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Dark Clouds on Spheres and Totally Non-spherical Bodies of Constant Breadth

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Abstract. In this paper, we show that for any dimension $d \geq 3$ there exists a body of constant breadth C, such that its projection onto any 2-plane is non-spherical. We call such a body *totally non-spherical*. The circumradius of the projection of any totally non-spherical body C of constant breadth onto any 2-plane is bigger than the half diameter of C. Showing the existence of such a body extends results of Eggleston [4] and Weissbach [2], who showed it in the case $d = 3$.

Keywords: radii, minimal projections, isoperimetric inequalities, dark clouds, constant breadth, constant width, non-spherical

1. Introduction

This paper deals with the existence of convex bodies of constant breadth (sometimes also called bodies of constant width) with a very special property: that is, on whichever 2-space one (orthogonally) projects them, the projection will not be a disc (but surely again of constant breadth). If $d = 2$ this is obviously the whole class of constant breadth sets, except the disc itself. In 3-space however the most considered constant breadth bodies (bodies of revolution of a 2-dimensional constant breadth body and the Meissner bodies) do have spherical projections. Eggleston and Weissbach [4, 2] describe d-dimensional bodies of

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constant breadth without spherical $(d-1)$-projections (which are totally non-spherical if $d = 3$) and the totally isoradial body described by Brandenberg, Dattasharma, and Gritzmann [5] can not be projected onto a disc either. But so far nothing was known about totally non-spherical bodies in dimension $d \geq 4$. Here we will show that totally non-spherical bodies do exist in any dimension $d \geq 2$ and therefore we complete the diagram about the general '$\leq$'-relations between the different radii given in [5].

In the construction of the non-spherical bodies in dimensions $d \geq 5$ we use the concept of dark clouds. This is work based on unpublished work of Danzer [6], who described the concept in a much more general way. Here we only introduce it as much as needed.

2. Dark clouds

Let $\mathbb{E}^d = (\mathbb{R}^d, \|\cdot\|)$ denote the d-dimensional Euclidean space and $\mathbb{B}$, $\mathbb{S}$ the unit ball and the unit sphere in $\mathbb{E}^d$, respectively. We call a set $C \subset \mathbb{E}^d$ with a non-empty interior a *body* if it is bounded, closed, and convex. For $j \in \{1, \ldots, d\}$ the inner j-radius of a body C is the maximum $r_j(C)$ of the radii of j-balls of radius ρ which fit into C and the outer j-radius is the minimum $R_j(C)$ of numbers $\rho \geq 0$ such that there exists a $(d-j)$-flat F in $\mathbb{E}^d$ for which $C \subset F + \rho\mathbb{B}$. Here the '+' denotes the usual Minkowski sum. In this terms the bodies of constant breadth are exactly the bodies with equal inner and outer 1-radius.

Definition 2.1. *Suppose G is a lattice in $\mathbb{R}^d$, $r\mathbb{B}$ is a ball of radius $r \leq \frac{1}{2}$, such that $r\mathbb{B} + G$ forms a packing of $\mathbb{R}^d$. Let $\alpha > 0$, $a_i \in \mathbb{R}^d$, $i = 0, \ldots, n-1$. A dark cloud in $\mathbb{R}^{d+1}$ is a packing $\bigcup_{i=0}^{n-1}(a_i, \alpha i) + r\mathbb{B} + G$ such that no line, which meets the hyperplane $x_{d+1} = 0$ in a single point, can miss all these translations. We call αn the width of the dark cloud.*

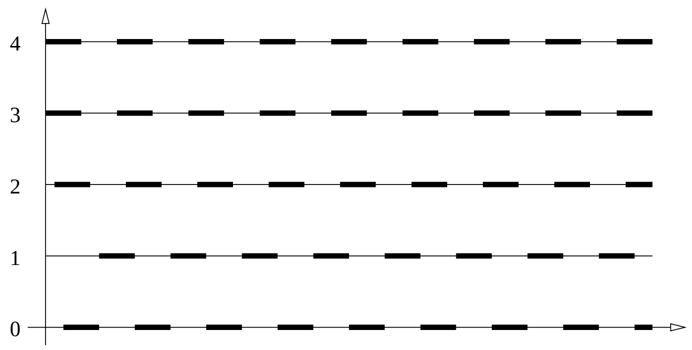


Figure 1. A sketch of a portion of a dark cloud for $d = 1$ and $n = 5$.

