.$$

It follows from the Lemma 1.1 in [3] that $HB_n(sA) = (\mathbb{Z}/s\mathbb{Z})^{|I_2^n \bigcup I_3^n|}$. $\square$

## References


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