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A Gel'fand Model for a Weyl Group of Type B_n

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Abstract. A *Gel'fand model* for a finite group G is a complex representation of G which is isomorphic to the direct sum of all the irreducible representation of G (see [9]). Gel'fand models for the symmetric group and the linear group over a finite field can be found in [2] and [8]. Using the same ideas as in [2], in this work we describe a Gel'fand model for a Weyl group of type B_n. When K is a field of characteristic zero and $\mathfrak{G}$ is a Weyl group of type B_n, we give a finite dimensional K-subspace $\mathcal{N}$ of the polynomial ring $K[x_1, \dots, x_n]$. If K is the field of complex numbers, then $\mathcal{N}$ provides a Gel'fand model for $\mathfrak{G}$.
The space $\mathcal{N}$ can be defined in a more general way (see [3]), obtained as the zeros of certain differential operators (symmetrical operators) in the Weyl algebra. However, in the case of a group G of type D_n (n even), $\mathcal{N}$ is not a Gel'fand model for G.

1. Symmetrical operators and the space $\mathcal{N}$

Let K be a field of characteristic zero. Fix a natural number n. We will denote by $\mathcal{A}$ the polynomial ring $K[x_1, \dots, x_n]$ and by $\mathcal{W}$ the Weyl algebra of K-linear *differential operators* $K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n\rangle$ generated by the multiplication operators x_i and the differential operators $\partial_i = \dfrac{\partial}{\partial x_i}$ where $i = 1, \dots, n$. The angular brackets are used to indicate that the generators do not commute, indeed, $\partial_i x_i = 1 + x_i \partial_i$ for each $i = 1, \dots, n$. We will make use of some basic properties of the algebra $\mathcal{W}$, which are proved in [4].
Let $\mathbb{I}_n = \{1, 2, \dots, n\}$ and $\mathcal{M}$ be the set of functions $\alpha : \mathbb{I}_n \to \mathbb{N}_0$, where $\mathbb{N}_0$ denotes the set of non-negative integers. Such a function is called a *multiindex*, and we put $\alpha_i = \alpha(i)$ and $\alpha = (\alpha_1, \dots, \alpha_n)$.

For two multiindexes α, β in $\mathcal{M}$ we will use the following notations:

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \qquad \alpha! = \prod_{i=1}^{n} \alpha_i!\,, \qquad \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \beta! \prod_{i=1}^{n} \binom{\alpha_i}{\beta_i}$$

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \qquad \partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}.$$

We will denote by $\mathfrak{S}_n$ the symmetric group of order n and by $\mathcal{C}_2$ the cyclic group of order two given by $\mathcal{C}_2 = \{\pm 1\}$. A group $\mathfrak{G}$ of type B_n can be presented as follows:

$$\mathfrak{G} = \mathcal{C}_2^n \times_s \mathfrak{S}_n$$

where the semidirect product is induced by the natural action of $\mathfrak{S}_n$ on $\mathcal{C}_2^n = \mathcal{C}_2 \times \cdots \times \mathcal{C}_2$ (n factors), i.e.

$$\sigma \cdot (\omega_1, \omega_2, \ldots, \omega_n) = \left(\omega_{\sigma(1)}, \omega_{\sigma(2)}, \ldots, \omega_{\sigma(n)}\right), \ (\omega_i \in \mathcal{C}_2)\,.$$

$\mathfrak{S}_n$ acts on $\mathcal{M}$ by

$$\sigma \cdot \alpha = \alpha \circ \sigma^{-1} \text{ if } \sigma \in \mathfrak{S}_n \text{ and } \alpha \in \mathcal{M}.$$

Then, we have a natural homomorphism of $\mathfrak{G}$ in $Aut\,(\mathcal{A})$, given by

$$(\omega, \sigma) \left(\sum_{\alpha \in \mathcal{M}} \lambda_\alpha\, x^\alpha \right) = \sum_{\alpha \in \mathcal{M}} \lambda_\alpha \, (\omega x)^{\sigma \cdot \alpha}$$

where $\lambda_\alpha \in K$, and

$$(\omega x)^{\sigma \cdot \alpha} = \prod_{i=1}^{n} (\omega_i \cdot x_i)^{(\sigma \cdot \alpha)_i}\,.$$

Let $\mathcal{Z}$ be the centralizer of $\mathfrak{G}$ in $\mathcal{W}$. Then $\mathcal{Z}$ is a subalgebra of $\mathcal{W}$. The elements of $\mathcal{Z}$ will be called *symmetrical operators.*

We know that each operator $\mathcal{D} \in \mathcal{W}$ can be written in a unique way as a finite sum

$$\mathcal{D} = \sum_{\alpha, \beta \in \mathcal{M}} \lambda_{\alpha,\beta}\, x^\alpha\, \partial^\beta \quad \text{, where } \lambda_{\alpha,\beta} \in K$$

where α and β are multiindexes (see [4]).

