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Non Standard Metric Products

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Abstract. We consider a fairly general class of natural non standard metric products and classify those amongst them, which yield a product of certain type (for instance a length space) for all possible choices of factors of this type (length spaces). We further prove the additivity of the Minkowski rank for a large class of metric products.

1. Introduction

Given a finite number (X_i, d_i), $i = 1, \dots, n$, of metric spaces there are different possibilities to define a metric d on the product $\Pi_{i=1}^n X_i$. The standard choice is of course the Euclidean product metric

$$d\Big((x_1, \dots, x_n), (y_1, \dots, y_n)\Big) \;=\; \Big(\sum_{i=1}^n d_i^2(x_i, y_i)\Big)^{\frac{1}{2}}.$$

A generalization of this construction is given by warped products ([1],[5]), which have been proven to be useful for the construction of Hadamard spaces. In this paper we consider another generalization, namely the following class of product metrics:
Let (X_i, d_i), $i = 1, \dots, n$, be metric spaces and denote the product set by $X = \Pi_{i=1}^n X_i$. It is natural to define a metric product d on X of the form $d = d_\Phi$,

$$d_\Phi\Big((x_1, \dots, x_n), (y_1, \dots, y_n)\Big) \;=\; \Phi\Big(d_1(x_1, y_1), \dots, d_n(x_n, y_n)\Big),$$

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where $\Phi : Q^n \longrightarrow [0,\infty)$ is a function defined on the quadrant $Q^n = [0,\infty)^n$.
Note that for Banach spaces a class of product spaces of this type appears in [6].

The function Φ has to satisfy certain natural conditions ((A) and (B) in Lemma 1) in order that d_Φ is a metric. These conditions still allow strange metrics on the product (even the trivial product when $n = 1$). In particular, Φ does not have to be continuous.

However, once we require for example that $\Phi : Q^n \longrightarrow [0,\infty)$ yields a length space (X, d_Φ) for all possible choices of length spaces (X_i, d_i), the conditions on Φ become very rigid. In fact, those conditions imply that Φ now has to be continuous.

In order to state the corresponding theorem we consider the function

$$\Psi : \mathbb{R}^n \longrightarrow [0,\infty), \qquad \Psi\Big(\sum_{i=1}^n x_i\, e_i\Big) := \Phi\Big(\sum_{i=1}^n |x_i|\, e_i\Big)$$

and say that Φ is induced by a norm if and only if the function Ψ is a norm.

We say that Φ preserves length spaces if the product of length spaces, endowed with the metric d_Φ, again is a length space (and correspondingly for other types of metric spaces).

Theorem 1. *Let $\Phi : Q^n \longrightarrow [0,\infty)$ be a function satisfying conditions* (A) *and* (B) *of Lemma* 1*. Then Φ preserves*
i) *length spaces,*

ii) *geodesic spaces,*

iii) *uniquely geodesic spaces,*

iv) *metric spaces of non-positive Busemann curvature,*

v) *metric spaces of curvature bounded from above (or below),*

if and only if

i),ii) Φ *is induced by a norm.*

iii),iv) Φ *is induced by a norm with strictly convex norm ball.*

v) Φ *is induced by a scalar product.*

In Section 3 we consider the behaviour of the Minkowski rank under non standard metric products. The Minkowski rank, $rank_M(X, d)$, of a metric space (X, d) is the supremum of the dimensions of normed vector spaces isometrically embedded into X. We generalize results of [7] to those metric products (X, d_Φ) where Φ is induced by a norm with strictly convex unit ball:

Theorem 2. *Let $\Phi : Q^n \longrightarrow [0,\infty)$ be a function such that $\Psi : \mathbb{R}^n \longrightarrow [0,\infty)$, defined as above, is a norm with a strictly convex norm ball. Let (X_i, d_i), $i = 1, \ldots, n$, be metric spaces and $X = \Pi_{i=1}^n X_i$. Then*

$$rank_M\Big(X, d_\Phi\Big) = \sum_{i=1}^n rank_M\Big(X_i, d_i\Big).$$

This shows one advantage of the Minkowski rank over the Euclidean rank, since the latter one is not additive, even with respect to the standard product ([7]).

A geodesic metric space (X, d) is called convex, if for every pair $c_1 : [a_1, b_1] \longrightarrow X$ and $c_2 : [a_2, b_2] \longrightarrow X$ of constant speed geodesics the function $d \circ (c_1, c_2) : [a_1, b_1] \times : [a_2, b_2] \longrightarrow \mathbb{R}$ is convex.

