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Zero-Dimensional Pairs

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Abstract. If $\{(R_i, T_i)\}_{i=1}^n$ is a finite family of zero-dimensional pairs, then $(\prod_{i=1}^n R_i, \prod_{i=1}^n T_i)$ is zero-dimensional pair but this result fails for an infinite family of zero-dimensional pairs. We give necessary and sufficient conditions in order that an infinite product $(\prod_{\alpha\in A} R_\alpha, \prod_{\alpha\in A} T_\alpha)$ of zero-dimensional pairs $\{(R_\alpha, T_\alpha)\}_{\alpha\in A}$ is zero-dimensional pair.

1. Introduction

All rings considered in this paper are assumed to be commutative and unitary. If R is a subring of a ring S, we assume that the unity element of S belongs to R, and hence is the unity of R. We use the term dimension of R, denoted $\dim R$, to refer to the Krull dimension of R. Thus $\dim R$ is the non negative integer n if there exists a chain $P_0 \subset P_1 \subset \cdots \subset P_n$ of proper prime ideals of R, but no longer such chain; if there is no upper bound on the lengths of such chains, we write $\dim R = \infty$. This paper is concerned with zero-dimensional rings in which each proper prime ideal is maximal, and zero-dimensional pairs. We frequently use the fact that dimension is preserved under integral extensions (cf. [1, (11.8)]). In particular, an integral extension ring of a zero-dimensional ring is zero-dimensional. Let R be a ring, R is said to be hereditarily zero-dimensional if each subring of R is zero-dimensional. We also consider this zero-dimensionality condition in a relative context: if R is a subring of T, we say that (R, T) is a zero-dimensional pair if each intermediate ring between R and T is zero-dimensional. R. Gilmer and W. Heinzer have given in [4, Theorem 4.9], the necessary and sufficient conditions under what an arbitrary product of infinite hereditarily zero-dimensional rings is hereditary zero-dimensional. Since the notion of zero-dimensional pair is more general than the hereditarily zero-dimensionality, then one may ask:

(Q) Let $\{(R_\alpha, T_\alpha)\}_{\alpha\in A}$ be a family of zero-dimensional pairs. Under what conditions is $(\prod_{\alpha\in A} R_\alpha, \prod_{\alpha\in A} T_\alpha)$ a zero-dimensional pair?

In Section 2, we give necessary and sufficient conditions to question **(Q)**.

2. Zero-dimensional pairs

The main purpose of this section is to give an answer to question **(Q)**. But first, we show that the homomorphic image of a zero-dimensional pair is a zero-dimensional pair and a finite product $(\prod_{i=1}^n R_i, \prod_{i=1}^n T_i)$ is a zero-dimensional pair if and only if each (R_i, T_i) is a zero-dimensional pair. Moreover, R. Gilmer and W. Heinzer have established that if $\dim R = 0$, then (R, T) is a zero-dimensional pair if and only if $R \hookrightarrow T$ is an integral extension [4, Corollary 4.2]. Before starting, we recall the polynomial over infinite product of rings. Let $\{R_\alpha\}_{\alpha\in A}$ be a family of rings and X be an indeterminate over $\prod_{\alpha\in A} R_\alpha$. Given $F \in (\prod_{\alpha\in A} R_\alpha)[X]$, then $F = f_n X^n + \cdots + f_1 X + f_0$, such that $f_i \in \prod_{\alpha\in A} R_\alpha$, as a function, for $i = 1, \ldots, n$; $f_i \in \prod_{\alpha\in A} R_\alpha$ means that $f_i(\alpha) \in R_\alpha$ for each $\alpha \in A$, for any $i \in \{1, \ldots, n\}$. We denote $F_\alpha = f_n(\alpha)X^n + \cdots + f_1(\alpha)X + f_0(\alpha) \in R_\alpha[X]$, for this reason we can regard each polynomial over an infinite product as $F = \{F_\alpha\}_{\alpha\in A} \in \prod_{\alpha\in A} R_\alpha[X]$.

