Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 44 (2003), No. 2, 575-580.

The Generating Rank of the Symplectic Line-Grassmannian

Rieuwert J. Blok

Department of Mathematics, Michigan State University
East Lansing, MI 48824-1027, USA
e-mail: blokr@member.ams.org

Abstract. We prove that the grassmannian of lines of the polar space associated to $\mathrm{Sp}_{2n}(\mathbb{F})$ has generating rank $2n^2 - n - 1$ when $\mathrm{Char}(\mathbb{F}) \neq 2$.

MSC 2000: 51E24 (primary); 51A50, 51A45 (secondary)
Keywords: symplectic geometry, grassmannian, generating rank

1. Introduction

In Cooperstein [4] the author determines the generating rank of the long-root geometries associated to a classical group over a prime field. The case of arbitrary fields for these geometries is first studied in Blok and Pasini [2] who give sharp bounds on these ranks. In addition they prove that the line-grassmannian of the symplectic polar space associated to the group $\mathrm{Sp}_{2n}(\mathbb{F})$, which is not the long-root geometry of that group, over a prime field of characteristic not 2 has generating rank $2n^2 - n - 1$. The bounds given by Blok and Pasini still involve the field, namely its degree over the prime field. Our result is the following.

Theorem 1. *The line-grassmannian of the polar space associated to* $\mathrm{Sp}_{2n}(\mathbb{F})$ *has generating rank* $2n^2 - n - 1$ *if* $\mathbb{F}$ *is a field with* $\mathrm{Char}(\mathbb{F}) \neq 2$.

2. Preliminaries

A *point-line geometry* is a pair $\Gamma = (\mathcal{P}, \mathcal{L})$ where $\mathcal{P}$ is a set whose elements are called 'points' and $\mathcal{L}$ is a collection of subsets of $\mathcal{P}$ called 'lines' with the property that any two points belong to at most one line. If $\mathcal{P}$ and $\mathcal{L}$ are not mentioned explicitly, the sets of points and lines of a point-line geometry Γ are denoted $\mathcal{P}(\Gamma)$ and $\mathcal{L}(\Gamma)$.

A *subspace* of Γ is a subset $X \subseteq \mathcal{P}$ such that any line containing at least two points of X entirely belongs to X. A *hyperplane* of Γ is a subspace that meets every line.

The *span* of a set $S \subseteq \mathcal{P}$ is the smallest subspace containing S; it is the intersection of all subspaces containing S and is denoted by $\langle S\rangle_\Gamma$. We say that S is a *generating set* (or *spanning set*) for Γ if $\langle S\rangle_\Gamma = \mathcal{P}$.

For a vector space W over some field $\mathbb{F}$, the *projective geometry* associated to W is the point-line geometry $\mathbb{P}(W) = (\mathcal{P}(W), \mathcal{L}(W))$ whose points and lines are the 1-spaces of W and the sets of 1-spaces contained in some 2-space.

A *projective embedding* of a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ is a pair (ϵ, W), where ϵ is an injective map $\mathcal{P} \xrightarrow{\epsilon} \mathcal{P}(W)$ that sends every line of $\mathcal{L}$ onto a line of $\mathcal{L}(W)$, and with the property that

$$\langle \epsilon(\mathcal{P})\rangle_{\mathbb{P}(W)} = \mathcal{P}(W).$$

The *dimension* of the embedding is the dimension of the vector space W. It is rather easy to verify that for any generating set S and any embedding (ϵ, W) we have

$$\dim(W) \leq |S|.$$

In case of equality S has minimal size and we then call $|S|$ the *generating rank* of Γ. At the same time then W provides the largest embedding for Γ.

We briefly describe the particular geometries we will discuss in this paper. Let V be a vector space over some field $\mathbb{F}$. The *projective line-grassmannian* associated to V is the point-line geometry $\mathrm{Gr}(V, 2)$ whose points are the 2-spaces of V and whose lines are the sets of lines l such that $p \subseteq l \subseteq u$ for some 1-space p and 3-space u.

Now suppose that V has dimension $2n$ and is endowed with a non-degenerate symplectic form $(\cdot, \cdot)$. A subspace U of V is called *totally isotropic* (t.i.) with respect to the form $(\cdot, \cdot)$ if $(u, v) = 0$ for any two vectors $u, v \in U$. The *symplectic polar space* is the point-line geometry Π whose points are the t.i. 1-spaces of V and whose lines are the sets of t.i. 1-spaces contained in some t.i. 2-space. We sometimes call t.i. 3-spaces *planes*.

