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Well Centered Spherical Quadrangles

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Abstract. We introduce the notion of a well centered spherical quadrangle or WCSQ for short, describing a geometrical method to construct any WCSQ. We shall show that any spherical quadrangle with congruent opposite internal angles is congruent to a WCSQ. We may classify them taking in account the relative position of the spherical moons containing its sides. Proposition 2 describes the relations between well centered spherical moons and WCSQ which allow the refereed classification.

Let L be a spherical moon. We shall say that L is *well centered* if its vertices belong to the great circle $S^2 \cap \{(x,y,z) \in \mathbb{R}^3 : x = 0\}$ and the semi-great circle bisecting L contains the point $(1,0,0)$.

If L_1 and L_2 are two spherical moons with orthogonal vertices then L_1 and L_2 are said to be *orthogonal*.

Let us consider the class Ω of all spherical quadrangles with all congruent internal angles or with congruent opposite internal angles.

Proposition 1. *$Q \in \Omega$ if and only if Q has congruent opposite sides.*

Proof. It is obvious that any spherical quadrangle, Q, with congruent opposite sides is an element of Ω.

Suppose now, that Q is an arbitrary element of Ω. Then Q has congruent opposite internal angles say, in cyclic order, $(\alpha_1, \alpha_2, \alpha_1, \alpha_2)$, with $\alpha_i \in (0, \pi)$, $i = 1, 2$, $\alpha_1 + \alpha_2 > \pi$. Lengthening two opposite sides of Q we get a spherical moon, L, as illustrated in Figure 1.

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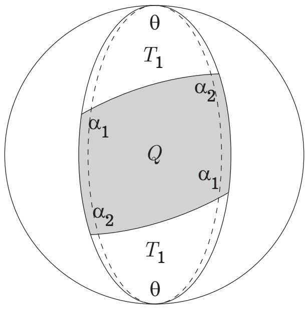

Figure 1

The moon L includes Q and two spherical triangles, T_1 and T_2. As T_1 and T_2 have congruent internal angles, they are congruent, and so the sides of Q common to respectively T_1 and T_2 are congruent. The result now follows applying the same reasoning to the other pair of opposite sides of Q. $\square$

Proposition 2. *Let L_1 and L_2 be two well centered spherical moons with distinct vertices of angle measure θ_1 and θ_2, respectively and let Q be the spherical quadrangle $Q = L_1 \cap L_2$. Then Q has internal angles and sides in cyclic order of the form, $(\alpha_1, \alpha_2, \alpha_1, \alpha_2)$ and (a, b, a, b), respectively. Moreover, L_1 and L_2 are orthogonal if and only if $\alpha_1 = \alpha_2$, and $\theta_1 = \theta_2$ if and only if $a = b$.*

Proof. Let L_1 and L_2 be two well centered spherical moons with distinct vertices of angle measure θ_1 and θ_2, respectively. L_1 and L_2 divide the semi-sphere into 8 spherical triangles, labelled as indicated in Figure 2, T_i, $i = 1, \ldots, 8$ and a spherical quadrangle $Q = L_1 \cap L_2$.

Let E and N be vertices of L_1 and L_2, respectively, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and a, b, c, d be, respectively, the angles and sides of Q in cyclic order (see Figure 2).

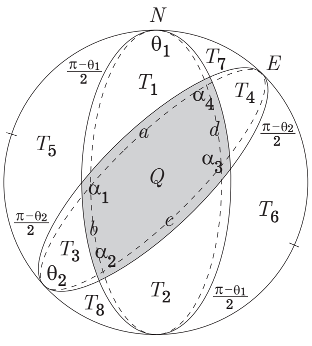

Figure 2

The triangles T_5 and T_6 are congruent (it is enough to verify that they have one congruent side and two congruent angles) and so $\alpha_1 = \alpha_3$. Also T_7 and T_8 are congruent and so $\alpha_2 = \alpha_4$. Since T_5 and T_6 are congruent and T_7 and T_8 are congruent we may conclude that T_1 and T_2 are congruent as well as T_3 and T_4 and so $a = c$ and $b = d$.

Now, $\theta_1 = \theta_2$ if and only if T_1 and T_4 are congruent, that is, if and only if $a = b$.

