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Partial Intersections and Graded Betti Numbers

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Abstract. It is well known that for 2-codimensional aCM subschemes of $\mathbb{P}^r$ with a fixed Hilbert function H there are all the possible graded Betti numbers between suitable bounds depending on H. For aCM subschemes of codimension $c \geq 3$ with Hilbert function H it is just known that there are upper bounds for the graded Betti numbers depending on H and these can be reached; but what are the graded Betti numbers which can be realized is not yet completely understood. The aim of the paper is to construct c-codimensional subschemes of $\mathbb{P}^r$ which could recover as many graded Betti numbers as possible generalizing both the 2-codimensional case and the maximal case.

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Introduction

In the last few years a large number of researchers in algebraic geometry made deep investigations about Hilbert functions and graded Betti numbers of projective schemes. In particular, to try to find all the possible Hilbert functions and all the possible graded Betti numbers of c-codimensional (reduced) schemes of $\mathbb{P}^r$ is surely one of the most important questions but not so easy to solve. On the other hand, the problem of finding all the possible graded Betti numbers for schemes with an assigned Hilbert function seems just as difficult. Many results are available on this field but they deal always with particular situations as for the aCM schemes of codimension 2 or, more recently, for the arithmetically Gorenstein schemes of codimension 3 (see, for instance, papers by Gaeta [7], Buchsbaum-Eisenbud [1], Stanley [17], Campanella [3], Maggioni-Ragusa [14], Bigatti [2], Hulett [12], De Negri-Valla [5], Diesel [6], Geramita-Migliore [11], Ragusa-Zappalà [16], and many others).

The following observation can be considered the motivation of the project that we intend to develope and of which this paper is in some sense a starting point: for aCM schemes of codimension 2 with an assigned Hilbert function H the possible graded Betti numbers have both a maximum and a minimum value, in the sense that there exist integers a_{ij}, b_{ij} for $i = 1, 2$ and $j \geq 1$, depending only on H, such that any minimal free resolution of such a scheme must have graded Betti numbers α_{ij} with

$$a_{ij} \leq \alpha_{ij} \leq b_{ij}$$

for $i = 1, 2$ and $j \geq 1$; moreover, there are resolutions with graded Betti numbers a_{ij} and b_{ij}, respectively, and any possibility satisfying the previous bounds can be realized (see for instance [3] or [14]). This is no longer true for aCM schemes of codimension $c \geq 3$; for such schemes with a fixed Hilbert function there is still a maximum for the graded Betti numbers but there is not necessarily a minimum (see the example by E.G. Evans in the paper [12]). Recently, A. Geramita, T. Harima, and Y.S. Shin in [9] construct suitable reduced 0-dimensional subschemes of $\mathbb{P}^r$ which realize the maximum for the graded Betti numbers with respect to an assigned Hilbert function. On the other hand, it is possible to construct reduced zero-dimensional schemes which realize the maximum for the graded Betti numbers with respect to a given Hilbert function simply lifting, by Hartshorne method, the corresponding lex-segment ideal.

Our project is to find methods to build aCM schemes of any codimension which could realize all the possible graded Betti numbers relative to a given Hilbert function. Here we develope a construction which generalizes both the codimension 2 case of [14] and the maximal situation as in [9]. Even if this method does not permit to obtain all the wanted graded Betti numbers it recovers a large part of the possibilities and, moreover, one can use this construction to easily build schemes with assigned generators' degrees or last syzygies' degrees (and consequently Cohen Macaulay type). Our intent is to develope in the near future other methods to try to recover all the remaining cases.

All this can be done in a general setting or applying suitable conditions either on the schemes or on the Hilbert function. For instance, it seems of great interest to solve the same questions for Gorenstein schemes (of codimension > 3), level schemes, schemes lying on an irreducible hypersurface of minimal degree, schemes in uniform position, or for maximal Hilbert functions, Hilbert functions generic with respect to the minimal degree and so on.

Now we give a sketch of the paper. The first section is initially devoted to prepare the machinery we need to define the "partial intersection" subschemes of $\mathbb{P}^r$; in particular, we study some properties of the "left segments". Then, after defining such subschemes we prove that they are reduced aCM schemes consisting of union of linear varieties. In Section 2 we compute the Hilbert function of c-codimensional partial intersections in terms of their left segment support and provide a free resolution for these schemes. Section 3 is dedicated to compute both the minimal generators' degrees and the last syzygies' degrees in terms of their support. This will complete all the graded Betti numbers for partial intersection of codimension 3. In the last section, using a "linear decomposition" for O-sequences, we perform all the partial intersections (or equivalently all the left segments) having an assigned Hilbert function.

1. Partial intersections in codimension c

Throughout this paper k will denote an algebraically closed field, $\mathbb{P}^r$ the r-dimensional projective space over k, $R=k[x_0, x_1, \ldots, x_r]=\bigoplus_{n\in\mathbf{Z}} H^0(\mathscr{O}_{\mathbb{P}^r}(n))$.