The *symplectic line-grassmannian* is the point-line geometry Λ whose points are the t.i. 2-spaces and whose lines are the sets of t.i. 2-spaces l such that $p \subseteq l \subseteq u$ for some t.i. 1-space p and t.i. 3-space u. We often identify the line with the pair (p, u). We will call the points and lines of Λ Points and Lines to distinguish them from the points and lines of Π.

3. Proof of Theorem 1

We first recall a result on the generating rank of Π and then define our minimal generating set for Λ. Both are related to the apartments of Π.

Let $\mathbb{E} = \{e_i \mid i = 1, 2, \ldots, 2n\}$ be a *hyperbolic basis* for V, i.e. we have $(e_i, e_j) = \delta_{n+i,j}$ where δ is the Kronecker delta. The *apartment* $\mathcal{A}(\mathbb{E})$ corresponding to $\mathbb{E}$ is the collection of t.i. subspaces of V whose basis is a subset of $\mathbb{E}$. For $I, J \subseteq [n]$, introduce the following notation

$$E_{I,J} = \langle e_i, e_{n+j} \mid i \in I, j \in J\rangle_V.$$

Then $E_{I,J}$ is t.i. if and only if $I \cap J = \emptyset$. In fact

$$\mathcal{A}(\mathbb{E}) = \{E_{I,J} \mid I, J \subseteq [n], I \cap J = \emptyset\}.$$

In the sequel we will drop $\mathbb{E}$ from the notation if no confusion can arise.

Theorem 3.1. (Blok and Brouwer [1], Cooperstein and Shult [3]) *The generating rank of the polar space associated to* $\mathrm{Sp}_{2n}(\mathbb{F})$ *is* $2n$ *if* $\mathrm{Char}(\mathbb{F}) \neq 2$.

The minimal generating set exhibited in both papers is simply the set of points in an apartment. Note that the conclusion of the theorem is false if $\mathbb{F}$ has even characteristic.

Our minimal generating set S for the symplectic line-grassmannian Λ is defined as follows.

Let e be a point of Π contained in $E_{I,\emptyset}$ but not in $E_{J,\emptyset}$ for any $J \subset I$. Then S is the collection of lines of $\mathcal{A}$, together with any $n-1$ lines on e that span a t.i. n-space meeting $E_{I,\emptyset}$ only in e.

More explicitly, let $e = e_1 + e_2 + \cdots + e_n$. Then, for S take

$$S = \{\langle e_i, e_j\rangle_V \mid 1 \leq i < j \leq 2n, n+i \neq j\} \cup \{\langle e, e_{n+i+1} - e_{n+i}\rangle_V \mid 1 \leq i < n\}.$$

Note that S is a set of $2n^2 - n - 1$ t.i. 2-spaces.

The first step in proving Theorem 1 is to show that Λ has a projective embedding of the right dimension. The following result is well-known (for a generalization see e.g. Shult [6]).

Lemma 3.2. *The line-grassmannian of the polar space associated to* $\mathrm{Sp}_{2n}(\mathbb{F})$ *(any characteristic) has a projective embedding of dimension* $2n^2 - n - 1$.

Proof. The embedding is afforded by a hyperplane in the exterior square $\wedge^2 V$ of the vector space V underlying the polar space. The hyperplane corresponds to the symplectic form for which all embedded polar lines are isotropic.

Let us make this more explicit. It is well-known and easy to verify that the projective line-grassmannian $\mathrm{Gr}(V, 2)$ has a projective embedding $(\varphi, \wedge^2 V)$

$$\langle x, y\rangle_V \quad \mapsto \quad \langle x \wedge y\rangle_{\wedge^2 V}.$$

By definition of $\wedge^2 V$, the φ-image of $\mathrm{Gr}(V, 2)$ spans $\mathbb{P}(\wedge^2 V)$.