Besides, L_1 and L_2 are orthogonal iff $E \cdot N = 0$, where $\cdot$ denotes the usual inner product in $\mathbb{R}^3$, iff T_6 and T_7 are congruent iff $\alpha_1 = \alpha_2$. $\square$

Corollary 1. *Using the same terminology as before one has*

i) *If* $\theta_1 = \theta_2$ *and* $E \cdot N = 0$ *then* $Q = L_1 \cap L_2$ *has congruent internal angles and all congruent sides;*

ii) *If* $\theta_1 = \theta_2$ *and* $E \cdot N \neq 0$ *then* $Q = L_1 \cap L_2$ *has all congruent sides and distinct congruent opposite pairs of angles;*

iii) *If* $\theta_1 \neq \theta_2$ *and* $E \cdot N = 0$ *then* $Q = L_1 \cap L_2$ *has congruent internal angles and distinct congruent opposite pairs of sides;*

iv) *If* $\theta_1 \neq \theta_2$ *and* $E \cdot N \neq 0$ *then* $Q = L_1 \cap L_2$ *has distinct congruent opposite pairs of angles and distinct congruent opposite pairs of sides.*

By a *well centered spherical quadrangle* (WCSQ) we mean a spherical quadrangle which is the intersection of two well centered spherical moons with distinct vertices.

Proposition 3. *Let* Q *be a spherical quadrangle with congruent internal angles, say* $\alpha \in (\frac{\pi}{2}, \pi)$, *and with congruent sides,* a. *Then* a *is uniquely determined by* α.

Proof. The diagonal of Q divides Q in two congruent isosceles triangles of angles $(\alpha, \frac{\alpha}{2}, \frac{\alpha}{2})$. Thus, if a is the side of Q one has

$$\cos a = \frac{\cos\frac{\alpha}{2}(1+\cos\alpha)}{\sin\frac{\alpha}{2}\sin\alpha} = \frac{1+\cos\alpha}{1-\cos\alpha}$$

We can observe that this relation defines an increasing continuous bijection between $\alpha \in (\frac{\pi}{2}, \pi)$ and $a \in (0, \frac{\pi}{2})$. ☐

Proposition 4. *Let* Q *be a spherical quadrangle with congruent internal angles, say* $\alpha \in (\frac{\pi}{2}, \pi)$, *and with congruent sides. Then* Q *is congruent to a WCSQ.*

Proof. Let Q be a spherical quadrangle with congruent internal angles, $\alpha \in (\frac{\pi}{2}, \pi)$, and with all congruent sides.

Consider two spherical moons well centered and orthogonal, L_1 and L_2 with the same angle measure $\theta \in (0, \pi)$ such that $\cos\theta = 2\cos\alpha + 1$ and $Q^\star = L_1 \cap L_2$, see Figure 3. Let us show that Q is congruent to $Q^\star$. By Corollary 1, $Q^\star$ has congruent internal angles and congruent sides.

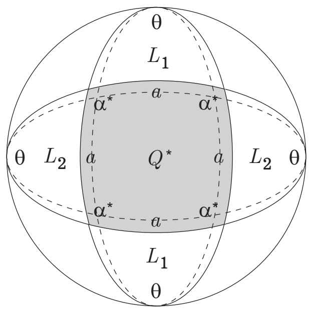

Figure 3

Denoting by $\alpha^\star \in (\frac{\pi}{2}, \pi)$ the internal angle of $Q^\star$ one has,

$$\cos\alpha^\star = -\cos^2\frac{(\pi-\theta)}{2} + \sin^2\frac{(\pi-\theta)}{2}\cos\frac{\pi}{2} = -\sin^2\frac{\theta}{2} = \frac{\cos\theta - 1}{2} = \cos\alpha$$

Thus $\alpha = \alpha^\star$ and consequently Q and $Q^\star$ are congruent, since they have internal congruent angles and by the previous proposition they also have congruent sides. It should be pointed out that the relation $\cos\theta = 2\cos\alpha + 1$ defines an increasing continuous bijection between $\alpha \in (\frac{\pi}{2}, \pi)$ and $\theta \in (0, \pi)$. □

Proposition 5. *Let Q be a spherical quadrangle with congruent internal angles, say $\alpha \in (\frac{\pi}{2}, \pi)$, and with distinct congruent opposite pairs of sides, say a and b. Then anyone of the parameters α, a or b is completely determined by the other two.*