Proposition 2.1. *Let $\{R_i\}_{i=1}^n$ and $\{T_i\}_{i=1}^n$ be two finite families of rings. Then $(\prod_{i=1}^n R_i, \prod_{i=1}^n T_i)$ is a zero-dimensional pair if and only if each (R_i, T_i) is a zero-dimensional pair.*

Proof. It is well-known that $Spec(\prod_{i=1}^n T_i) = \{\prod_{i=1}^n S_i : S_{j_0} = M_{j_0} \in Spec(T_{j_0})$ and $S_i = T_i$ for each $i \in \{1, \ldots, n\} \setminus \{j_0\}\}$. Since each (R_i, T_i) is a zero-dimensional pair, according to [2, Result 1.6], $\prod_{i=1}^n T_i$ is an integral extension of $\prod_{i=1}^n R_i$. Moreover, $\dim\prod_{i=1}^n T_i = \dim\prod_{i=1}^n R_i = 0$; on account of [4, Corollary 4.2], $(\prod_{i=1}^n R_i, \prod_{i=1}^n T_i)$ is a zero-dimensional pair. Conversely, T_i is integral over R_i for each $i = 1, \ldots, n$; because $\prod_{i=1}^n T_i$ is integral over $\prod_{i=1}^n R_i$ (cf. [2, Result 1.6]) and $\dim\prod_{i=1}^n R_i = 0$ imply $\dim R_i = 0$. According to [4, Corollary 4.2], each (R_i, T_i) is a zero-dimensional pair. □

On the other hand, Proposition 2.1 fails for an infinite family of zero-dimensional rings as shows the next example.

Example 2.2. Let $\mathbb{Q}$ be the field of rational numbers and p be a prime integer. Let ξ_i be a primitive p^i-th root of 1 for each $i \in \mathbb{Z}^+$. We have $(\mathbb{Q}, \mathbb{Q}(\xi_i))$ is zero-dimensional pair for each $i \in \mathbb{Z}^+$. Nevertheless, $(\prod_{i\in\mathbb{N}^*} \mathbb{Q}, \prod_{i\in\mathbb{N}^*} \mathbb{Q}(\xi_i))$ is not zero-dimensional pair. In fact, let $\xi = \{\xi_i\}_{i\in\mathbb{N}^*}$ be an element of $\prod_{i\in\mathbb{N}^*} \mathbb{Q}(\xi_i)$. Since there exists no monic polynomial of $\prod_{i\in\mathbb{N}^*} \mathbb{Q}[X]$ that vanishes ξ, we have ξ is transcendental over $\prod_{i\in\mathbb{N}^*} \mathbb{Q}$.

Let T be an integral extension of a ring R, and x be an element of T. We denote $I_x = \{f \in R[X] : f$ is a monic polynomial which vanishes on $x\}$. We use also $degI_x$ to denote $\text{Min}\{deg.f : f \in I_x\}$, where $deg.f$ is the degree of f. Next, we give our main result in this section which answers question **(Q)**. Initially we note that if $R = \prod_{\alpha\in A} R_\alpha$ is the direct product of zero-dimensional rings R_α, by [5, Proposition 2.5], R need not be zero-dimensional.

Theorem 2.3. *Let $\{(R_\alpha, T_\alpha)\}_{\alpha\in A}$ be a family of zero-dimensional pairs. If $dim\prod_{\alpha\in A} T_\alpha = 0$, then the following statements are equivalent.*

(i) $(\prod_{\alpha\in A} R_\alpha, \prod_{\alpha\in A} T_\alpha)$ *is a zero-dimensional pair;*

(ii) *For each* $x = \{x_\alpha\}_{\alpha\in A} \in \prod_{\alpha\in A} T_\alpha$, *there exists* $k_x \in \mathbb{N}^*$ *such that* $\{\alpha \in A : degI_{x_\alpha} > k_x\}$ *is finite.*

Before proving this theorem, we establish the following lemma.

Lemma 2.4. *Let* R *be a subring of a ring* T *and* $\varphi : T \longrightarrow T$ *be a ring-homomorphism. If* (R, T) *is a zero-dimensional pair, then so is* $(\varphi(R), \varphi(T))$.