The embedding φ restricts to an embedding of the symplectic line grassmannian Λ into some hyperplane of $\wedge^2 V$. The vector space $\wedge^2 V$ has a basis $\{e_i \wedge e_j \mid 1 \leq i < j \leq 2n\}$. Suppose $x = \sum_{i=1}^{2n} x_i e_i$ and $y = \sum_{i=1}^{2n} y_i e_i$. Then

$$x \wedge y = \sum_{1 \leq i < j \leq 2n} (x_i y_j - x_j y_i) e_i \wedge e_j.$$

Now our symplectic form looks like

$$(x, y) = \sum_{i=1}^{n} (x_i y_{n+i} - x_{n+i} y_i).$$

Hence a 2-space of V is t.i. if and only if its φ-image belongs to the hyperplane

$$H = \{ \sum_{1\le i<j\le 2n} u_{i,j} e_i \wedge e_j \mid \sum_{i=1}^{n} u_{i,n+i} = 0\}.$$

We only have to show that $\langle\varphi(\Lambda)\rangle_H = H$. This is true because the images of the elements in S are linearly independent.

Thus (φ, H) is a projective embedding for Λ of dimension $2n^2 - n - 1$. □

For the moment let $S \subseteq \mathcal{P}(\Lambda)$ be an arbitrary set of Points. A point p of Π is called *S-full* whenever all lines on p are contained in $\langle S\rangle_\Lambda$. The following lemma is essentially proved in Blok and Pasini [2, Lemma 5.1], but we will prove it here for the reader's convenience.

We will denote the orthogonality relation between subspaces of V with respect to the symplectic form by $\perp$. Two subspaces X and Y of V with $\dim(X) \le \dim(Y)$ are called *opposite* if $\dim(X^\perp \cap Y) = \dim(Y) - \dim(X)$.

Lemma 3.3. *Suppose that a line l contains two S-full points.*

(a) *If s is S-full and r is the unique point on l collinear to s, then r is S-full.*

(b) *In particular, if there exists a line m opposite to l all points of which are S-full, then all points in l are S-full.*

Proof. Suppose that p and q are S-full points on l and that s is an S-full point on m. Let r be the point $s^\perp \cap l$.

The subgeometry Λ_r of Λ consisting of lines and planes on r is isomorphic to a symplectic polar space of type $\mathrm{Sp}_{2(n-1)}(\mathbb{F})$.

The set H of lines in $l^\perp$ containing r forms a hyperplane of Λ_r. Now H is a maximal subspace of Λ_r and so together with the line rs, which doesn't belong to H, it generates Λ_r. Thus in order to show that r is S-full it suffices to show that both H and rs belong to $\langle S\rangle_\Lambda$.

Clearly rs belongs to $\langle S\rangle_\Lambda$ because s is S-full.

As for H, let k be any line on r contained in $l^\perp$ and let u be the plane on l and k. Then, for any point $t \ne r$ on k, the lines tp and tq belong to $\langle S\rangle_\Lambda$ (because p and q are S-full) hence so does k, as these three Points lie on the Line (t, u).

We are done. □

Proof (of Theorem 1). We will first prove that the set S defined at the beginning of this section is a generating set for Λ. For $n = 2$ this is easy to verify. For $n \ge 3$ we do this by showing that all points of Π are S-full. In the following 'points', 'lines', and 'planes' refer to points, lines, and planes of Π unless otherwise specified. First we note that, since $\mathrm{Char}(\mathbb{F}) \ne 2$, the points of Π contained in a given apartment span Π by Theorem 3.1. As this also applies to the symplectic polar space of t.i. lines and planes on a given point we get that all points of $\mathcal{A}$ are S-full. In particular, all lines $\langle e_i, e\rangle_V$ are in the span of S. In turn, by the same principle, also e is S-full. Similarly, every point of Π contained in $E_{i,i}$ is S-full, for every $i \in I$.

We now show that every point on every line of $\mathcal{A}$ is S-full. Consider $i < j \in I$. Call $x = E_{\emptyset,i}$, $\hat{x} = E_{i,\emptyset}$, $y = E_{\emptyset,j}$, $\hat{y} = E_{j,\emptyset}$, and let y_0 be any point on $E_{j,j} \setminus \{y, \hat{y}\}$. Denote $xy = E_{\emptyset,\{i,j\}}$ and $\hat{x}\hat{y} = E_{\{i,j\},\emptyset}$.

Note that e is not collinear to x or y since it is not contained in $E_{J,\emptyset}$ for any proper subset $J \subseteq I$. Thus by Lemma 3.3 with $l = xy$ and $s = e$ we see that there is a point $z_0 \in xy \setminus \{x, y\}$ that is S-full.