Putting

$$\mathcal{W}_i = \left\{ \sum_{\alpha, \beta \in \mathcal{M}} \lambda_{\alpha,\beta}\, x^\alpha\, \partial^\beta : |\alpha| - |\beta| = i \right\}$$

we have that

$$\mathcal{W} = \bigoplus_{i \in \mathbb{Z}} \mathcal{W}_i.$$

When $\mathcal{D} \neq 0$, starting from this expression for $\mathcal{D}$, we define *the degree of* $\mathcal{D}$ by:

$$\deg(\mathcal{D}) = max\{|\alpha| - |\beta| : \lambda_{\alpha,\beta} \neq 0\}.$$

Let $\mathcal{Z}^-$ be the subspace of $\mathcal{Z}$ defined by:

$$\mathcal{Z}^- = \{\mathcal{D} \in \mathcal{Z} : \deg(\mathcal{D}) \leq -1\}$$

and let $\mathcal{N}$ be the subspace of $\mathcal{A}$ defined by

$$\mathcal{N} = \left\{P \in \mathcal{A} : \mathcal{D}(P) = 0, \forall \mathcal{D} \in \mathcal{Z}^-\right\}.$$

Using the results in [3] and the fact that $\mathfrak{G}$ has a subgroup of type B_{n-1}, we have

$$\dim(\mathcal{N}) \leq (2n)^n,$$

also, we have that every simple $K[\mathfrak{G}]$-module is isomorphic to a $K[\mathfrak{G}]$-submodule of $\mathcal{N}$.
It is clear that

$$\mathcal{Z}^- \supseteq \bigoplus_{i \leq -1} \mathcal{Z}_i$$

where $\mathcal{Z}_i = \mathcal{Z}^- \cap \mathcal{W}_i$.

2. Minimal orbits

Let $\mathcal{O}$ be the orbit space of $\mathfrak{S}_n$ in $\mathcal{M}$. For each γ in $\mathcal{O}$ we put

$$\mathcal{S}_\gamma = \left\{\sum_{\alpha \in \gamma} \lambda_\alpha x^\alpha : \lambda_\alpha \in K\right\}.$$

Given $\alpha, \beta \in \mathcal{M}$, we put $\alpha \equiv \beta$ if and only if for every $i \in \mathbb{I}_n$, α_i and β_i both have the same parity. Two orbits γ and μ in $\mathcal{O}$ are said to be *equivalent* if there are $\alpha \in \gamma$ and $\beta \in \mu$ such that $\alpha \equiv \beta$.
It is not difficult to prove that: γ and μ are equivalent if and only if there exists a bijection $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ which satisfies:

i) $\varphi(k)$ and k both have the same parity, $\forall k \in \mathbb{N}_0$.

ii) $\mu = \{\varphi \circ \alpha : \alpha \in \gamma\}$.

When γ and μ are equivalent, we write $\gamma \sim \mu$.
We observe that if α and β are in a given orbit γ, then we have $|\alpha| = |\beta|$ and $\alpha! = \beta!$. So, we will put $|\gamma|$ and $\gamma!$ respectively for these coincident values.
Let $\gamma \sim \mu$ be and φ as above. We define the operator

$$\partial_\gamma^\mu = \frac{1}{\mu!} \sum_{\alpha \in \gamma} x^\alpha \partial^{\varphi \circ \alpha}.$$

An orbit γ in $\mathcal{O}$ is called *minimal* if $|\gamma| \leq |\mu|$ for all μ in $\mathcal{O}$ such that $\mu \sim \gamma$.

Proposition 2.1.

i) $\mathcal{Z}^- = \bigoplus_{i \leq -1} \mathcal{Z}_i$.

ii) ∂_γ^μ *is a symmetrical operator of degree* $|\gamma| - |\mu|$.

iii) $\partial_\gamma^\mu : \mathcal{S}_\mu \to \mathcal{S}_\gamma$ *is a* $\mathfrak{G}$*-isomorphism.*

Proof. First, we observe that for β, δ in $\mathcal{M}$ and σ in $\mathfrak{S}_n$ we have:
a) If $\delta - \beta$ is in $\mathcal{M}$, then $\sigma \cdot (\delta - \beta) = \sigma \cdot \delta - \sigma \cdot \beta$.
b) Since the same factors occur in both numbers $\begin{bmatrix} \delta \\ \beta \end{bmatrix}$ and $\begin{bmatrix} \sigma\cdot\delta \\ \sigma\cdot\beta \end{bmatrix}$, we have that $\begin{bmatrix} \delta \\ \beta \end{bmatrix} = \begin{bmatrix} \sigma\cdot\delta \\ \sigma\cdot\beta \end{bmatrix}$.
Now, we can establish the identities:

$$\begin{aligned} \partial^\beta \circ \omega \left(x^\delta\right) &= \omega^\delta \, \partial^\beta \left(x^\delta\right) \quad (\omega \in \mathcal{C}^n) \\ \sigma \circ \partial^\beta &= \partial^{\sigma\cdot\beta} \circ \sigma \end{aligned} \tag{1}$$

In fact, the first identity is clear. For the second one, by using a) and b), we have

$$\begin{aligned} \sigma\left(\partial^\beta\left(x^\delta\right)\right) &= \sigma\left(\begin{bmatrix} \delta \\ \beta \end{bmatrix} x^{\delta-\beta}\right) = \begin{bmatrix} \delta \\ \beta \end{bmatrix} x^{\sigma\cdot\delta-\sigma\cdot\beta} \\ &= \begin{bmatrix} \delta \\ \beta \end{bmatrix} \begin{bmatrix} \sigma\cdot\delta \\ \sigma\cdot\beta \end{bmatrix}^{-1} \partial^{\sigma\cdot\beta}\left(x^{\sigma\cdot\delta}\right) = \partial^{\sigma\cdot\beta} \circ \sigma\left(x^\delta\right) \end{aligned}$$

It follows that

$$\sigma \circ \left(x^\alpha \, \partial^\beta\right) \circ \sigma^{-1}\left(x^\delta\right) = \sigma\left(x^\alpha \sigma^{-1}\left(\partial^{\sigma\cdot\beta}\left(x^\delta\right)\right)\right) = x^{\sigma\cdot\alpha} \, \partial^{\sigma\cdot\beta}\left(x^\delta\right)$$

that is

$$\sigma \circ \left(x^\alpha \, \partial^\beta\right) \circ \sigma^{-1} = x^{\sigma\cdot\alpha} \, \partial^{\sigma\cdot\beta} \tag{2}$$