Proof. Let Q be a spherical quadrangle in the above conditions. For $\alpha \in (\frac{\pi}{2}, \pi)$ and $a \in (0, \pi)$, b is determined by the system of equations:

$$\begin{cases} \cos b &= \cos^2 \frac{(\pi-a)}{2} + \sin^2 \frac{(\pi-a)}{2} \cos\theta \\ \cos\theta &= -\cos^2(\pi-\alpha) + \sin^2(\pi-\alpha)\cos b \end{cases}$$

where θ is the angle indicated in Figure 4.

Therefore,

$$\cos b = -1 + \frac{2}{1 + \cot^2 \frac{a}{2} \cos^2 \alpha}$$

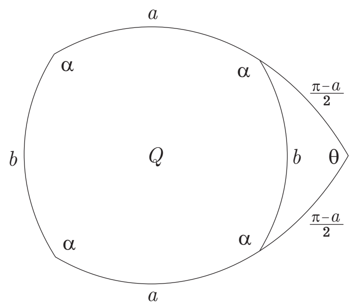

Figure 4

In a similar way, a can be expressed as a function of b and α.

We shall show in next lemma that α can also be expressed as a function of a and b. □

Lemma 1. *Let Q be a spherical quadrangle with distinct congruent opposite pairs of sides, say a and b and with congruent internal angles, say α. Then*

$$\cos\alpha = -\tan\frac{a}{2}\,\tan\frac{b}{2}$$

Proof. Let Q be a spherical quadrangle in the above conditions. Lengthening the vertices of two adjacent edges, one gets two isosceles triangles with sides a, $\frac{\pi-b}{2}$, and b, $\frac{\pi-a}{2}$ respectively, see Figure 5.

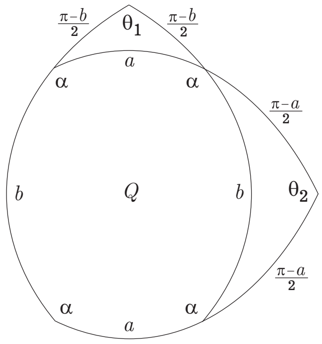

Figure 5

Let θ_1 and θ_2 be the internal angle measure of these triangles, see Figure 5. Then

$$\cos\alpha = -\cos\frac{\pi-\theta_1}{2}\cos\frac{\pi-\theta_2}{2} + \sin\frac{\pi-\theta_1}{2}\sin\frac{\pi-\theta_2}{2}\cos\frac{\pi}{2} = -\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}.$$

On the other hand,

$$\cos\theta_1 = \frac{\cos a - \cos^2\frac{\pi-b}{2}}{\sin^2\frac{\pi-b}{2}} = \frac{\cos a - \sin^2\frac{b}{2}}{\cos^2\frac{b}{2}}$$

and

$$\cos\theta_2 = \frac{\cos b - \cos^2\frac{\pi-a}{2}}{\sin^2\frac{\pi-a}{2}} = \frac{\cos b - \sin^2\frac{a}{2}}{\cos^2\frac{a}{2}}.$$

Thus

$$\cos\alpha = -\sqrt{\frac{1-\cos\theta_1}{2}}\sqrt{\frac{1-\cos\theta_2}{2}} = -\tan\frac{a}{2}\tan\frac{b}{2}.$$

□

Proposition 6. *Let Q be a spherical quadrangle with congruent internal angles, say $\alpha \in (\frac{\pi}{2}, \pi)$ and with distinct congruent opposite sides, say $a \in (0, \pi)$ and $b = b(a, \alpha)$. Then Q is congruent to a WCSQ.*

Proof. Suppose that Q is a spherical quadrangle in the above conditions. We shall show that for two orthogonal well centered moons of angle measure, respectively, θ_1 and θ_2, the unique solution of the system of equations,

$$\begin{cases} \cos\alpha &= -\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} \\ \cos a &= \frac{\cos\theta_1+\cos^2\alpha}{\sin^2\alpha} \end{cases} \tag{1}$$

defines a well centered quadrangle (the moon's intersection) congruent to Q. In fact if a such well centered quadrangle exists then by Corollary 1 it has to be the intersection of two orthogonal moons L_1 and L_2 of angles $\theta_1 \in (0, \pi)$ and $\theta_2 \in (0, \pi)$, respectively.