If $V \subset \mathbb{P}^r$ is a subscheme, I_V will denote its defining ideal and $H_V(n) = \dim_k R_n - \dim_k(I_V)_n$ its Hilbert function. Moreover, if $V \subset \mathbb{P}^r$ is a c-codimensional aCM scheme with minimal free resolution

$$0 \longrightarrow \oplus R(-j)^{\alpha_{cj}} \cdots \longrightarrow \oplus R(-j)^{\alpha_{2j}} \longrightarrow \oplus R(-j)^{\alpha_{1j}} \longrightarrow I_V \longrightarrow 0$$

then the integers $\{\alpha_{ij}\}_j$ will denote the i-th graded Betti numbers.

In this section we construct suitable c-codimensional aCM subschemes of $\mathbb{P}^r$ for which we will be able to compute both Hilbert functions and the first and the last graded Betti numbers.

In order to define these subschemes of $\mathbb{P}^r$, we need some elementary properties of particular posets.

Let $(\mathcal{P}, \leq)$ be a poset. We denote, for every $H \in \mathcal{P}$,

$$\mathcal{S}_H = \{K \in \mathcal{P} \mid K < H\}, \quad \overline{\mathcal{S}}_H = \{K \in \mathcal{P} \mid K \leq H\}.$$

Definition 1.1. *A subset $\mathcal{A}$ of the poset $\mathcal{P}$ is said to be a left segment if for every $H \in \mathcal{A}$, $\mathcal{S}_H \subseteq \mathcal{A}$. In particular, when $\mathcal{P} = \mathbb{N}^c$ with the order induced by the natural order on $\mathbb{N}$, a finite left segment will be mentioned as a c-left segment.*

If U is a finite subset of $\mathbb{N}^c$ then the c-left segment

$$< U >= \bigcup_{H\in U} \overline{\mathcal{S}}_H$$

will be called the *c-left segment generated* by U. Note that every c-left segment $\mathcal{A}$ has sets of generators and among those there is a unique minimal set of generators consisting of the maximal elements of $\mathcal{A}$, which will be denoted by $G(\mathcal{A})$.

In the sequel $\alpha_i : \mathbb{N}^c \to \mathbb{N}$ will denote the projection to the i-th component; moreover, $\hat{\alpha}_i : \mathbb{N}^c \to \mathbb{N}^{c-1}$ will indicate the map $\hat{\alpha}_i(m_1, \ldots, m_c) = (m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_c)$.

If $\mathcal{A}$ is a c-left segment, we set $a_i = \max\{\alpha_i(H) \mid H \in \mathcal{A}\}$, for $1 \leq i \leq c$. The c-tuple $T = (a_1, \ldots, a_c)$ will be called the *size* of $\mathcal{A}$.

A c-left segment is said to be *degenerate* if $a_i = 1$ for some i.

If $\mathcal{A}$ is a c-left segment, $F(\mathcal{A})$ will denote the set of minimal elements of $\mathbb{N}^c \setminus \mathcal{A}$, i.e.

$$F(\mathcal{A}) = \{H \in \mathbb{N}^c \setminus \mathcal{A} \mid \mathcal{S}_H \subseteq \mathcal{A}\}.$$

Note that, if $H = (m_1, \ldots, m_c)$ is in $F(\mathcal{A})$ and $m_i > 1$, then $H_i = (m_1, \ldots, m_i-1, \ldots, m_c) \in \mathcal{A}$. Moreover, the elements

$$T_1 = (a_1 + 1, 1, \ldots, 1), \ldots, T_c = (1, 1, \ldots, a_c + 1)$$

always belong to $F(\mathcal{A})$.

In the sequel we denote the c-tuple $(1,\dots,1)$ by I and, for every subset Z of $\overline{\mathcal{S}}_T$, we denote

$$C_T(Z)=\{T+I-H \mid H\in Z\}.$$

Finally, for every c-left segment $\mathcal{A}$ we define

$$\mathcal{A}^*=C_T(\overline{\mathcal{S}}_T\setminus\mathcal{A}).$$

Observe that $\mathcal{A}^*$ is a c-left segment: namely, if $T+I-H\in\mathcal{A}^*$ and $K<T+I-H$, we see that $H<T+I-K$ hence $T+I-K\notin\mathcal{A}$, therefore $K=T+I-(T+I-K)\in\mathcal{A}^*$.

For this c-left segment we set $a_i^*=\max\{\alpha_i(H)\mid H\in\mathcal{A}^*\}$ and $T_i^*=(1,\dots,a_i^*+1,\dots,1)$ for $1\le i\le c$.