For $\omega \in \mathcal{C}_2^n$ and $\alpha, \beta, \delta \in \mathcal{M}$, we have

$$\begin{aligned} \left(\omega \circ \left(x^\alpha \partial^\beta\right) \circ \omega^{-1}\right)\left(x^\delta\right) &= \omega\left(\omega^\delta \cdot \begin{bmatrix} \delta \\ \beta \end{bmatrix} x^{\alpha+\delta-\beta}\right) = \omega^{\beta-\alpha} \cdot \begin{bmatrix} \delta \\ \beta \end{bmatrix} x^{\alpha+\delta-\beta} = \\ &= \omega^{\beta-\alpha} \cdot \left(\left(x^\alpha \partial^\beta\right)\left(x^\delta\right)\right) = \left(\omega^{\beta-\alpha} \cdot \left(x^\alpha \partial^\beta\right)\right)\left(x^\delta\right). \end{aligned}$$

In particular, when $\alpha \equiv \beta$, we have that

$$\omega \circ \left(x^\alpha \partial^\beta\right) \circ \omega^{-1} = x^\alpha \partial^\beta. \tag{3}$$

Using the identities in (2) and (3), it follows that

$$\tau \circ \mathcal{D} \circ \tau^{-1} \in \mathcal{W}_i \ , \ \forall \tau \in \mathfrak{G} \ , \forall \mathcal{D} \in \mathcal{W}_i.$$

On other hand, every $\mathcal{D} \in \mathcal{Z}^-$ can be written in a unique way as

$$\mathcal{D} = \mathcal{D}_1 + \cdots + \mathcal{D}_k \ , \ \mathcal{D}_i \in \mathcal{W}_{-i}.$$

Given $\tau \in \mathfrak{G}$, from the identity

$$\tau \circ \mathcal{D} \circ \tau^{-1} = \mathcal{D}$$

we have

$$\tau \circ \mathcal{D}_1 \circ \tau^{-1} + \cdots + \tau \circ \mathcal{D}_k \circ \tau^{-1} = \mathcal{D}_1 + \cdots + \mathcal{D}_k$$

that is

$$\tau \circ \mathcal{D}_i \circ \tau^{-1} = \mathcal{D}_i \;\; (i = 1, \ldots, k)\,.$$

It follows that

$$\mathcal{Z}^- \subseteq \bigoplus_{i \le -1} \mathcal{Z}_i$$

and we have i).

ii) follows from the preceding identities (2), (3) and the fact that $\gamma \sim \mu$. On the other hand, it is clear that $\deg\left(\partial_\gamma^\mu\right) = |\gamma| - |\mu|$.

iii) For $\beta \in \mu$, let $\delta \in \gamma$ be such that $\beta = \varphi \circ \delta$. We have

$$\partial_\gamma^\mu\left(x^\beta\right) = \frac{1}{\mu!} \sum_{\alpha \in \gamma} x^\alpha\, \partial^{\varphi \circ \alpha}\left(x^\beta\right) = \frac{\beta!}{\mu!}\, x^\delta = x^\delta.$$

It follows that ∂_γ^μ is an isomorphism. □

Corollary 2.2. *If $\mu \in \mathcal{O}$ is non-minimal then $\mathcal{N} \cap \mathcal{S}_\mu = 0$.*

Proof. Let γ in $\mathcal{O}$ be such that $\gamma \sim \mu$ with $|\gamma| < |\mu|$. Then $\deg\left(\partial_\gamma^\mu\right) \le -1$, and so

$$\partial_\gamma^\mu\left(\mathcal{N} \cap \mathcal{S}_\mu\right) = 0.$$

Now, Corollary 2.2 follows from ii) and iii) of Proposition 2.1. □

Corollary 2.3. *Let $P \in \mathcal{N}$, then the homogeneous components of P are also in $\mathcal{N}$.*

Proof. Assume that:

$$P = P_1 + \cdots + P_m \quad , \quad \deg\left(P_i\right) = i$$

where $P_1, \ldots, P_m$ are the homogeneous components of P. On the other hand, for every $\mathcal{D} \in \mathcal{Z}_j$ we have

$$0 = \mathcal{D}\left(P\right) = \mathcal{D}\left(P_1\right) + \cdots + \mathcal{D}\left(P_m\right).$$

Since the $\mathcal{D}\left(P_i\right)$ are zero if $i < j$ or they are in homogeneous components of degree $i - j$, it follows that

$$\mathcal{D}\left(P_i\right) = 0 \;\; \forall \mathcal{D} \in \mathcal{Z}_j.$$

Using 2.1 i), we have that $P_i \in \mathcal{N}$. □

Corollary 2.4.

$$\mathcal{N} = \bigoplus_{\gamma \text{ minimal}} \mathcal{N} \cap \mathcal{S}_\gamma.$$

Proof. It is clear that

$$\mathcal{N} \supseteq \bigoplus_{\gamma \text{ minimal}} \mathcal{N} \cap \mathcal{S}_\gamma .$$

By Corollary 2.3 we have that the homogeneous components of an element P in $\mathcal{N}$, are also in $\mathcal{N}$. We assume that P is a nonzero homogeneous polynomial and write

$$P = P_1 + \cdots + P_m$$

where the P_i are nonzero polynomials in $\mathcal{S}_{\gamma_i}$, and $|\gamma_i| = \deg(P)$ for $i = 1, \dots, m$. It follows from Proposition 2.1 that the operator

$$\partial_{\gamma_i} = \frac{1}{\gamma_i!} \sum_{\alpha \in \gamma_i} x^\alpha \, \partial^\alpha$$

is symmetrical and has degree zero.
Observe that if α and β are multiindexes such that $|\alpha| = |\beta|$ then

$$\partial^\alpha \left(x^\beta\right) = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \alpha! & \text{if } \alpha = \beta \end{cases}$$