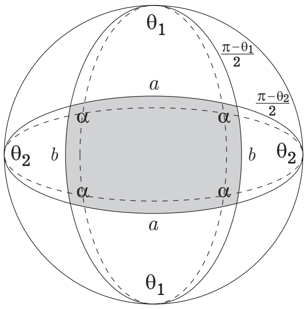

Figure 6

With the Figure 6 annotation, one has

$$\cos\alpha = -\sin\frac{\theta_1}{2}\,\sin\frac{\theta_2}{2},$$

as we have seen before. And

$$\cos a = \frac{\cos\theta_1 + \cos^2(\pi-\alpha)}{\sin^2(\pi-\alpha)} = \frac{\cos\theta_1 + \cos^2\alpha}{\sin^2\alpha}.$$

It is a straightforward exercise to show that the system of equations (1) has a unique solution and that $L1 \cap L2$ is congruent to Q. Observe that $\theta_1 \in (0,\,\pi)$, where the cosine function is injective and $\frac{\theta_2}{2} \in (0,\,\frac{\pi}{2})$, where the sine function is also injective. □

Remark 1. Let $\alpha : (0,\,\pi) \times (0,\,\pi) \to (\frac{\pi}{2},\,\pi)$ and $a : (0,\,\pi) \times (0,\,\pi) \to (0,\,\pi)$ be such that

$$\alpha(\theta_1,\theta_2) = \arccos(-\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}) \text{ and } a(\theta_1,\theta_2) = \arccos\frac{\cos\theta_1 + \sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}}{1-\sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}}.$$

The contour levels of α and a are illustrated in Figure 7 (done by Mathematica).

We may observe that the intersection of any two contour levels of α, and a determine a unique pair of angles $(\theta_1,\,\theta_2) \in (0,\,\pi) \times (0,\,\pi)$, which means that a spherical quadrangle in the conditions of the last proposition is congruent to a well centered spherical quadrangle (the intersection of two orthogonal well centered spherical moons of angles θ_1 and θ_2).

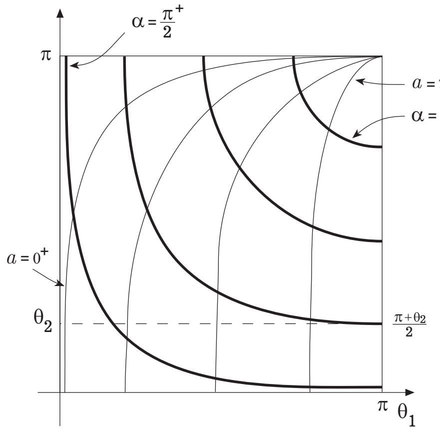

Figure 7

Proposition 7. *Let Q be a spherical quadrangle with all congruent sides, say $a \in (0, \frac{\pi}{2})$ and with congruent opposite angles, say α_1, α_2, $\alpha_1 \geq \alpha_2$. Then $\alpha_1 \geq \arccos(1 - \frac{2}{1+\cos a})$ and anyone of the parameters a, α_1 and α_2 is completely determined by the other two.*

Proof. Suppose that Q is in the above conditions.

1. If $\alpha_1 = \alpha_2 = \alpha$ then as seen in proposition 3, $\cos a = \frac{1+\cos\alpha}{1-\cos\alpha}$, that is, $\cos\alpha = 1 - \frac{2}{1+\cos a}$;
2. If $\alpha_1 > \alpha_2$ then a continuity argument allows us to conclude that $\alpha_1 > \arccos(1 - \frac{2}{1+\cos a}) > \alpha_2$. This can be seen dragging two opposite vertices of Q along the diagonal of Q containing them, see Figure 8.