Kleiner ([8],Theorem D) proved that for a locally compact, convex, geodesic metric space (X, d) with cocompactly acting isometry group the Minkowski rank coincides with a number of other invariants, one of which is the quasi-Euclidean rank. The quasi-Euclidean rank, $rank_{qE}(X, d)$, is defined as

$$\sup\{k \mid \text{there is a quasi-isometric embedding } f : \mathbb{E}^k \longrightarrow X\}.$$

As a corollary of our results, we show that these invariants are additive under certain products. In particular we obtain the

Corollary 1. *Let Φ be as in Theorem 2 and (X_i, d_i), $i = 1, \ldots, n$, be locally compact convex metric spaces with cocompactly acting isometry group. Then for the quasi-Euclidean rank, $rank_{qE}$, one has*

$$rank_{qE}\Big(X, d_\Phi\Big) \;=\; \sum_{i=1}^{n} rank_{qE}\Big(X_i, d_i\Big).$$

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2. Non standard metric products

On Q^n we define a partial ordering $\leq$ in the following way: if $q^1 = (q^1_1, \ldots, q^1_n)$ and $q^2 = (q^2_1, \ldots, q^2_n)$ then

$$q^1 \;\leq\; q^2 \quad :\Longleftrightarrow \quad q^1_i \;\leq q^2_i \;\; \forall i \in \{1, 2, \ldots, n\}.$$

Let $\Phi : Q^n \longrightarrow [0, \infty)$ be a function and consider the function $d_\Phi : X \times X \longrightarrow [0, \infty)$,

$$d_\Phi\Big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\Big) \;=\; \Phi\Big(d_1(x_1, y_1), \ldots, d_n(x_n, y_n)\Big).$$

In order that d_Φ will be a metric, we clearly have to assume

(A) $\Phi(q) \;\geq\; 0 \;\; \forall q \in Q$ and $\Phi(q) \;=\; 0 \;\; \Leftrightarrow \;\; q = 0.$

The symmetry of d_Φ is obvious. We now translate the triangle inequality for d_Φ into a condition on Φ.

Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $z = (z_1, \ldots, z_n) \in X$ and consider the "distance vectors"

$$\begin{aligned} q^1 \;&:=\; \Big(d_1(x_1, z_1), \ldots, d_n(x_n, z_n)\Big), \\ q^2 \;&:=\; \Big(d_1(x_1, y_1), \ldots, d_n(x_n, y_n)\Big) \quad \text{and} \\ q^3 \;&:=\; \Big(d_1(y_1, z_1), \ldots, d_n(y_n, z_n)\Big) \end{aligned}$$

in Q^n. Since for every $i \in \{1,\dots,n\}$, x_i, y_i, z_i are points in X_i we see that $q^j \leq q^k + q^l$ for every permutation $\{j,k,l\}$ of $\{1,2,3\}$.
Now d_Φ satisfies the triangle inequality if Φ satisfies
(B) for all points $q^1, q^2, q^3 \in Q^n$ with $q^j \leq q^k + q^l$ we have

$$\Phi(q^j) \;\leq\; \Phi(q^k) \;+\; \Phi(q^l).$$

Remark.
i) Note that for q^1, q^2, q^3 one can always take a triple of the form $p, q, p+q$, hence (B) implies in particular $\Phi(p+q) \leq \Phi(p) + \Phi(q)$, which will be called the sub-additivity of Φ in the following.
ii) The condition (B) can be applied for the triple p, q, q in the case that $p \leq 2q$. Then $\Phi(p) \leq 2\Phi(q)$.

It is now easy to prove the following result

Lemma 1. *Let $\Phi : Q^n \longrightarrow [0,\infty)$ be a function. Then d_Φ is a metric on X for all possible choices of metric spaces (X_i, d_i), $i = 1,\dots,n$, if and only if Φ satisfies* (A) *and* (B).

This Lemma still allows strange metrics on a product (even the trivial product $n = 1$). Let for example $\Phi : Q^n \longrightarrow [0,\infty)$ be an arbitrary function with $\Phi(0) = 0$ and $\Phi(q) \in \{1,2\}$, $\forall q \in Q^n \setminus \{0\}$. Then d_Φ is a metric.

If we, however, require for example that the product metric space X is always a length space in the case the X_i are, the conditions on Φ are very rigid.

For the convenience of the reader we recall the notion of a length space (compare e.g. [3] I.3). Let (X,d) be a metric space. For a continuous path $c : [0,1] \longrightarrow X$ one defines as usual the length

$$L(c) \;:=\; \sup\Big\{\sum_{j=1}^{k} d\Big(c(t_{j-1}), c(t_j)\Big)\Big\}, \tag{1}$$

where the *sup* is taken over all subdivisions

$$0 \;=\; t_0 \;\leq\; t_1 \;\leq \cdots \leq\; t_k \;=\; 1 \quad \text{of } [0,1].$$

(X,d) is called a length space if for all $x, y \in X$, $d(x,y) = \inf L(c)$, where the inf is taken over all continuous paths from x to y. The curve c is called rectifiable if $L(c)$ is finite.