Proof. Let S be a ring such that $\varphi(R) \subseteq S \subseteq \varphi(T)$. Let A be the inverse image of S by φ, so $R \subseteq A \subseteq T$. Since $\dim(S) \leq \dim(A) = 0$, then $\dim(S) = 0$. □

Proof of Theorem 2.3. (i) $\Longrightarrow$ (ii). Let $x = \{x_\alpha\}_{\alpha\in A} \in \prod_{\alpha\in A} T_\alpha$, then x is integral over $\prod_{\alpha\in A} R_\alpha$, i.e., there exists a monic polynomial $G = \{f_\alpha\}_{\alpha\in A} \in (\prod_{\alpha\in A} R_\alpha)[X]$ such that $G(x) = 0$; and $\deg.G = k \in \mathbb{N}^*$, then $\deg I_{x_\alpha} \leq k$ for each $\alpha \in A$.

(ii)$\Longrightarrow$ (i). Let $x = \{x_\alpha\}_{\alpha\in A} \in \prod_{\alpha\in A} T_\alpha$, since every T_α is integral over R_α there exists $f_\alpha \in I_{x_\alpha}$ such that $f_\alpha(x_\alpha) = 0$, for each $\alpha \in A$. We denote $B = \{\alpha \in A : degI_{x_\alpha} > k_x\} = \{\alpha_1, \dots, \alpha_n\}$; we put $\deg.f_\alpha = n_\alpha$ for each $\alpha \in A$. Let $F = \{f_\alpha\}_{\alpha\in A} \in (\prod_{\alpha\in A} R_\alpha[X])$ and $s = Sup\{deg.f_\alpha : \alpha \in A\}$. Since B is finite, s is a finite integer. Let $g_\alpha = X^{s-n_\alpha} f_\alpha$, $G = \{g_\alpha\}_{\alpha\in A} \in \prod_{\alpha\in A}(R_\alpha[X])$ be a monic polynomial of degree equal to s with $G(x) = 0$. Therefore, x is integral over $\prod_{\alpha\in A} R_\alpha$. On the other hand, $\dim\prod_{\alpha\in A} R_\alpha = 0$ [3, Theorem 3]. By [4, Corollary 4.2], $(\prod_{\alpha\in A} R_\alpha, \prod_{\alpha\in A} T_\alpha)$ is a zero-dimensional pair, and the proof is complete. □

Example 2.5. Let $\{p_i\}_{i\in\mathbb{N}^*}$ be a family of prime positive integers, X be an indeterminate and m be a positive integer. We consider $R_i = (\mathbb{Z}/p_i\mathbb{Z})^i \otimes_{\mathbb{Z}/p_i\mathbb{Z}} GF(p_i^2)(X)$, the tensor product, where $GF(p_i^2)$ is the finite field with p_i^2 elements, for each $i \in \mathbb{Z}^+$; and $T_i = (\mathbb{Z}/p_i\mathbb{Z})^i \otimes_{\mathbb{Z}/p_i\mathbb{Z}} GF(p_i^{2m})(X)$, where $GF(p_i^{2m})$ is the finite field with p_i^{2m} elements. According to [8, Theorem 3.7], $\dim R_i = \dim T_i = 0$. Since $GF(p_i^2)(X) \hookrightarrow GF(p_i^{2m})(X)$ is an algebraic extension, the ring T_i is integral over R_i for each $i \in \mathbb{Z}^+$. We remark that $GF(p_i^{2m}) = GF(p_i^2)(\xi_i)$, where ξ_i is a generator of the cyclic group $GF(p_i^{2m})\setminus(0)$ [7, Théorème 2.2, page 75], for each $i \in \mathbb{Z}^+$; and we have $\deg I_x \leq m$ for each $x \in T_i$. We see via Theorem 2.3, that $(\prod_{i\in\mathbb{N}^*} R_i, \prod_{i\in\mathbb{N}^*} T_i)$ is a zero-dimensional pair.

If $x \in N(R)$, we denote by $\eta(x)$ the index of nilpotency of x – that is, $\eta(x) = k$ if $x^k = 0$ but $x^{k-1} \neq 0$. We define $\eta(R)$ to be $\mathrm{Sup}\{\eta(x) : x \in N(R)\}$; if the set $\{\eta(x) : x \in N(R)\}$ is unbounded, then we write $\eta(R) = \infty$. From [4, Theorem 3.4], we have $\dim\prod_{\alpha\in A} T_\alpha = 0$ if and only if $\{\alpha \in A : \eta(T_\alpha) > k\}$ is finite for some $k \in \mathbb{Z}^+$, where $\{T_\alpha\}_{\alpha\in A}$ is a family of zero-dimensional rings.