Since $|\gamma_i| = \deg(P_j)$ for all i, j, it follows that

$$\partial_{\gamma_i}(P_j) = \begin{cases} 0 & \text{if } j \neq i \\ P_i & \text{if } j = i \end{cases}$$

Since $\mathcal{W}$ has no divisors of zero, for every $\mathcal{D} \in \mathcal{Z}^-$ we have $\mathcal{D} \circ \partial_{\gamma_i} \in \mathcal{Z}^-$. Hence

$$0 = \mathcal{D} \circ \partial_{\gamma_i}(P) = \mathcal{D}(P_i).$$

That is $P_i \in \mathcal{N} \cap \mathcal{S}_{\gamma_i}$. Since $P_i \neq 0$, it follows from Corollary 2.2 that γ_i is minimal. □

The following proposition will be used for the characterization of the minimal orbits.

Proposition 2.5. *Let $k_1 \geq k_2 \geq \cdots \geq k_n$ be a sequence of natural numbers. If $e_1, \dots, e_n$ are distinct non-negative integers, then the minimal value of the sum*

$$\sum_{i=1}^{n} k_i e_i$$

occurs only when $e_i = i - 1$.

Proof. Fix a sum $\sum_{i=1}^{n} k_i e_i$. For $i < j$ such that $k_i = k_j$ we can assume that $e_i < e_j$. Let π be a permutation of $\mathbb{I}_n$ such that the sequence $e_{\pi(1)}, \dots, e_{\pi(n)}$ is increasing. Suppose that there exists j such that

$$e_j \neq e_{\pi(j)}$$

we can assume that j is minimal in the inequality above. It follows that

$$e_j > e_{\pi(j)} \text{ and } \pi(j) > j$$

Putting

$$f_i = \begin{cases} e_i & \text{if } \ i \neq j, \pi(j) \\ e_{\pi(j)} & \text{if } \ i = j \\ e_j & \text{if } \ i = \pi(j) \end{cases}$$

we have

$$\sum_{i=1}^{n} k_i e_i = \sum_{i=1}^{n} k_i f_i + \left(k_j - k_{\pi(j)}\right)\left(e_j - e_{\pi(j)}\right) \geq \sum_{i=1}^{n} k_i f_i.$$

Hence we may consider only the sums where the sequence $e_1, \dots, e_n$ is increasing. In this case, we have that

$$e_i \geq i - 1$$

hence, the minimal value is

$$\sum_{i=1}^{n} k_i (i-1)$$

and it is clear that it occurs only when $e_i = i - 1$. □

We will denote by $|A|$ the cardinality of a set A.

Proposition 2.6. *Given an orbit γ we have*

i) *γ is minimal if and only if for every $\alpha \in \gamma$ the following holds: Given $i, j \in \mathbb{N}_0$ is such that $i < j$ and i, j both have the same parity, then $|\alpha^{-1}(i)| \geq |\alpha^{-1}(j)|$.*

ii) *There is a unique minimal orbit which is equivalent to γ.*

Proof. i) Let $\alpha \in \gamma$. We put:

$$\operatorname{Im}(\alpha)_0 = \{p \in \operatorname{Im}(\alpha) : p \text{ is even}\} = \{p_1, p_2, \dots, p_s\}$$
$$\operatorname{Im}(\alpha)_1 = \{q \in \operatorname{Im}(\alpha) : q \text{ is odd}\} = \{q_1, q_2, \dots, q_t\}$$

and assume that $k_1 \geq k_2 \geq \cdots \geq k_s$ and $h_1 \geq h_2 \geq \cdots \geq h_t$ where $k_i = |\alpha^{-1}(p_i)|$ and $h_i = |\alpha^{-1}(q_i)|$, therefore, when $k_i = k_{i+1}$ (respectively $h_i = h_{i+1}$) we assume $p_i < p_{i+1}$ (respectively $q_i < q_{i+1}$).

Let $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ be a bijection such that

$$\varphi(p_i) = 2(i-1) \quad \text{and} \quad \varphi(q_i) = 2i - 1$$

It is clear that there exists such a function. We put $\alpha^* = \alpha \circ \varphi$ and denote by γ^* the orbit of α^*. Notice that γ^* is uniquely determined by s, t and the sequences $k_1, \dots, k_s, h_1, \dots, h_t$.

We claim that γ^* is minimal. In fact, putting $p_i = 2e_i$, $q_i = 2f_i+1$ and using the Proposition 2.5, we have

$$\begin{aligned}|\gamma| &= \sum_{i=1}^{s} k_i p_i + \sum_{i=1}^{t} h_i q_i = 2\sum_{i=1}^{s} k_i e_i + 2\sum_{i=1}^{t} h_i f_i + \sum_{i=1}^{t} h_i \\ &\geq 2\sum_{i=1}^{s} k_i (i-1) + 2\sum_{i=1}^{t} h_i (i-1) + \sum_{i=1}^{t} h_i \\ &= \sum_{i=1}^{s} k_i \varphi(p_i) + \sum_{i=1}^{t} h_i \varphi(q_i) = |\gamma^*| .\end{aligned}$$

This inequality becomes an equality only if $p_i = 2(i-1)$ and $q_i = 2i-1$. It follows that γ is minimal if and only if $\gamma = \gamma^*$.

ii) Let us suppose that γ and μ are equivalent, and that the values h_j and $k_1, \ldots, k_{h_j}$ given in i) are the same for γ and μ. Then we must have $\gamma^* = \mu^*$. Therefore, if γ and μ are minimal, from i) we have $\gamma = \gamma^* = \mu^* = \mu$. □

3. The Laplacian

We denote by Δ the Laplace's operator given by:

$$\Delta = \sum_{i=1}^{n} \partial_i^2 .$$

It is clear that Δ is a symmetrical operator.