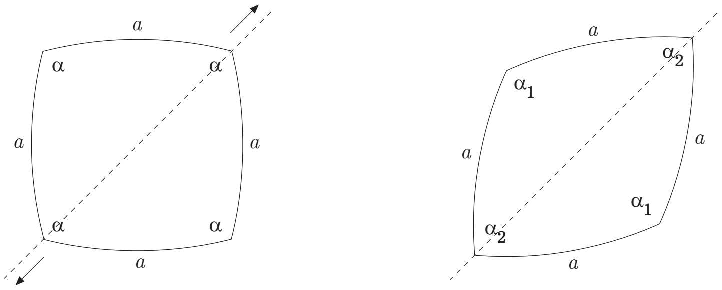

Figure 8

Now, given a and α_1, α_2 is completely determined by the system of equations,

$$\begin{cases} \cos\alpha_2 & = -\cos^2\frac{\alpha_1}{2} + \sin^2\frac{\alpha_1}{2}\cos l \\ \cos l & = \cos^2 a + \sin^2 a \,\cos\alpha_2 \end{cases}$$

where l denotes the diagonal of Q bisecting α_1, see Figure 9.

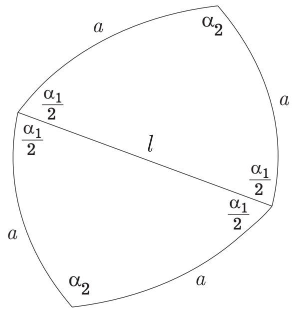

Figure 9

Thus,

$$\cos\alpha_2 = 1 - \frac{2}{1+\tan^2\frac{\alpha_1}{2}\,\cos^2 a} \quad \text{and} \quad \cos a = \cot\frac{\alpha_1}{2}\,\cot\frac{\alpha_2}{2}.$$

□

Proposition 8. *Let Q be a spherical quadrangle with all congruent sides, say $a \in (0, \frac{\pi}{2})$ and with congruent opposite pairs of angles, α_1, α_2, $\alpha_1 > \alpha_2$, with $\alpha_2 = \alpha_2(\alpha_1, a)$ and $\alpha_1 > \arccos(1 - \frac{2}{1+\cos a})$. Then Q is congruent to a WCSQ.*

Proof. Let Q be a spherical quadrangle as indicated above. Let us show first that when two well centered spherical moons with congruent angles, say θ, and $\frac{\pi}{2} - x$, $x \in (0, \frac{\pi}{2})$ as the angle measure between them, then the following system of equations

$$\begin{cases} \cos\alpha_1 &= -\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}\sin x \\ \cos a &= \frac{\cos\theta + \cos\alpha_1\cos\alpha_2}{\sin\alpha_1\sin\alpha_2} \end{cases}$$

has a unique solution which defines a WCSQ congruent to Q.

As seen in Corollary 1, if such well centered spherical quadrangle exists then it has to be the intersection of two well centered spherical moons with congruent angles $\theta \in (0, \pi)$, and such that the angle measure between them is $\frac{\pi}{2} - x$, $x \in (0, \frac{\pi}{2})$, see Figure 10.

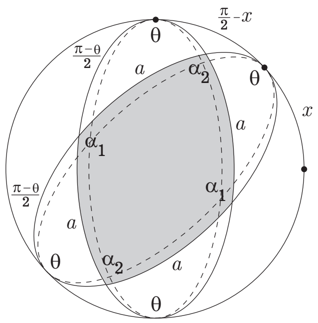

Figure 10

With the labelling of Figure 10 one has,

$$\cos\alpha_1 = -\cos^2\frac{\pi-\theta}{2} + \sin^2\frac{\pi-\theta}{2}\cos(\frac{\pi}{2}+x) = -\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}\sin x$$

and on the other hand,

$$\cos a = \frac{\cos\theta + \cos(\pi-\alpha_2)\cos(\pi-\alpha_1)}{\sin(\pi-\alpha_2)\sin(\pi-\alpha_1)} = \frac{\cos\theta + \cos\alpha_1\cos\alpha_2}{\sin\alpha_1\sin\alpha_2}.$$

Using a similar argument to the one used in proposition 6 it can be seen that the solution is unique and that Q is congruent to a WCSQ. □

Remark 2. Let $\alpha_1 : (0, \pi) \times (0, \frac{\pi}{2}) \to (\frac{\pi}{2}, \pi)$ and $a : (0, \pi) \times (0, \frac{\pi}{2}) \to (0, \frac{\pi}{2})$ be such that

$$\alpha_1(\theta, x) = \arccos(-\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}\sin x)$$

and

$$a(\theta, x) = \arccos\frac{\cos\theta + (-\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}\sin x)(-\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\sin x)}{\sqrt{1-(-\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}\sin x)^2}\sqrt{1-(-\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\sin x)^2}}.$$

The contour levels of α_1 and a are represented in Figure 11 (done by Mathematica).