Spaces of curvature bounded from above (resp. below) are defined via comparison triangles in appropriate space forms of constant curvature. We refer the reader to [3] and [4] for details. A space of non-positive Busemann curvature is a geodesic space such that the metric is locally convex ([3]). If the metric is even globally convex, the geodesic space is called convex. For instance each metric space of curvature ≤ 0 has non-positive Busemann curvature and each geodesic CAT(0)-space is convex (but not vice-verse).

We further need the following two Lemmata:

Lemma 2. *For $\Phi : Q^n \longrightarrow [0,\infty)$ the function $\Psi : \mathbb{R}^n \longrightarrow [0,\infty)$ defined via*

$$\Psi\Big(\sum_{i=1}^{n} x_i\, e_i\Big) \;:=\; \Phi\Big(\sum_{i=1}^{n} |x_i|\, e_i\Big)$$

is a norm on $\mathbb{R}^n$ if and only if Φ satisfies the following conditions:
(1) $\Phi(q) \geq 0\ \forall q \in Q^n$ *and* $\Phi(q) = 0\ \Leftrightarrow q = 0$,
(2) Φ *is monotone, i.e.* $q \leq p \implies \Phi(q) \leq \Phi(p) \forall p, q \in Q^n$,
(3) $\Phi(p+q) \;\leq\; \Phi(p) \;+\; \Phi(q)$,
(4) $\Phi(\lambda q) \;=\; \lambda \Phi(q)\ \forall p \in Q^n, \lambda \geq 0$.

Proof. $\Longrightarrow$: Let Φ satisfy $(1)-(4)$. Then $\Psi \geq 0$, $\Psi(x) = 0 \iff x = 0$ and $\Psi(\lambda x) = |\lambda| \Psi(x)$ directly follow from the definition of Ψ. In order to verify the subadditivity, note that for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$

$$\begin{aligned}
\Psi(x \;+\; y) \;&=\; \Phi\Big(\sum_{i=1}^{n} |x_i + y_i|\, e_i\Big) \\
&\overset{(2)}{\leq}\; \Phi\Big(\sum_{i=1}^{n} (|x_i| + |y_i|)\, e_i\Big) \\
&\overset{(3)}{\leq}\; \Phi\Big(\sum_{i=1}^{n} |x_i|\, e_i\Big) \;+\; \Phi\Big(\sum_{i=1}^{n} |y_i|\, e_i\Big) \\
&=\; \Psi(x) \;+\; \Psi(y).
\end{aligned}$$

$\Longleftarrow$: Assume now that Ψ is a norm. Then Φ clearly satisfies $(1), (3), (4)$. To prove (2) it is enough to show that $\Phi(p + \lambda e_i) \geq \Phi(p)$ for any unit vector e_i and $\lambda \geq 0$. Assume that $\Phi(p + \lambda e_i) < \Phi(p)$. Write $p = (p_1, \ldots, p_n) \in Q^n$, let $q = (p_1, \ldots, p_{i-1}, -p_i - \lambda, p_{i+1}, \ldots, p_n) \in \mathbb{R}^n$. Then $\Psi(q) = \Psi(p + \lambda e_i) < \Psi(p)$ but p is on the segment between q and $p + \lambda e_i$. This contradicts the subadditivity of Ψ. $\square$

Lemma 3. *Let $\Phi : Q^n \longrightarrow [0,\infty)$ satisfy $(1)-(4)$ as in Lemma 2. Then Ψ as in Lemma 2 is induced by a scalar product g_Ψ on $\mathbb{R}^n$ if and only if Φ satisfies the property*

$$(5) \qquad \Phi^2\Big(\sum_{i=1}^{n} \lambda_i\, e_i\Big) \;=\; \sum_{i=1}^{n} \Phi^2(\lambda_i\, e_i) \quad \forall \lambda_i > 0.$$

In this case the set $\{e_1, \ldots, e_n\}$ is an orthogonal system of g_Ψ.

Proof. From Lemma 2 we know that Ψ is a norm if and only if conditions $(1)-(4)$ hold. Now we show that Ψ satisfies the parallelogram equation if and only if condition (5) also holds:
$\Longrightarrow$: Suppose that Φ satisfies condition (5). Then for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ the parallelogram equation is equivalent to

$$\sum_{i=1}^{n} \Big[|x_i + y_i|^2 \;+\; |x_i - y_i|^2 \;-\; 2\,[|x_i|^2 \;+\; |y_i|^2]\Big]\Phi^2(e_i) \;=\; 0,$$

which holds trivially.