For $\gamma \in \mathcal{O}$, let $\mathcal{S}_\gamma^o$ be the subspace of $\mathcal{S}_\gamma$ defined by:

$$\mathcal{S}_\gamma^o = \{P \in \mathcal{S}_\gamma : \Delta(P) = 0\} .$$

Given $\alpha \in \gamma$, we denote by $\mathcal{H}$ the isotropy group of x^α in $\mathfrak{G}$. We have a projector in $End_K(\mathcal{A})$ given by

$$\Delta_{\mathcal{H}} = \frac{1}{|\mathcal{H}|} \sum_{\eta \in \mathcal{H}} \eta .$$

Proposition 3.1. *Suppose* $\tau \in \mathcal{H}$ *and* $P \in \mathcal{A}$ *such that* $\tau(P) = \lambda \cdot P$ *where* $\lambda \in K$ *is different from* 1. Then $\Delta_{\mathcal{H}}(P) = 0$.

Proof. It is not difficult to see that

$$\tau\, \Delta_{\mathcal{H}} = \Delta_{\mathcal{H}} = \Delta_{\mathcal{H}}\, \tau \quad (\tau \in \mathcal{H})$$

hence

$$\Delta_{\mathcal{H}}(P) = \Delta_{\mathcal{H}}\, \tau(P) = \lambda\, \Delta_{\mathcal{H}}(P)$$

Since $\lambda \neq 1$, we have $\Delta_{\mathcal{H}}(P) = 0$. □

Proposition 3.2. *Let γ, α and $\Delta_{\mathcal{H}}$ be as before. If $\beta \in \gamma$ is such that $\beta \not\equiv \alpha$, then we have* $\Delta_{\mathcal{H}}\left(x^{\beta}\right) = 0$.

Proof. Since $\alpha \not\equiv \beta$, there exists $i \in \mathbb{I}_n$ such that α_i is even and β_i is odd. Let $\omega \in \mathcal{C}_2^n$ be given by $\omega_i = -1$ and $\omega_j = 1$ for $j \neq i$. We have that

$$\omega\left(x^{\alpha}\right) = x^{\alpha} \text{ and } \omega\left(x^{\beta}\right) = -\beta.$$

Using the Proposition 3.1 for $\lambda = -1$ and $\tau = \omega$, we obtain $\Delta_{\mathcal{H}}\left(x^{\beta}\right) = 0$. □

Lemma 3.3. *For a minimal orbit γ we have*

$$\dim\left(\Delta_{\mathcal{H}}\left(S_{\gamma}^{\circ}\right)\right) \leq 1.$$

Proof. We denote by $\widetilde{\alpha}$ the set of $\beta \in \gamma$ such that $\beta \equiv \alpha$. Using the Proposition 3.2, we note that

$$\Delta_{\mathcal{H}}\left(S_{\gamma}\right) = \left\langle \Delta_{\mathcal{H}}\left(x^{\beta}\right) : \beta \in \gamma \right\rangle = \left\langle \Delta_{\mathcal{H}}\left(x^{\beta}\right) : \beta \in \widetilde{\alpha} \right\rangle.$$

On the other hand, for any $\eta \in \mathcal{H}$ and $\beta \in \widetilde{\alpha}$, we have

$$\eta\left(x^{\beta}\right) = x^{\mu}$$

where $\mu \in \widetilde{\alpha}$. Put $\eta = \omega\,\pi$, $\omega \in \mathcal{C}_2^n$ and $\pi \in \mathfrak{S}_n$. Since $\eta\left(x^{\alpha}\right) = x^{\alpha}$, we have that α_i and $\alpha_{\pi(i)}$ have both the same parity, therefore, the number of the indices i such that α_i is odd is an even number. Then, for $\beta \in \widetilde{\alpha}$ and $i \in \mathbb{I}_n$, we have

$$\beta_i \equiv \alpha_i \equiv \alpha_{\pi(i)} \equiv \beta_{\pi(i)} \equiv (\pi \cdot \beta)_i \mod 2, \text{ and } \omega \cdot \beta = \beta.$$

It follows that

$$\Delta_{\mathcal{H}}\left(S_{\gamma}\right) \subseteq \left\langle x^{\beta} : \beta \in \widetilde{\alpha} \right\rangle.$$

Now, we put

$$h = \max\left\{k : k \in \operatorname{Im}(\alpha)\right\}.$$

For every $\beta \in \gamma$ we define the vector

$$\widehat{\beta} = \left(\beta^0, \ldots, \beta^h\right) \text{ where } \beta^l = \sum_{k \in \alpha^{-1}(l)} \beta_k.$$

It is clear that for every $\tau \in \mathfrak{S}_n \cap \mathcal{H}$ the identity $\widehat{\tau \cdot \beta} = \widehat{\beta}$ holds. We order the vectors $\widehat{\beta}$ according to the lexicographical order, so that $\widehat{\alpha}$ is the minimum element.
Let $\beta \in \gamma$ and suppose that there are two indices $i, j \in \mathbb{I}_n$ such that $\beta_i = \beta_j + 2$ and $\alpha_i < \alpha_j$. Let $\tau \in \mathfrak{S}_n$ be the transposition (i, j), then

$$\widehat{\tau \cdot \beta} < \widehat{\beta}.$$

In fact, from the identities

$$(\tau\cdot\beta)_k = \begin{cases} \beta_k & \text{if} \quad k\neq i,j \\ \beta_j & \text{if} \quad k=i \\ \beta_i & \text{if} \quad k=j \end{cases}$$

it follows that

$$(\tau\cdot\beta)^l = \begin{cases} \beta^l & \text{if} \quad l\neq \alpha_i,\alpha_j \\ \beta^l-\beta_i+\beta_j=\beta^l-2 & \text{if} \quad l=\alpha_i \\ \beta^l-\beta_j+\beta_i=\beta^l+2 & \text{if} \quad l=\alpha_j \end{cases}$$

Hence $l=\alpha_i$ is the first index where $\widehat{\tau\cdot\beta}$ and $\widehat{\beta}$ do not coincide, since $(\tau\cdot\beta)^l=\beta^l-2$ we have that $\widehat{\tau\cdot\beta}<\widehat{\beta}$.