Lemma 2.2. *Dark clouds exist for any $d \in \mathbb{N}$ and any radius $r \leq \frac{1}{2}$.*

Proof. As every line intersecting $x_{d+1} = 0$ in a single point can be determined by a pair of points $(x, 0), (y, 1) \in \mathbb{R}^{d+1}$, we want to investigate sets of the form

$$K(\lambda, a, i) := \{(x, y) \in \mathbb{R}^{2d} : (i(y - x) + x, i) \in (a, i) + \lambda r\mathbb{B} + G\},$$

where $\lambda \in \{\frac{1}{2}, 1\}$, $a \in \mathbb{R}^d$, $i \in 0, \dots, n-1$ for some $n \in \mathbb{N}$, and G is the unit lattice. Hence $K(\lambda, a, i)$ is the subset of $\mathbb{R}^{2d}$ of all points (x, y) such that the line through $(x, 0)$ and $(y, 1)$ meets the packing $(a, i) + \lambda r\mathbb{B} + G$. Because G is the unit lattice we are able to restrict our attention to $x, y \in I^d$. So the density of our sets in $\mathbb{R}^d$ or $\mathbb{R}^{2d}$ are simply their volumes in $\mathbb{R}^d$ or $\mathbb{R}^{2d}$, respectively.

Note that the probability that $(x, y) \in \mathbb{R}^{2d}$ is in $K(\lambda, a, i)$ for any a is $\lambda^d \rho$, where ρ is the volume of $r\mathbb{B}$. So, if $a_0, \dots, a_{n-1}$ are chosen at random in $\mathbb{R}^d$, the probability that $(x, y) \notin \bigcup_{i=1}^n K(\lambda, a_i, i)$ is $(1 - \lambda^d \rho)^n$. Consequently there must exist $a_0, \dots, a_{n-1} \in \mathbb{R}^d$ such that the density of $\mathbb{R}^{2d} \setminus \bigcup_{i=1}^n K(\lambda, a_i, i)$ is at most $(1 - \lambda^d \rho)^n$.

Now for $(x_0, y_0) \in \mathbb{R}^{2d}$ consider the subset

$$T(x_0, y_0) := \{(x, y) : x \in x_0 + \frac{1}{15n} r\mathbb{B} + G, y \in y_0 + \frac{1}{15n} r\mathbb{B} + G\}.$$

If $T(x_0, y_0) \cap K(\frac{1}{2}, a_i, i) \neq \emptyset$ then there exist x, y, with $\|x - x_0\| < \frac{r}{15n}$, $\|y - y_0\| < \frac{r}{15n}$, and $(iy - (i-1)x, i) \in (a_i, i) + \frac{1}{2} r\mathbb{B} + G$, i.e. $\|a_i - iy + (i-1)x\| < \frac{1}{2} r, \mod G$. So if $(x', y') \in T(x_0, y_0)$, then $\|x' - x\| < \frac{2r}{15n}$, $\|y' - y\| < \frac{2r}{15n}$, and therefore $\|i(y - y') - (i-1)(x - x')\| < \frac{4}{15} r$. Hence $\|a_i - (iy' - (i-1)x')\| < r$, i.e. $T(x_0, y_0) \subset K(1, a_i, i)$. Now, because the density of $T(x_0, y_0) = \frac{4\rho^2}{225n^2}$, by choosing $\lambda = \frac{1}{2}$ and n large enough, we get $(1 - \frac{1}{2^d}\rho)^n < \frac{4\rho^2}{225n^2}$. Hence, for all $(x_0, y_0) \in \mathbb{R}^{2d}$ there exist a_i, $i = 0, \dots, (n-1)$, such that $T(x_0, y_0) \cap K(\frac{1}{2}, a_i, i) \neq \emptyset$ for at least one i. But this means that $T(x_0, y_0) \subset K(1, a_i, i)$. In particular $(x_0, y_0) \in K(1, a_i, i)$, so $\bigcup_{i=1}^n K(1, a_i, i)$ covers $\mathbb{R}^{2d}$ and that means that the sets $\bigcup_{i=1}^n (a_i, i) + r\mathbb{B} + G$ form a dark cloud. □