Now each point x' on $E_{i,i} \setminus \{x\}$ lies on a line with y_0 which is opposite to xy. Let z' be the unique point on $x'y_0$ collinear to z_0. Then by Lemma 3.3 with $l = x'y_0$ and $s = z_0$, since x', y_0, and z_0 are S-full, also z' is S-full. For $x' = x$ we set $z' = x$ which is also S-full.

Let $H = \{z' \mid x' \in E_{i,i}\}$. Clearly $H = \{z_0, y_0\}^\perp \cap E_{\{i,j\},\{i,j\}}$ is a hyperbolic line. Note that for all $x' \in E_{i,i} \setminus \{\hat{x}\}$ the line $x'y_0$ is opposite to $\hat{x}\hat{y}$. For each $z' \in H$ let $\hat{z}$ be the unique point on $\hat{x}\hat{y}$ collinear to z'. Again by Lemma 3.3 with $l = \hat{x}\hat{y}$ and $s = z'$ we find that $\hat{z}$ is S-full. For $x' = \hat{x}$ the unique point on $\hat{x}\hat{y}$ collinear to z' is $\hat{z} = \hat{x}$, which is also S-full. The hyperbolic line H is opposite to $\hat{x}\hat{y}$ in the sense that $H \cap \hat{x}\hat{y} = \emptyset$ and $H^\perp \cap \hat{x}\hat{y} = \emptyset$. This is because $z' \neq x'$ except if $x' = x$ and because if $x' = x$, then ${z'}^\perp \cap \hat{x}\hat{y} = \hat{y}$ whereas if $x' = \hat{x}$, then ${z'}^\perp \cap \hat{x}\hat{y} = \hat{x}$. Therefore the map $z' \mapsto \hat{z}$ is a bijection between the points of H and the points of $\hat{x}\hat{y}$. Hence, all points of $\hat{x}\hat{y} = E_{\{i,j\},\emptyset}$ are S-full.

Now by Lemma 3.3 applied with $l = xy$ and $m = \hat{x}\hat{y}$ we find that all points of $xy = E_{\emptyset,\{ij\}}$ are S-full.

By the same token, all points contained in any line of $\mathcal{A}$ are S-full.

As the points of $\mathcal{A}$ span Π by Theorem 3.1, and every line of Π is opposite to some line of $\mathcal{A}$, using Lemma 3.3 repeatedly, we find that in fact all points of Π are S-full. Thus S is a generating set for Λ of size $2n^2 - n - 1$.

Since Λ has a natural embedding of dimension $2n^2 - n - 1$ by Lemma 3.2, it follows that its generating rank is $2n^2 - n - 1$. □

Note. An alternative generating set for the symplectic line-grassmannian was found by Cooperstein [5]. He constructs a generating set for $n = 2, 3$ and then inductively defines one for all $n > 3$. The main ingredient is the observation that, given a subspace U of V of dimension $2(n-1)$ on which the symplectic form is non-degenerate, the set of lines meeting U forms a geometric hyperplane of the symplectic line-grassmannian. Again, this construction only works in odd characteristic since it also uses Theorem 3.1.

By Lemma 3.2 the generating rank in any characteristic is at least $2n^2 - n - 1$. Note that for $n = 2$ this is the actual generating rank, even in even characteristic.

Acknowledgement. The author would like to express his gratitude to B. Cooperstein for his comments on the manuscript.

References

[1] Blok, R. J.; Brouwer, A. E.: *Spanning point-line geometries in buildings of spherical type.* J. Geometry **62** (1998), 26–35. Zbl 0915.51004

[2] Blok, R. J.; Pasini, A.: *Point-line geometries with a generating set that depends on the underlying field.* In: Finite Geometries. Proceedings of the Fourth Isle of Thorns Conference 2000. Dordrecht Kluwer Academic Publishers Dev. Math. **3**, 1–25. Zbl pre01795236

[3] Cooperstein, B. N.; Shult, E. E.: *Frames and bases of Lie incidence geometries*. J. Geometry **60** (1997), 17–46. Zbl 0895.51004

[4] Cooperstein, B. N.: *Generating long root subgroup geometries of classical groups over finite prime fields*. Bull. Belg. Math. Soc. – Simon Stevin **5** (1998), 531–548. Zbl 0923.51002

[5] Cooperstein, B. N.: personal communication.

[6] Shult, E. E.: *Geometric Hyperplanes of Embeddable Grassmannians*. J. Algebra **145** (1992), 55–82. Zbl 0751.51002

Received November 11, 2002