For any β in γ, we fix $i\in\mathbb{I}_n$ such that $\beta_i>0$. For each j in $\mathbb{I}_n$ such that $\beta_i=\beta_j+2$ consider the transposition τ_j in $\mathfrak{S}_n$ that switches i and j.

Let P be in $\triangle_{\mathcal{H}}\left(S_\gamma^\circ\right)$. We write

$$P=\sum_{\beta\in\widetilde{\alpha}} a_\beta\ x^\beta .$$

Since Δ is symmetrical, and Δ commutes with $\triangle_{\mathcal{H}}$, we have

$$\Delta\left(P\right)=0.$$

From this identity it follows that

$$a_\beta+\sum_j a_{\tau_j\cdot\beta}=0.$$

In fact, the left member of the equality from above is, except for a constant factor, the coefficient of the monomial $x^{\widetilde{\beta}}$ in $\Delta\left(P\right)$, where $\widetilde{\beta}$ is given by

$$\widetilde{\beta}_k=\begin{cases} \beta_k \text{ if } k\neq i \\ \beta_j \text{ if } k=i\,. \end{cases}$$

Since $a_{\tau\beta}=a_\beta\ \ \forall\,\tau\in\mathcal{H}$, the relationship between the coefficients can be written as

$$a_\beta+\sum_{\tau_j\in\mathcal{H}} a_{\tau_j\cdot\beta}+\sum_{\tau_j\notin\mathcal{H}} a_{\tau_j\cdot\beta}=0.$$

Observing that τ_j is in $\mathcal{H}$ if and only if $\alpha_j=\alpha_i$, the preceding identity takes the form

$$(1+m)\ a_\beta+\sum_{\alpha_j\neq\alpha_i} a_{\tau_j\cdot\beta}=0$$

where $m\in\mathbb{N}_0$.

We will prove Lemma 3.3 by showing that the linear functional

$$\varphi : \triangle_{\mathcal{H}} \left(S_\gamma^\circ\right) \to K$$

defined by

$$\varphi\left(P\right) = a_\alpha$$

is injective.
Let us suppose that $a_\alpha = 0$. If $P \neq 0$, we choose β in $\widetilde{\alpha}$ such that $a_\beta \neq 0$ and $\widehat{\beta}$ minimal. Since $\widehat{\alpha} < \widehat{\beta}$, there is an index k in $\mathrm{Im}\left(\alpha\right)$ such that

$$\beta^k > \alpha^k \text{ and } \beta^l = \alpha^l \text{ if } l < k$$

From these conditions we infer that

$$\beta^l = l \cdot \left|\alpha^{-1}\left(l\right)\right| \text{ if } l < k$$

But this is only possible if β coincides with α in $\alpha^{-1}\left\{0, 1, \ldots, k-1\right\}$. On the other hand, from the fact that $\beta^k > \alpha^k = k \cdot \alpha^{-1}\left(k\right)$, there exists i in $\alpha^{-1}\left(k\right)$ such that $\beta_i > k \geq 0$. Since $\beta \in \widetilde{\alpha}$ we have that β_i and k both have the same parity, then $\beta_i - 2 \geq k$. The indices j for which $\beta_j = \beta_i - 2 \geq k$, belong to $\alpha^{-1}\left\{k, \ldots, h\right\}$, and this set is non-empty because γ is minimal. For these indices, the transpositions τ_j previously defined, satisfy

$$\begin{aligned} a_{\tau_j \cdot \beta} = a_\beta \quad &\text{if} \quad \alpha_j = k \\ \widehat{\tau_j \cdot \beta} < \widehat{\beta} \quad &\text{if} \quad \beta_j > k. \end{aligned}$$

If $m = \left|\left\{j : \beta_j = \beta_i - 2 \text{ and } \alpha_j = k\right\}\right|$, from the relations obtained for the coefficients of P, it follows that

$$\left(1+m\right) a_\beta + \sum_{\widehat{\tau_j \cdot \beta} < \widehat{\beta}} a_{\tau_j \cdot \beta} = 0.$$

Since $\widehat{\beta}$ is minimal, we obtain $a_\beta = 0$, a contradiction. □

4. The structure of $\mathcal{N}$

Let $\mathcal{F} \subseteq \mathbb{I}_n$, $\mathcal{F} = \left\{f_1, f_2, \ldots, f_k\right\}$ where $f_i < f_{i+1}$. Given a function $\mu : \mathcal{F} \to \mathbb{N}_0$, we denote by

$$e_{\mathcal{F}}^\mu = \det\left[x_{f_i}^{\mu_j}\right] \tag{4}$$

where $\mu_j = \mu\left(f_j\right)$.
Putting $x^\mu = x_{f_1}^{\mu_1} . x_{f_2}^{\mu_2} \cdots x_{f_k}^{\mu_k}$, it is clear that the coefficient of x^μ in $e_{\mathcal{F}}^\mu$ equals 1. Therefore, we remark that $e_{\mathcal{F}}^\mu = 0$ if μ is not injective.