Lemma 2.3. *Let $\alpha \in (0, 1)$, and $\beta, \gamma > 0$. Then there exists a dark cloud in the region $0 \leq x_{d+1} \leq \alpha$, such that each ball in the cloud has radius $r < \beta$ and any pair of balls is at least $e > \gamma r$ apart.*

Proof. By Lemma 2.2 there exists a dark cloud with n layers at 1 apart consisting of balls of radius $r < \beta$ in these layers. Now reduce everything by a factor $\frac{\alpha}{n}$. The layers are then in the region $0 \leq x_{d+1} \leq \alpha$ and their distance apart is $\frac{\alpha}{n}$. The balls are now of radius $\frac{\alpha}{n} r$ and in their layers they are $\frac{\alpha}{n}(2 - 2r)$ apart, while the balls in different layers are $\frac{\alpha}{n}(1 - 2r)$ apart. Hence the balls have radius $\frac{\alpha}{n} r < r < \beta$ and their distance apart is at least $\frac{\alpha}{n}(1 - 2r)$. So picking r such that $\frac{1}{r} - 2 > \gamma$ we get the desired result. □

Lemma 2.4. *Suppose A is the annulus $1 \leq \|x\| \leq 1 + \epsilon$, $\epsilon > 0$. Then there exists a collection of dark clouds $\mathcal{C}$ such that any line meeting $\mathbb{B}$ meets at least one of the balls of $\mathcal{C}$ within A.*

Proof. Suppose P is a polytope such that $\mathbb{B} \subset P$ and all vertices of $(1 + \alpha)P$ are contained in $(1 + \epsilon)\mathbb{B}$ for some α, with $0 < \alpha < \epsilon$. Now we place dark clouds of width α along all of the facets of P. Hence every line meeting $\mathbb{B}$ meets also P and because the vertices of $(1 + \alpha)P$ are lying in the annulus every such line cuts through one of the dark clouds touching a ball in the cloud within the annulus. □

Definition 2.5. *Any packing of caps on the d-dimensional sphere $\mathbb{S}$ within the region $\alpha - \epsilon \leq x_{d+1} \leq \alpha$, $0 < \alpha < 1$, is called a spherical dark cloud of width ϵ, if any great 2-circle on $\mathbb{S}$ which meets the cap $x_{d+1} \geq \alpha$ intersects at least one cap in the packing.*

Lemma 2.6. *Every cap of* $\mathbb{S}$ *of the form* $x_{d+1} \geq \alpha$, $0 < \alpha < 1$ *can be blocked by a spherical dark cloud of any width* $0 < \epsilon < \alpha$.

Proof. If we project the region $\alpha - \epsilon \leq x_{d+1} \leq \alpha$ from 0 onto the hyperplane $x_{d+1} = 2$, it forms an annulus.