$\Longleftarrow$: Now suppose that the parallelogram equation holds. For $x = (x_1, \dots, x_{n-1}, 0)$ and $y = (0, \dots, 0, y_n)$ it takes the form

$$\Phi^2\Big(\sum_{i=1}^{n-1} |x_i|\, e_i \;+\; |y_n|\, e_n\Big) \;=\; \Phi^2\Big(\sum_{i=1}^{n-1} |x_i|\, e_i\Big) \;+\; \Phi^2\Big(|y_n|\, e_n\Big).$$

The same computation for $x = (x_1, \dots, x_{n-2}, 0, 0)$, $y = (0, \dots, 0, y_{n-1}, 0)$ and so on finally yields condition (5). □

From Lemmata 2 and 3 we easily conclude the following propositions:

Proposition 1. *Let $(V_i, ||\cdot||_i)$, $i = 1, \dots, k$ be normed vector spaces and $\Phi : Q^n \longrightarrow [0, \infty)$ be a function. Define the function $||\cdot||_\Phi : V = V_1 \times \dots \times V_k \longrightarrow [0, \infty)$ through*

$$\Big|\Big|(v_1, \dots, v_k)\Big|\Big|_\Phi \;:=\; \Phi\Big(\sum_{i=1}^{k} ||v_i||_i\, e_i\Big).$$

Then $(V, ||\cdot||_\Phi)$ is a normed vector space for all possible choices of normed vector spaces $(V_i, ||\cdot||_i)$ if and only if Ψ as defined in Lemma 2 is a norm.

and

Proposition 2. *Let $(V_i, ||\cdot||_i)$, $i = 1, \dots, k$, be normed vector spaces the norms of which are induced by scalar products $< \cdot, \cdot >_i$ on V_i and $\Phi : Q^n \longrightarrow [0, \infty)$ be a function.*
Then the norm $||\cdot||_\Phi$ on $V = V_1 \times \dots \times V_k$ as in Proposition 1 is induced by a scalar product $< \cdot, \cdot >_\Phi$ for all choices of vector spaces V_i with scalar products $< \cdot, \cdot >_i$, if and only if the norm Ψ as defined in Lemma 2 is induced by a scalar product g_Ψ on $\mathbb{R}^n$.
Thus for two vectors $v = (v_1, \dots, v_k), w = (w_1, \dots, w_k) \in v$ one always has

$$< v, w >_\Phi = \sum_{i=1}^{k} \Phi^2(e_i) \;\; < v_i, w_i > ,$$

which is the usual Euclidean product up to a scale of the scalar products on the factors.

Note that the degree to which $\{e_1, \dots, e_n\}$ fails to be an orthonormal basis of g_Ψ is the degree to that $< \cdot, \cdot >_\Phi$ differs from the standard scalar product of Euclidean products.

Outline of the proofs of Propositions 1 *and* 2. The "if parts" are obvious consequences of the Lemmata 2 and 3. For the only if part one considers special settings in which all of the factors are the reals with the standard norm (scalar product, respectively). While conditions (1) and (4) from Lemma 2 hold trivially, the conditions (2) and (3) now easily follow by simple constructions in $\mathbb{R}$. Finally, for Theorem 2, condition (5) follows just as in Lemma 3.

We are now able to give the

Proof of Theorem 1. $\Longrightarrow$: For the "only if" part we will restrict our attention to appropriately chosen factors (X_i, d_i). Of course we do so by assuming that all the factors are the reals with the standard metric: $(X_i, d_i) = (\mathbb{R}, d_e)$, $i = 1, \ldots, k$.

Similar as in Lemma 1 we see that Φ satisfies $(A) = (1)$ and (B) which implies the subadditivity of Φ.

We first show that in all the cases i) - v) Φ must be induced by a norm. To start with, let us show that Φ has to be continuous. Suppose it is not. By subadditivity, one easily gets that the restriction of Φ to one of the coordinate lines (say the first one) is discontinuous at 0. Then any two points with different projections on X_1 can not be joined by a continuous curve in the product space, which therefore can not be a length space.

That Φ is induced by a norm now follows immediately from Theorem 6 in [2], which implies that any homogeneous length metric on $\mathbb{R}^n$ that induces the same topology as the standard metric must come from a norm.

v) now follows from the fact that a normed vector space has a lower or upper curvature bound if and only if it is Euclidean. Thus, due to Proposition 2, Φ must be induced by a scalar product.