Let γ be a minimal orbit and take α in γ. We write $\mathbb{I}_n = \mathcal{P} \cup \mathcal{Q}$ where $\mathcal{P}$ and $\mathcal{Q}$ are given by

$$\mathcal{P} = \{i \in \mathbb{I}_n : \alpha_i \text{ is even}\} \quad \text{and} \quad \mathcal{Q} = \{i \in \mathbb{I}_n : \alpha_i \text{ is odd}\}.$$

An *α-partition* $\mathcal{B}$ of $\mathbb{I}_n$ is a pair of partitions of $\mathcal{P} = \cup_i P_i$ and $\mathcal{Q} = \cup_i Q_i$ respectively which satisfies that the restrictions $\alpha|_{P_i}$ and $\alpha|_{Q_i}$ of α to P_i and α to Q_i are minimal and injective. Given a α-partition $\mathcal{B}$, we put

$$e_{\mathcal{B}} = \left(\prod_i e^{\alpha}_{P_i}\right)\left(\prod_i e^{\alpha}_{Q_i}\right).$$

To obtain the coefficient of x^{α} in $e_{\mathcal{B}}$, we need to multiply the coefficients of

$$x^{\alpha|_{P_i}} \text{ in } e^{\alpha}_{P_i} \quad \text{and} \quad x^{\alpha|_{Q_i}} \text{ in } e^{\alpha}_{Q_i}$$

so that the coefficient of x^{α} in $e_{\mathcal{B}}$ equals 1. On the other hand, notice that $e_{\mathcal{B}}$ is the product of the factors of the form

$$(x_i \pm x_j) \text{ where } i,j \in P_k \text{ or } i,j \in Q_k \quad \text{and} \quad x_i \text{ where } i \in Q_k.$$

All these factors occur with multiplicity 1 in $e_{\mathcal{B}}$.
Let τ be a reflection in $\mathfrak{G}$ associated to one of these factors, that is, the reflection whose hyperplane of fixed points is given by the equations $x_i = \pm x_j$ or $x_i = 0$. We have the following

Proposition 4.1. *Let τ be as above, then*

$$\tau\left(e_{\mathcal{B}}\right) = -e_{\mathcal{B}}. \tag{5}$$

Proof. Let l be the factor associated to τ. Using (4) we have the following
If l is not a factor of $e^{\alpha}_{P_i}$ or $e^{\alpha}_{Q_i}$, then

$$\tau\left(e^{\alpha}_{P_k}\right) = e^{\alpha}_{P_k} \quad \text{and} \quad \tau\left(e^{\alpha}_{Q_k}\right) = e^{\alpha}_{Q_k}.$$

If l is a factor of $e^{\alpha}_{P_i}$ or $e^{\alpha}_{Q_i}$, then

$$\tau\left(e^{\alpha}_{P_k}\right) = -e^{\alpha}_{P_k} \quad \text{and} \quad \tau\left(e^{\alpha}_{Q_k}\right) = -e^{\alpha}_{Q_k}.$$

Since the multiplicity of l in $e_{\mathcal{B}}$ is 1, it follows (5). □

We denote by δ_{α} the polynomial in $\mathcal{S}_{\gamma}$ given by

$$\delta_{\alpha} = \sum_{\mathcal{B}} e_{\mathcal{B}}$$

where $\mathcal{B}$ runs through all α-partitions. The coefficient of x^{α} in δ_{α} is equal to the number of partitions $\mathcal{B}$ satisfying the required conditions, that is $\delta_{\alpha} \neq 0$.

Proposition 4.2. *Let $\tau \in \mathfrak{G}$ be a reflection and r be a root of τ. If $P \in \mathcal{A}$ is such that $\tau(P) = -P$, then the linear form given by $\phi(x) = \sum_{i=1}^{n} r_i x_i$ is a factor of P.*

Proof. Because K is infinite, we may see P as polynomial function on K^n.
Let $\{\varphi_1 = \phi, \varphi_2, \dots, \varphi_n\}$ be a basis of the dual space $(K^n)^*$, such that $\tau(\varphi_i) = \varphi_i$ if $i \neq 1$. For $x \in K^n$ we can write

$$P(x) = \sum_{\beta} \lambda_\beta \, y^\beta$$

where $\lambda_\beta \in K$, $y^\beta = y_1^{\beta_1} \cdots y_n^{\beta_n}$ and $y_i = \varphi_i(x)$. From the condition $\tau(P) = -P$ it follows that $\beta_1 > 0$ when $\lambda_\beta \neq 0$. Then $\phi(x)$ is a factor of P. □

Lemma 4.3. *Let γ be a minimal orbit. Then*
i) $e_{\mathcal{B}} \in \mathcal{N}$.
ii) $\triangle_{\mathcal{H}}(S_\gamma^\circ) = K \cdot \delta_\alpha$.