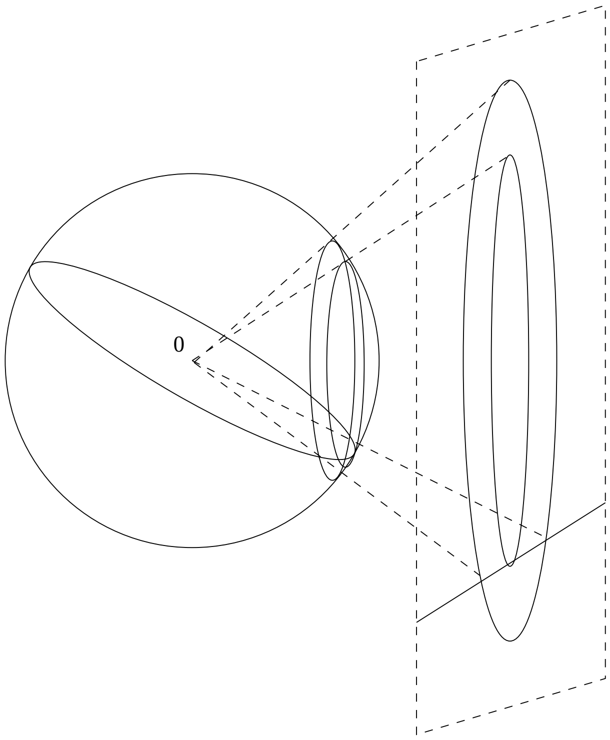


Figure 2. Projecting the region between two parallel caps onto an annulus

Now we apply Lemma 2.4 to obtain a collection of dark clouds which blocks every line meeting the ball surrounded by the annulus. But, because every great 2-circle on $\mathbb{S}$ which meets the cap $x_{d+1} \geq \alpha$ is projected onto such a line on $x_{d+1} = 2$, we receive, by back projection, a blocking of great 2-circles on the sphere. So far the projected collection of dark clouds does not necessarily consist of disjoint caps, but because of Lemma 2.3 we can choose the distance of the balls within one cloud to be arbitrary large. So, by replacing the disjoint parts of the projection onto the sphere by equally sized disjoint caps we receive our spherical dark cloud. □

3. Totally non-spherical bodies

Lemma 3.1. *For any dimension* $d \geq 3$ *there exists a finite set of closed caps* $\pm C_1, \ldots, \pm C_m$ *on* $\mathbb{S}$ *with disjoint relative interior such that every great 2-circle on* $\mathbb{S}$ *(and therefore any great* j*-circle with* $2 \leq j \leq d-1$*) meets the relative interior of at least one pair* $\pm C_i$.

Proof. Every point x on $\mathbb{S}$ has $\|x\| = 1$. Hence every great 2-circle meets the hyperplane $x_i = \frac{1}{\sqrt{d}}$ for some i. Now we block all these hyperplanes, as described in Lemma 2.6 in the region $\frac{1}{\sqrt{d}} - \epsilon \leq x_i \leq \frac{1}{\sqrt{d}}$ and handle overlapping caps as we handled it already in the proof of that lemma. But now, as no great 2-circle can be parallel or approximately parallel to all of this hyperplanes they all hit at least one antipodal pair of the clouds and therefore at least one antipodal pair of caps within the clouds. $\square$

Also the above proof holds for all $d \geq 3$ we will give a special one for $d \in \{3, 4\}$:

Proof. This time we start with the caps $\pm C_i$, $i = 1, \dots, d$ as follows:

$$C_i := \{x \in \mathbb{S} : x_i \geq \frac{1}{\sqrt{2}}\}.$$

If $d = 3$ every great circle must intersect through this caps as the biggest disc which fits into a cube of edge length $\sqrt{2}$ has radius $\frac{\sqrt{3}}{2}$ [3], which is strictly less than 1.

Hence we can assume that $d \geq 4$ and concentrate on great circles which do not entirely lay in a hyperplane of the form $x_i = 0$ (otherwise we can reduce the problem to the $d = 3$ case).

The intersection sets of $\pm C_i \cap \pm C_j$ are only the points with i-th and j-th coordinate $\pm\frac{1}{\sqrt{2}}$ and the rest zero.