In order to complete the proof in the cases iii) and iv) it suffices to remark that if Φ does not admit a strictly convex unit ball, then the product space is generally not uniquely geodesic. This can be seen for instance in the case $(\mathbb{R}^n, \Psi)$.

$\Longleftarrow$: Let now Φ be induced by a norm. Note that from Lemma 2 it follows immediately that Φ satisfies the conditions $(1) - (4)$ of Lemma 2. In order to show that for any choices of length (geodesic) spaces $X_1, \ldots, X_k$ the product (X, d_Φ) is a length (geodesic) space we prove the following

Lemma 4. *Let (X_i, d_i) be metric spaces and $c_i : [0,1] \longrightarrow X_i$ be continuous curves parameterized by arclength connecting $p_i \in X_i$ with $q_i \in X_i$, $i = 1, \ldots, k$. Denote by l_i the (X_i, d_i)-length of c_i and suppose that Φ satisfies conditions $(1) - (4)$. Then the (X, d_Φ)-length of the product curve $c = (c_1, \ldots, c_k) : [0,1] \longrightarrow X$ is $L(c) = \Phi(l_1, \ldots, l_k)$.*

Furthermore c is also parameterized by arclength.

Proof. Note that the (X_i, d_i)-length $L(c_i)$ of c_i is given through

$$l_i \;=\; L(c_i) \;=\; \lim_{N \longrightarrow \infty} \sum_{j=1}^{N} d_i\Big(c_i(\frac{j-1}{N}), c_i(\frac{j}{N})\Big),$$

where $d_i(c_i(\frac{j-1}{N}), c_i(\frac{j}{N})) \leq \frac{l_i}{N}$.

For the (X, d_Φ)-length $L(c)$ of c one has

$$\begin{aligned} L(c) &= \lim_{N \longrightarrow \infty} \sum_{j=1}^{N} d_\Phi\Big(c(\frac{j-1}{N}), c(\frac{j}{N})\Big) \\ &= \lim_{N \longrightarrow \infty} \sum_{j=1}^{N} \Phi\Big(d_1\Big(c_1(\frac{j-1}{N}), c_1(\frac{j}{N})\Big), \ldots, d_k\Big(c_k(\frac{j-1}{N}), c_k(\frac{j}{N})\Big)\Big) \end{aligned}$$

$$\begin{aligned}
&\overset{(2)}{\leq} \lim_{N\longrightarrow\infty}\sum_{j=1}^{N}\Phi\Big(\frac{l_1}{N},\dots,\frac{l_k}{N}\Big)\\
&\overset{(4)}{=} \lim_{N\longrightarrow\infty}\sum_{j=1}^{N}\frac{1}{N}\Phi(l_1,\dots,l_k)\\
&= \Phi(l_1,\dots,l_k).
\end{aligned}$$

On the other hand the continuity and subadditivity of Φ yield:

$$\begin{aligned}
L(c) &= \lim_{N\longrightarrow\infty}\sum_{j=1}^{N} d_\Phi\Big(c(\frac{j-1}{N}),c(\frac{j}{N})\Big)\\
&= \lim_{N\longrightarrow\infty}\sum_{j=1}^{N}\Phi\Big(d_1\Big(c_1(\frac{j-1}{N}),c_1(\frac{j}{N})\Big),\dots,d_k\Big(c_k(\frac{j-1}{N}),c_k(\frac{j}{N})\Big)\Big)\\
&\overset{(3)}{\geq} \lim_{N\longrightarrow\infty}\Phi\Big(\sum_{j=1}^{N} d_1\Big(c_1(\frac{j-1}{N}),c_1(\frac{j}{N})\Big),\dots,\sum_{j=1}^{N} d_k\Big(c_k(\frac{j-1}{N}),c_k(\frac{j}{N})\Big)\Big)\\
&= \Phi(l_1,\dots,l_k),
\end{aligned}$$

where the last equality is due to the continuity of Φ. □

Let now (X_i,d_i), $i=1,\dots,k$, be length spaces. Then the distance of any two points $p_i,q_i\in X_i$ may be approximated arbitrarily well by the lengths of continuous curves in (X_i,d_i) joining p_i and q_i. Thus (X,d_Φ) turns out to be a length space itself, due to the definition of d_Φ, the validity of Lemma 4 and the continuity of Φ.
This completes the proof in the cases i) and ii).

v) now just follows from Proposition 2 and the fact that stretching the factor metrics d_i by $\Phi(e_i)$ and then taking the standard product yields as product a space with an upper (resp. lower) curvature bound. This last fact can be seen by an easy comparison argument using the fact that the Euclidean product of two constant curvature space forms has upper and lower curvature bounds. See also [3], II 1.16. and [4].