Proof. i) Suppose that $\mathcal{D} \in \mathcal{Z}^-$ and that $\tau \in \mathfrak{G}$ is a reflection such that $\tau(e_{\mathcal{B}}) = -e_{\mathcal{B}}$. We have

$$\tau \cdot (\mathcal{D}(e_{\mathcal{B}})) = \mathcal{D}(\tau \cdot e_{\mathcal{B}}) = -\mathcal{D}(e_{\mathcal{B}}).$$

Proposition 4.2 shows that all the linear factors of $e_{\mathcal{B}}$ are factors of $\mathcal{D}(e_{\mathcal{B}})$, but any two of these factors being non-proportional, we infer that $e_{\mathcal{B}}$ is a factor of $\mathcal{D}(e_{\mathcal{B}})$. Furthermore, if $\mathcal{D}(e_{\mathcal{B}}) \neq 0$, we have that $\deg(\mathcal{D}(e_{\mathcal{B}})) < \deg(e_{\mathcal{B}})$, and so we conclude that $\mathcal{D}(e_{\mathcal{B}}) = 0$.
ii) Let $\mathcal{B}$ be as before. For $\tau \in \mathcal{H}$ we put $\tau = \omega \cdot \pi$ where $\omega \in \mathcal{C}_2^n$ and $\pi \in \mathfrak{S}_n$. Denoting by $\mathcal{B}^\tau$ the bipartition defined by

$$\mathcal{P} = \bigcup_k \pi(P_k) \quad \text{and} \quad \mathcal{Q} = \bigcup_k \pi(Q_k).$$

It is clear that $\mathcal{B}^\tau$ satisfies the required conditions for a bipartition. From the identities

$$\det\left[x_j^{(\alpha\circ\pi)_i}\right] = \pi^{-1}\left(\det\left[x_j^{\alpha_i}\right]\right) \quad \text{and} \quad \omega(e_{\mathcal{B}}) = e_{\mathcal{B}}$$

we obtain

$$e_{\mathcal{B}^\tau} = \tau^{-1}(e_{\mathcal{B}}).$$

It follows that τ permutes the terms of δ_α, and so

$$\tau \cdot \delta_\alpha = \delta_\alpha \;\; \forall \tau \in \mathcal{H}$$

that is

$$\triangle_{\mathcal{H}}(\delta_\alpha) = \delta_\alpha.$$

Thus Lemma 3.1 implies ii). □

Theorem 4.4. *Let γ be a minimal orbit and α in γ. Then*
i) *The $\mathfrak{G}$-module $\langle \delta_\alpha \rangle$ generated by δ_α is simple.*

ii) $\mathcal{S}_\gamma^o = \mathcal{N} \cap \mathcal{S}_\gamma = \langle \delta_\alpha \rangle$.
iii) *The multiplicity of* $\mathcal{S}_\gamma^o$ *in* $\mathcal{N}$ *is* 1.

Proof. We will make use of the fact that when the base field K has characteristic zero all the K-linear representations of a finite group are completely reducible.
i) If $\mathcal{S}$ and $\mathcal{T}$ are submodules of $\langle \delta_\alpha \rangle$ such that

$$\langle \delta_\alpha \rangle = \mathcal{S} \oplus \mathcal{T}$$

writing $\delta_\alpha = s + t$ where $s \in \mathcal{S}$ and $t \in \mathcal{T}$, we have

$$\delta_\alpha = \Delta_{\mathcal{H}}(\delta_\alpha) = \Delta_{\mathcal{H}}(s) + \Delta_{\mathcal{H}}(t).$$

It follows that at least one of terms in the sum is not zero. In conclusion, from ii) of Lemma 4.3, we have that $\delta_\alpha \in \mathcal{S}$ or $\delta_\alpha \in \mathcal{T}$, that is $\mathcal{S} = 0$ or $\mathcal{T} = 0$.
ii) From i) of the Lemma 4.3 we have

$$\langle \delta_\alpha \rangle \subseteq \mathcal{N} \cap \mathcal{S}_\gamma \subseteq \mathcal{S}_\gamma^o.$$

Let $\mathcal{T}$ be a submodule of $\mathcal{S}_\gamma^o$ such that

$$\mathcal{S}_\gamma^o = \langle \delta_\alpha \rangle \oplus \mathcal{T}.$$

Let $0 \neq P \in \mathcal{T}$. Replacing P by $\sigma \cdot P$ with σ in $\mathfrak{S}_n$, if necessary, we may suppose that the coefficient a_α of x^α in P is different from zero. Since the coefficient of x^α in $\Delta_{\mathcal{H}}(P)$ is a_α, by Lemma 4.3, we have that there exists in K a non-zero element λ such that $\delta_\alpha = \lambda\, \Delta_{\mathcal{H}}(P)$. Thus $\delta_\alpha \in \mathcal{T}$, but this is a contradiction.
iii) Let $\theta : \mathcal{S}_\gamma^o \to \mathcal{S}_\mu^o$ be an isomorphism of $\mathfrak{G}$-modules, where γ and μ are minimal orbits. We can assume that $|\mu| \leq |\gamma|$. Consider α in γ and $e_{\mathcal{B}}$ as above. With similar arguments as in i) of Lemma 4.3, we obtain that $e_{\mathcal{B}}$ is a factor of $\theta(e_{\mathcal{B}})$, so that $|\gamma| = |\mu|$ and there is a $\lambda \neq 0$ in K such that

$$\theta(e_{\mathcal{B}}) = \lambda\, e_{\mathcal{B}}.$$

That is, $\mathcal{S}_\gamma \cap \mathcal{S}_\mu \neq 0$, therefore $\gamma = \mu$. □

Remark. As stated earlier in [3] we defined the space $\mathcal{N}$ for a finite group $G \subset GL_n(K)$, and we showed that every simple $K[\mathfrak{G}]$-module is isomorphic to a $K[\mathfrak{G}]$-submodule of $\mathcal{N}$. When K is the complex number field and $\mathcal{N}$ is a multiplicity-free direct sum of simple $K[G]$-modules, we have that $\mathcal{N}$ is a Gel'fand model for G. Hence, the following corollary can be obtained by using Corollary 2.4 and Theorem 4.4.

Corollary 4.5. *If K is the complex number field, then $\mathcal{N}$ is a Gel'fand Model for $\mathfrak{G}$. In particular, the number of minimal orbits coincides with the number of conjugacy classes of $\mathfrak{G}$.*

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