Let us now seek a great circle Σ not cutting through the relative interior of the 8 caps. Suppose Σ meets the hyperplane $x_4 = 0$ at the points $\pm(x_1, x_2, x_3, 0)$. So, we have $|x_i| \leq \frac{1}{\sqrt{2}}$, $i = 1, 2, 3$. Now let $\pm y$ be the points on Σ perpendicular to $\pm x$. Then $|y_i| \leq \frac{1}{\sqrt{2}}$, $i = 1, \dots, 4$. Now, every point $z \in \Sigma$ is given by $z = x\cos\theta + y\sin\theta$ with $\theta \in [0, 2\pi)$, and we require $|x_i\cos\theta + y_i\sin\theta| \leq \frac{1}{\sqrt{2}}$, $i = 1, \dots, 4$ for all θ. But as $|x_i\cos\theta + y_i\sin\theta| \leq \sqrt{x_i^2 + y_i^2}$ for all θ it must hold $\sqrt{x_i^2 + y_i^2} \leq \frac{1}{\sqrt{2}}$ and therefore $x_i^2 + y_i^2 \leq \frac{1}{2}$, $i = 1, \dots, 4$. By adding these inequalities over all i and using $x, y \in \mathbb{S}$ we receive that $x_i^2 + y_i^2 = \frac{1}{2}$, $i = 1, \dots, 4$. As $x_4 = 0$ it follows $|y_4| = \frac{1}{\sqrt{2}}$ and therefore that Σ touches $\pm C_4$ in $\pm y$. By symmetry, Σ touches each of $\pm C_i$, $i = 1, \dots, 4$. But this means that for all $i = 1, 2, 3$ there must also exist some θ_i such that $x_i\cos\theta_i + y_i\sin\theta_i = \frac{1}{\sqrt{2}}$. Without loss of generality we can assume that $x_i, y_i \geq 0$ for a fixed i. Hence $|x_i\cos\theta_i + y_i\sin\theta_i| < \max\{|x_i\cos\theta_i|, |y_i\sin\theta_i|\} \leq \max\{x_i, y_i\} < \frac{1}{\sqrt{2}}$, if $\theta \in (\frac{\pi}{2}, \pi) \cup (\frac{3\pi}{2}, 2\pi)$. On the other hand if $\theta \in [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}]$ then $|x_i\cos\theta_i + y_i\sin\theta_i| = |x_i\cos\theta_i| + |y_i\sin\theta_i|$ which is the 1-norm of the point in $\mathbb{R}^2$ with coordinates $x_i\cos\theta_i$ and $y_i\sin\theta_i$. But as the 2-norm of this point is $\frac{1}{\sqrt{2}}$ the only possibilities for θ_i are $\theta \in \{\frac{k\pi}{2} : k \in \mathbb{Z}\}$ and $x_i, y_i \in \{0, \frac{1}{\sqrt{2}}\}$ such that $x_i + y_i = \frac{1}{\sqrt{2}}$.

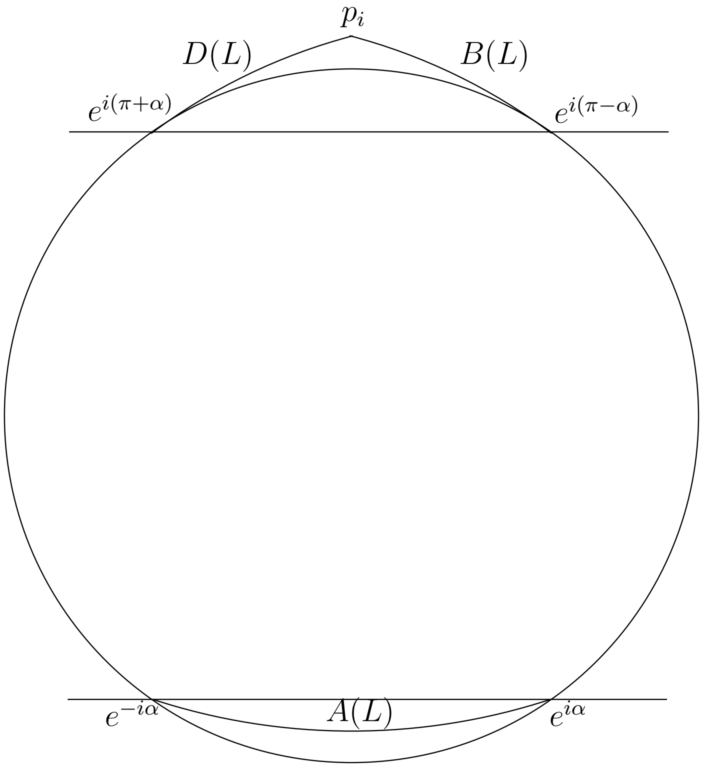
Now this means for all $i = 1, 2, 3$ the coordinates x_i, y_i have to be 0 or $\pm\frac{1}{\sqrt{2}}$ with one being 0 and the other one being $\pm\frac{1}{\sqrt{2}}$. But now as $x, y \in \mathbb{S}$ there can only be one $i \in \{1, 2, 3\}$ such that $y_i = \pm\frac{1}{\sqrt{2}}$. Hence there are only 6 different choices for Σ. But now as $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ all possible Σ run through 4 of the points $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ which are far away from the caps $x_i \geq \frac{1}{\sqrt{2}}$. So by adding caps $\pm C_i =$, $i = 5, \dots, 12$ of the form $\sum_{j=1}^{4} \pm x_j \geq 2 - \epsilon$, for a sufficiently small ϵ we get the desired set of closed caps. $\square$

As used in the second proof for $d \in \{3, 4\}$ it would be always possible to avoid a dark clouds construction if one knows a symmetric d-polytope P such that P does not contain a disc of radius 1 and the intersection of P and $\mathbb{B}$ does only contain points p on any $(d-2)$-face of P with $p \in \mathbb{S}$.