Observe that if $\alpha_1 \in (\frac{\pi}{2}, \pi)$ and $a \in (0, \frac{\pi}{2})$ such that $\alpha_1 \geq \arccos(1 - \frac{2}{1+\cos a})$ then the intersection of any two contour levels of α_1 and a is a unique point $(\theta, x) \in (0, \pi) \times (0, \frac{\pi}{2})$. In other words any spherical quadrangle in the conditions of the previous proposition is congruent to the intersection of two well centered spherical moons with the same angle measure, θ, and being $\frac{\pi}{2} - x$ the angle measure between them.

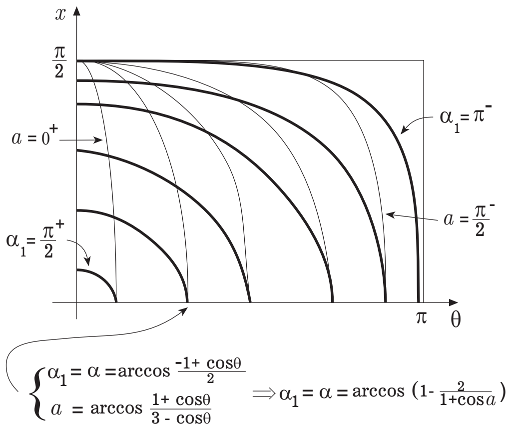

Figure 11

Proposition 9. *Let Q be a spherical quadrangle with congruent opposite sides, say a and b and with congruent opposite angles, say α_1 and α_2 with $\alpha_1 \geq \alpha_2$. Then,*

i) $a + b < \pi$;

ii) $\alpha_1 \geq \arccos(-\tan\frac{a}{2}\tan\frac{b}{2})$;

iii) *anyone of the parameters α_1, α_2, a or b is completely determined by the other three.*

Proof. If Q is quadrangle as described above then it follows that $0 < 2a + 2b < 2\pi$ and also $2\alpha_1 + 2\alpha_2 - 2\pi > 0$, with $\alpha_1 \in (0, \pi)$ and $\alpha_2 \in (0, \pi)$. That is, $0 < a + b < \pi$, $\alpha_1 + \alpha_2 > \pi$, $\alpha_2 \in (0, \pi)$ and $\alpha_1 \in (\frac{\pi}{2}, \pi)$, since $\alpha_1 \geq \alpha_2$.

Assume, in first place, that $\alpha_1 = \alpha_2 = \alpha$. Then, by lemma 1 we have $\cos\alpha = -\tan\frac{a}{2}\tan\frac{b}{2}$ and so $\alpha = \alpha_1 = \alpha_2 = \arccos(-\tan\frac{a}{2}\tan\frac{b}{2})$.

As before a continuity argument allows us to conclude that if $\alpha_1 > \alpha_2$, then $\alpha_1 > \arccos(-\tan\frac{a}{2}\tan\frac{b}{2}) > \alpha_2$.

Now, we show how to determine α_2 as a function of a, b and α_1. The diagonal l of Q through α_2 gives rise to two angles, x and y, ($\alpha_2 = x + y$) as illustrated in Figure 12.

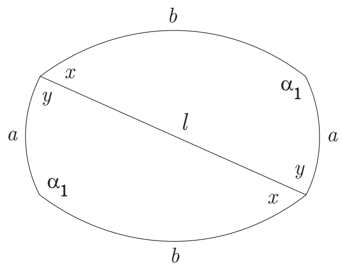

Figure 12

One has,

$$\cos l = \cos a \cos b + \sin a \sin b \cos \alpha_1.$$

Besides,

$$\cos x = \frac{\cos a - \cos b \cos l}{\sin b \sin l} \text{ and } \cos y = \frac{\cos b - \cos a \cos l}{\sin a \sin l}.$$

Since $\alpha_2 = x + y$, then α_2 is function of a, b and α_1.

We can also determine b as a function of a, α_1 and α_2 as follows. Let b_1, b_2 and θ be, respectively, the sides and the internal angle measure (to be determined) of the triangle obtained by lengthening the b sides of Q, see Figure 13.