The local convexity of the distance function in iv) is obvious. In order to finish the proof in the cases iii) and iv) we have to show that the product is uniquely geodesic again and we might as well only consider the product of two factors in order to avoid unneccesary index complications:
Let therefore $(x_1,x_2),(y_1,y_2)\in X$, set $D:=d_\Phi((x_1,x_2),(y_1,y_2))$ and let $c=(c_1,c_2):[0,D]\longrightarrow X$ be a geodesic in X joining (x_1,x_2) to (y_1,y_2). Set $v_1:=(d_1(x_1,c_1(t)),d_2(x_2,c_2(t)))\in Q^2$, $v_2:=(d_1(y_1,c_1(t)),d_2(y_2,c_2(t)))\in Q^2$ for $t\in[0,D]$ and $v:=(d_1(x_1,y_1),d_2(x_2,y_2))\in Q^2$. By the triangle inequality in each component, we have $v_1+v_2\geq v$. With (c_1,c_2) being a geodesic and by subadditivity, we must have

$$\Phi(v_1)+\Phi(v_2)=\Phi(v)\leq\Phi(v_1+v_2)\leq\Phi(v_1)+\Phi(v_2).$$

Since the norm ball associated to Φ is strictly convex, the conditions $\Phi(v_1)+\Phi(v_2)=\Phi(v)$ and $v\leq v_1+v_2$ can only be satisfied if $v_1=\lambda_1 v, v_2=\lambda_2 v$ with $\lambda_1+\lambda_2=1$. Since

$\Phi(v_1) = t, \Phi(v) = D$, we get $\lambda_1 = \frac{t}{D}, \lambda_2 = 1 - \frac{t}{D}$. The spaces X_1, X_2 are uniquely geodesic and therefore $c_1(t), c_2(t)$ are fixed by these equations. Hence there is a unique geodesic (c_1, c_2) joining (x_1, x_2) and (y_1, y_2). □

3. Minkowski rank of products

In this section we prove Theorem 2 and Corollary 1. As the proof of Theorem 2 is almost the same as the one of Theorem 2 of [7] we keep it fairly short.

In [7] we introduced the notions of the Euclidean and the Minkowski rank for arbitrary metric spaces as follows.

Definition 1. *Minkowski- and Euclidean rank for metric spaces:*
a) *For an arbitrary metric space (X, d) the Minkowski rank is*

$$rank_M(X,d) \ := \sup_{(V,||\cdot||)} \Big\{ dimV \ \Big| \ \exists \ isometric \ map \ \ i_V : (V, ||\cdot||) \longrightarrow (X,d) \Big\}.$$

b) *The Euclidean rank is defined as*

$$rank_E(X,d) \ := \ \sup \Big\{ n \in \mathbb{N} \ \Big| \ \exists \ isometric \ map \ \ i_{\mathbb{E}^n} : \mathbb{E}^n \longrightarrow (X,d) \Big\}.$$

The Minkowski rank was shown to be additive with respect to the standard product of arbitrary metric spaces, whereas a counterexample to the corresponding additivity of the Euclidean rank was provided. Since for metric spaces of locally one-side bounded Alexandrov curvature these two ranks coincide ([9],[7]), the additivity of the Euclidean rank with respect to the standard product follows for instance for metric spaces of non-positive or non-negative Alexandrov curvature.

In [8] Kleiner considered another notion of rank which we will refer to as quasi-Euclidean rank in the following. Recall that given two metric spaces $(X, d_X), (Y, d_Y)$, a map $f : X \longrightarrow Y$ is called quasi-isometric embedding of X in Y, if there exist $\lambda \geq 1$ and $\epsilon > 0$ such that for all $x_1, x_2 \in X$:

$$\lambda^{-1} d_X(x_1, x_2) - \epsilon \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2) + \epsilon$$

Definition 2. *The quasi-Euclidean rank is defined as*

$$rank_{qE}(X,d) \ := \ \sup \Big\{ n \in \mathbb{N} \ \Big| \ \exists \ quasi\text{-}isometric \ embedding \ \ f_{\mathbb{E}^n} : \mathbb{E}^n \longrightarrow (X,d) \Big\}.$$