Definition 3.2. *Suppose C is a d-dimensional body of constant breadth. If none of the orthogonal projections of C onto 2-planes are discs, we call C totally non-spherical.*

Theorem 3.3. *For all $d \geq 3$ there exists a totally non-spherical body.*

Proof. The basic idea of this proof was already used by Danzer [1] and it is to replace the pairs of caps C_i and $-C_i$, $i = 1, \ldots, m$ in Lemma 3.1 by asymmetric sets D_i^+ and D_i^- which preserve the constant breadth property for the resulting body. How to do this?



Consider any pair $\pm C_i$, their line of symmetry l_i passing through 0 (the center of $\mathbb{B}$), and a 2-plane L containing l_i. Let the bounding points of $-C_i \cap L$ be $e^{-i\alpha}$ and $e^{i\alpha}$. We construct the point p on l_i lying above 0 relative to $-C_i$, at distance 2 from both $e^{-i\alpha}$ and $e^{i\alpha}$. Hence $p = (\sqrt{2 - \sin^2 \alpha} - \cos \alpha)e^{i\alpha}$ but is the same for any choice of L through l_i. p lies outside $L \cap \mathbb{B}$ but below the intersection of the tangents to $L \cap \mathbb{B}$ at $e^{i(\pi+\alpha)}$ and $e^{i(\pi-\alpha)}$ respectively. Now consider the three circular arcs of radius 2

(i) $A(L)$ with center in p and end points $e^{-i\alpha}$ and $e^{i\alpha}$ within $-C_i \cap L$,

(ii) $B(L)$ with center in $e^{-i\alpha}$ and end points $e^{i(\pi-\alpha)}$ and p, and

(iii) $D(L)$ with center in $e^{i\alpha}$ and end points $e^{i(\pi+\alpha)}$ and p.

Now we define D_i^+ as the union over all 2-planes L of the regions bounded by $B(L)$, $D(L)$, and the arc on $\mathbb{S}$ between $e^{i(\pi+\alpha)}$ and $e^{i(\pi-\alpha)}$ and D_i^- as the union over all 2-planes L of the regions bounded by $A(L)$ and the arc on $\mathbb{S}$ between $e^{i\alpha}$ and $e^{-i\alpha}$.

Now the resulting body K is again of constant breadth and because of Lemma 3.1 every great 2-circle on $\mathbb{S}$ intersects at least one of the regions $\pm C_i$, $i = 1, \ldots, m$. Hence the orthogonal projection of K onto any 2-plane can not be a disc. □

Theorem 3.3 allows us also to state below a corollary which generalizes results from Eggleston [4] and Weissbach [2], who showed it for $d = 3$.

Corollary 3.4. *For all $d \geq 3$ there exists a convex body C such that*

$$r_d(C) \leq \ldots \leq r_2(C) < r_1(C) = R_1(C) < R_2(C) \leq \ldots \leq R_d(C).$$

Proof. Follows from Theorem 3.3 and that the circumradius of a non-spherical 2-dimensional body of constant breadth is bigger than its half diameter. □

Because of Corollary 3.4 the diagram from [5] (see Figure 3) is complete in the sense that for any two radii which are not connected by a directed path there are bodies where the '<'-relationship holds in one (totally non-spherical bodies) or the other (ellipsoids with all axis of different length) direction.

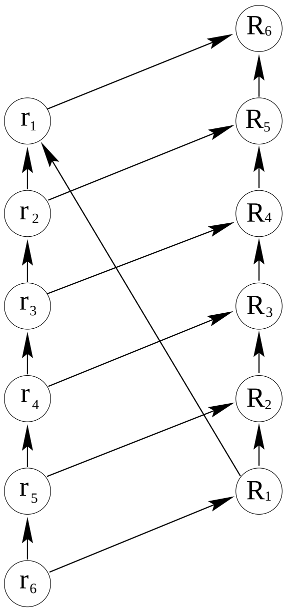


Figure 3. The edges imply a generally smaller-than relationship between the two corresponding radii.

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