Proposition 2.6. *Let* $\{R_\alpha\}_{\alpha\in A}$ *and* $\{T_\alpha\}_{\alpha\in A}$ *be two infinite families of rings such that* $R_\alpha \hookrightarrow T_\alpha$ *is a ring extension and for each* α *and each maximal ideal* M_α *of* T_α, T_α/M_α *is a finite separable algebraic field extension of* $R_\alpha/\mathfrak{m}_\alpha$, *where* $\mathfrak{m}_\alpha = M_\alpha \cap R_\alpha$. *If* $(\prod_{\alpha\in A} R_\alpha, \prod_{\alpha\in A} T_\alpha)$ *is a zero-dimensional pair, then each* (R_α, T_α) *is a zero-dimensional pair and there exists* $k \in \mathbb{Z}^+$ *such that* $\{\alpha \in A :$ *there exists* $M_\alpha \in Spec(T_\alpha)$ *with* $[T_\alpha/M_\alpha : R_\alpha/\mathfrak{m}_\alpha] > k\}$ *is a finite set.*

Proof. Since $(R_\alpha, T_\alpha) = (p_\alpha(\prod_{\alpha\in A} R_\alpha), p_\alpha(\prod_{\alpha\in A} T_\alpha))$, where $p_\alpha : \prod_{\alpha\in A} R_\alpha \to R_\alpha$ is the canonical projection homomorphism, by Lemma 2.4, (R_α, T_α) is a zero-dimensional pair, for each $\alpha \in A$. Assume that for each $k \in \mathbb{Z}^+$ the set $\{\alpha \in A :$ there exists $M_\alpha \in Spec(T_\alpha)$ with $[T_\alpha/M_\alpha : R_\alpha/\mathfrak{m}_\alpha] > k\}$ is infinite. Let $\{\alpha_i\}_{i\in\mathbb{Z}^+}$ be a countably infinite subset of A such that there exists $M_{\alpha_i} \in Spec(T_{\alpha_i})$ with $[T_{\alpha_i}/M_{\alpha_i} : R_{\alpha_i}/\mathfrak{m}_{\alpha_i}] > i$ for each $i \in \mathbb{Z}^+$. Set $L_i = T_{\alpha_i}/M_{\alpha_i}$ and $K_i = R_{\alpha_i}/\mathfrak{m}_{\alpha_i}$, then L_i is an algebraic extension of K_i. Since each L_i is separable of finite degree over K_i, by the Primitive Element Theorem [6, Theorem 5.6, page 55], there exists $x_i \in L_i$ such that $L_i = K_i(x_i)$ and hence $[K_i(x_i) : K_i] > i$ for each $i \in \mathbb{Z}^+$. It follows that $\prod_{i=1}^\infty K_i(x_i)$ is transcendental over $\prod_{i=1}^\infty K_i$, since there is no monic polynomial $f \in (\prod_{i=1}^\infty K_i)[X]$ such that $f(x) = 0$, where x is the element given by $\{x_i\}_{i=1}^\infty$, a contradiction with $(\prod_{i=1}^\infty R_i, \prod_{i=1}^\infty T_i)$ being a zero-dimensional pair. $\square$

Since the proof of the following corollary is the same as of Proposition 2.6, we omit it.

Corollary 2.7. *Let $\{(R_\alpha, T_\alpha)\}_{\alpha\in A}$ be a family of zero-dimensional pairs. If $(\prod_{\alpha\in A} R_\alpha, \prod_{\alpha\in A} T_\alpha)$ is a zero-dimensional pair, then there exists $k \in \mathbb{Z}^+$ such that $\{\alpha \in A :$ there exists $x_\alpha \in T_\alpha$ and there exists $M_\alpha \in Spec(T_\alpha)$ with $[R_\alpha/M_\alpha \cap R_\alpha(\overline{x}_\alpha) : R_\alpha/M_\alpha \cap R_\alpha] > k\}$ is a finite set.*

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