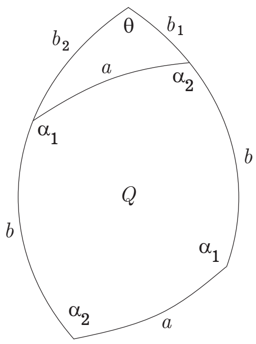

Figure 13

One has,

$$\cos \theta = -\cos \alpha_1 \cos \alpha_2 + \sin \alpha_1 \sin \alpha_2 \cos a$$

and

$$\cos b_1 = -\frac{\cos \alpha_1 + \cos \alpha_2 \cos \theta}{\sin \alpha_2 \sin \theta} \quad \wedge \quad \cos b_2 = -\frac{\cos \alpha_2 + \cos \alpha_1 \cos \theta}{\sin \alpha_1 \sin \theta}$$

Finally, $b = \pi - (b_1 + b_2)$ is function of a, α_1 and α_2. $\square$

Proposition 10. *Let Q be a spherical quadrangle with congruent opposite sides, say a and b such that $a + b < \pi$ and with congruent internal angles α_1, α_2, $\alpha_1 > \alpha_2$. Let us suppose also that $\alpha_1 > \arccos(-\tan \frac{a}{2} \tan \frac{b}{2})$ and $\alpha_2 = \alpha_2(a, b, \alpha_1)$. Then, Q is congruent to a WCSQ.*

Proof. Let Q be a spherical quadrangle in the above conditions. We shall show that when we have two well centered spherical moons with angle measure θ_1 and θ_2 and such that $\frac{\pi}{2} - x$, $x \in (0, \frac{\pi}{2})$ is the angle measure between them, see Figure 14, then the unique solution of the system of equations

$$\begin{cases} \cos \alpha_1 &= -\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} - \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \sin x \\ \cos a &= \frac{\cos \theta_1 + \cos \alpha_1 \cos \alpha_2}{\sin \alpha_1 \sin \alpha_2} \\ \cos b &= \frac{\cos \theta_2 + \cos \alpha_1 \cos \alpha_2}{\sin \alpha_1 \sin \alpha_2} \end{cases}$$

defines a well centered spherical quadrangle congruent to Q.

As seen in Corollary 1, if a such WCSQ exists it should be the intersection of two well centered spherical moons (not orthogonal) with angles measure θ_1 and θ_2, $0 < \theta_i < \pi$, $i = 1, 2$ and with $\frac{\pi}{2} - x$, $0 < x < \frac{\pi}{2}$ as the angle measure between them, see Figure 14.

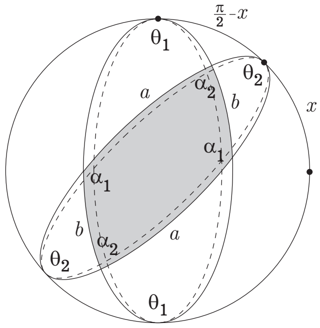

Figure 14

With the notation used in Figure 14 one has,

$$\begin{aligned}\cos\alpha_1 &= -\cos\frac{\pi-\theta_1}{2}\cos\frac{\pi-\theta_2}{2} + \sin\frac{\pi-\theta_1}{2}\sin\frac{\pi-\theta_2}{2}\cos(\frac{\pi}{2}+x)\\ &= -\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} - \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin x\end{aligned}$$

On the other hand,

$$\cos a = \frac{\cos\theta_1 + \cos(\pi-\alpha_1)\cos(\pi-\alpha_2)}{\sin(\pi-\alpha_1)\sin(\pi-\alpha_2)} = \frac{\cos\theta_1 + \cos\alpha_1\cos\alpha_2}{\sin\alpha_1\sin\alpha_2}$$

and

$$\cos b = \frac{\cos\theta_2 + \cos(\pi-\alpha_1)\cos(\pi-\alpha_2)}{\sin(\pi-\alpha_1)\sin(\pi-\alpha_2)} = \frac{\cos\theta_2 + \cos\alpha_1\cos\alpha_2}{\sin\alpha_1\sin\alpha_2}.$$

As before, it is a straightforward exercise to state the uniqueness of the solution. □

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