Scetch of proof of Theorem 2. The analogue of Proposition 2 in [7] is

Proposition 3. *Let A denote an affine space on which the normed vector space $(V, |\cdot|)$ acts simply transitively. Let further (X_i, d_i), $i = 1, \ldots, n$, be metric spaces, $\Phi : Q^n \longrightarrow [0, \infty)$ be a function satisfying conditions* (1) – (4) *such that the norm ball of Ψ is strictly convex and let $\varphi : (A, |\cdot|) \longrightarrow (\Pi_{i=1}^n X_i, d_\Phi)$ be an isometric map. Then there exist pseudonorms $||\cdot||_i$, $i = 1, \ldots, n$, on V such that*
i) $|v| \ = \ \Phi\Big(\sum\limits_{i=1}^n \, ||v||_i \, e_i \Big) \quad \forall v \in V \quad$ *and*

ii) $\varphi_i : (A, ||\cdot||_i) \longrightarrow (X_i, d_i)$, $i = 1, \dots, n$ *are isometric.*

Note that from Proposition 3 one achieves the subadditivity of the Minkowski rank, while the superadditivity just follows by considering the product of two isometric embeddings of normed vector spaces into the factors.

We define $\alpha_i : A \times V \longrightarrow [0, \infty)$, $i = 1, \dots, n$, via

$$\alpha_i(a, v) \ := \ d_i\Big(\varphi_i(a), \varphi_i(a + v)\Big).$$

Since φ is isometric we have

$$\Phi\Big(\sum_{i=1}^{n} \alpha_i(a, v)\, e_i\Big) \ := \ d_\Phi\Big(\varphi(a), \varphi(a + v)\Big) \ = \ |v|. \tag{2}$$

In order to prove Proposition 3, we might as well restrict to the case $n = 2$ and show that $||\cdot||_i : V \longrightarrow \mathbb{R}^+$, $||v||_i := \alpha_i(v)$ $\forall v \in V$, $i = 1, 2$, are pseudonorms on V. Therefore we establish the following properties of α:

1) $$\alpha_i(a, v) \ = \ \alpha_i(a + v, v), \quad i = 1, 2, \quad \forall a \in A, v \in V,$$

2) $$\alpha_i(a, tv) \ = \ |t|\alpha_i(a, v), \quad i = 1, 2, \quad \forall a \in A, v \in V, t \in \mathbb{R},$$

3) $$\alpha_i(a, v) \ = \ \alpha_i(b, v), \quad i = 1, 2, \quad \forall a, b \in A, v \in V \quad \text{and}$$

4) $$\alpha_i(v + w) \ \le \ \alpha_i(v) \ + \alpha_i(w), \quad i = 1, 2, \quad \forall v, w \in V,$$

where $\alpha_i(v) := \alpha_i(a, v)$ with $a \in A$ arbitrary (compare with 3)). In order to prove 1) we note that the d_i's triangle inequality yields

$$\alpha_i(a, v) \ + \ \alpha_i(a + v, v) \ \ge \ \alpha_i(a, 2v).$$

Therefore the monotonicity of Φ gives

$$\Phi\Big(\sum_{i=1}^{n} [\alpha_i(a, v) \ + \ \alpha_i(a + v, v)]\, e_i\Big) \ \ge \ \Phi\Big(\sum_{i=1}^{n} \alpha_i(a, 2v)\, e_i\Big) \ = \ |2v| \ = \ 2\,|v|. \tag{3}$$

Set $x := \sum_{i=1}^{n} \alpha_i(a, v) e_i$ and $y := \sum_{i=1}^{n} \alpha_i(a + v, v) e_i$ and note that with equations (2) and (3) one has

$$\Phi(x) \ = \ |v| \ = \ \Phi(y) \quad \text{and} \quad \Phi(x + y) \ = \ 2\,|v|$$

and hence

$$\Phi(x + y) \ = \ \Phi(x) \ + \ \Phi(y) \ = \ 2\Phi(x) \ = \ 2\Phi(y).$$

From this it follows with the strict convexity of Φ that

$$x \ = \ y \quad \Longleftrightarrow \quad \alpha_i(a, v) \ = \ \alpha_i(a + v, v) \quad \forall i = 1, \dots, n,$$

which proves 1).
In order to prove 2) we note that the d_i's triangle inequality yields for all $n \in \mathbb{N}$

$$\alpha_i(a, nv) \;\leq\; \sum_{k=0}^{n-1} \alpha_i(a + kv, v) \;=\; n\alpha_i(a, v),$$

where the last equation follows from 1) by induction. Thus we find $\forall n \in \mathbb{N}, v \in V, a \in A$:

$$n^2||v||^2 \;=\; \Phi\Big(\alpha_1(a, nv), \alpha_2(a, nv)\Big) \;\leq\; n^2\Phi\Big(\alpha_1(a, v), \alpha_2(a, v)\Big) \;=\; n^2||v||^2$$

and therefore

$$\alpha_i(a, nv) \;=\; n\alpha_i(a, v), \quad i = 1, 2, \quad \forall n \in \mathbb{N}, v \in V,\, a \in A.$$

The claim now follows by the usual extension to $n \in \mathbb{Q}$ and finally to $n \in \mathbb{R}$.
In order to prove 3) we observe that for $n \in \mathbb{N}$ we have

$$\begin{aligned}
\Big|\alpha_i(a, nv) \;-\; \alpha_i(b, nv)\Big| \;&=\; \Big|d_i\Big(\varphi_i(a), \varphi(a + nv)\Big) - d_i\Big(\varphi_i(b), \varphi(b + nv)\Big)\Big| \\
&\leq\; d_i\Big(\varphi_i(a), \varphi_i(b)\Big) \;+\; d_i\Big(\varphi_i(a + nv), \varphi_i(b + nv)\Big) \\
&\leq\; d\Big(\varphi(a), \varphi(b)\Big) \;+\; d\Big(\varphi(a + nv), \varphi(b + nv)\Big) \\
&=\; 2||b - a||, \qquad i = 1, 2,
\end{aligned}$$

and therefore

$$\alpha_i(a, v) \;=\; \lim_{n\longrightarrow\infty} \frac{1}{n}\alpha_i(a, nv) \;=\; \lim_{n\longrightarrow\infty} \frac{1}{n}\alpha_i(b, nv) \;=\; \alpha_i(b, v), \quad i = 1, 2.$$

Finally 4) follows from

$$\alpha_i(v + w) \;=\; \alpha_i(a, v + w) \;\leq\; \alpha_i(a, v) \;+\; \alpha_i(a + v, w) \;=\; \alpha_i(v) \;+\; \alpha_i(w),$$

where the inequality follows by the d_i's triangle inequality and the last equation is due to 3).

The following example shows that the strict convexity of Φ, assumed in Theorem 2, is a necessary condition. Take the interval $\mathbb{R}_+ := [0, \infty)$ with the metric induced from $\mathbb{R}$. By simple geometric arguments, $rank_M(\mathbb{R}_+) = 0$. However, with $\Phi(x_1, x_2) := x_1 + x_2$, the Φ-product of two copies of $\mathbb{R}_+$ admits the geodesic line $c(t) = (-t, 0)$ if $t \leq 0$, $c(t) = (0, t)$ if $t \geq 0$.

Proof of Corollary 1. It is evident that the product of locally compact, convex metric spaces with cocompactly acting isometry group also satisfies these conditions. From Theorem D of [8] it follows that under these conditions the quasi-Euclidean and the Minkowski rank coincide. Thus the validity of Corollary 1 is a consequence of Theorem 2. □

Remark. In general, the quasi-Euclidean rank is not additive with respect to the standard product of metric spaces. Just consider the above example of two copies of $\mathbb{R}_+$ with the standard product. The geodesic defined above is trivially a quasi-geodesic.

References

[1] Alexander, S. B.; Bishop, R.L.: *Warped products of Hadamard spaces.* Manuscripta Math. **96** (1998), 487–505. Zbl 0909.53050

[2] Berestovskij, V. N.: *Homogeneous manifolds with an intrinsic metric I.* Sib. Math J. **29** (1988), 887–897. Zbl 0686.53041

[3] Bridson, M.; Haefliger, A.: *Metric spaces of non-positive curvature.* Springer Verlag Berlin 1999. Zbl 0988.53001

[4] Burago, Yu.; Gromov, M.; Perelman, G.: *A.D.Alexandrov spaces with curvature bounded below.* Russian Math. Surveys **47** (1992), 1–58. Zbl 0802.53018

[5] Chen, C-H.: *Warped products of metric spaces of curvature bounded from above.* Trans. Amer. Math. Soc. **351** (1999), 4727–4740. Zbl 0979.53035

[6] Clarkson, J. A.: *Uniformly convex spaces.* Trans. Amer. Math. Soc. **40** (1936), 396–414. Zbl 0015.35604

[7] Foertsch, T.; Schroeder, V.: *Minkowski- Versus Euclidean Rank for Products of Metric Spaces.* Adv. Geom. **2** (2002), 123–131. Zbl 0988.53033

[8] Kleiner, B.: *The local structure of length spaces with curvature bounded above.* Math. Z. **231** (1999), 409–456. Zbl 0940.53024

[9] Rinow, W.: *Die innere Geometrie der metrischen Räume.* Die Grundlehren der Mathematischen Wissenschaften **105**. Berlin-Göttingen-Heidelberg: Springer-Verlag.XV, pp. 520 (1961). Zbl 0096.16302

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