Example 1.2. Let $\mathcal{A}=<(1,3,2),(2,2,1)>$; then
$T=(2,3,2)$ and $\overline{\mathcal{S}}_T\setminus\mathcal{A}=\{(2,3,1),(2,1,2),(2,2,2),(2,3,2)\}$, hence

$$\mathcal{A}^*=\{(1,1,2),(1,3,1),(1,2,1),(1,1,1)\}=<(1,1,2),(1,3,1)>.$$

Proposition 1.3. *If $\mathcal{A}$ is a c-left segment, then*

1. $F(\mathcal{A})=C_T(G(\mathcal{A}^*))\cup\{T_1,\dots,T_c\}$,
2. $F(\mathcal{A}^*)=C_T(G(\mathcal{A}))\cup\{T_1^*,\dots,T_c^*\}$.
3. *If $T_i^*\ne T_i$, for some i, then $T_i^*\in C_T(G(\mathcal{A}))$.*

Proof. 1. Let $H\in F(\mathcal{A})$, $H\notin\{T_1,\dots,T_c\}$, then $H\in\overline{\mathcal{S}}_T\setminus\mathcal{A}$: namely, if $H\notin\overline{\mathcal{S}}_T$, $H=(h_1,\dots,h_c)$, there is an index j such that $h_j>a_j$; on the other hand by assumption $H\ne T_j$ there is $h_t>1$ for some $t\ne j$; denoted $H'=(h'_1,\dots,h'_c)$, with $h'_t=h_t-1$ and $h'_i=h_i$ for $i\ne t$, we have $H'<H$ which should imply $H'\le T$ a contradiction with $h'_j>a_j$. Consequently, $T+I-H\in\mathcal{A}^*$; on the other hand, if $L>T+I-H$ then $H>T+I-L$ therefore $T+I-L\in\mathcal{A}$, i.e. $T+I-L\notin\overline{\mathcal{S}}_T\setminus\mathcal{A}$, which means $L\notin\mathcal{A}^*$. In conclusion $H\in C_T(G(\mathcal{A}^*))$.

Conversely, let $H\in C_T(G(\mathcal{A}^*))$, then $T+I-H\in G(\mathcal{A}^*)$. From this we get $T+I-H\in\mathcal{A}^*$, i.e. $H\notin\mathcal{A}$. Moreover, if $K<H$ then $T+I-K>T+I-H$ which implies $T+I-K\notin\mathcal{A}^*$ since $T+I-H$ belongs to $G(\mathcal{A}^*)$. Thus $K\in\mathcal{A}$ and consequently, $H\in F(\mathcal{A})$.

2. It works with the same argument as in 1.

3. Suppose that $T_i^*\ne T_i$, indeed $a_i^*<a_i$ for some i, then $T+I-T_i^*=(a_1,\dots,a_i-a_i^*+1,\dots,a_c)\in\mathcal{A}$; on the other hand, from $(1,\dots,a_i^*-1,\dots,1)\in\mathcal{A}^*$ we get that $(a_1,\dots,a_i-a_i^*+2,\dots,a_c)\notin\mathcal{A}$, i.e. $T+I-T_i^*$ is maximal in $\mathcal{A}$, so $T+I-T_i^*\in G(A)$. □

Fix a c-left segment $\mathcal{A}$ and consider c families of hyperplanes of $\mathbb{P}^r$, $c\le r$,

$$\{A_{1j}\}_{1\le j\le a_1},\quad \{A_{2j}\}_{1\le j\le a_2},\quad \dots,\quad \{A_{cj}\}_{1\le j\le a_c}$$

sufficiently generic, in the sense that $A_{1j_1}\cap\ldots\cap A_{cj_c}$ are $\prod_{i=1}^{c}a_i$ pairwise distinct linear varieties of codimension c.

For every $H = (j_1, \dots, j_c) \in \mathcal{A}$, we denote by

$$L_H = \bigcap_{h=1}^{c} A_{hj_h}.$$

With this notation we have the following

Definition 1.4. *The subscheme of* $\mathbb{P}^r$

$$V = \bigcup_{H \in \mathcal{A}} L_H$$

will be called a c-partial intersection with respect to the hyperplanes $\{A_{ij}\}$ *and support on the c-left segment* $\mathcal{A}$.

If $\mathcal{B} \subseteq \mathcal{A}$ *is a c-left segment,* $W = \bigcup_{H \in \mathcal{B}} L_H$ *will be called a sub partial intersection of* V.

Note that for $c = 2$ these are substantially the partial intersections studied in [14].

In the paper [13] the authors use schemes which are similar to partial intersections in order to lift Artinian monomial ideals. Nevertheless, this new approach will be crucial for our goals.

Let $\mathcal{A}_j = \hat{\alpha}_c(\alpha_c^{-1}(j))$, i.e. the set of $(c-1)$-tuples H such that $(H, j) \in \mathcal{A}$ $(c \geq 2)$. It is trivial to see that $\mathcal{A}_j$ is a $(c-1)$-left segment. Moreover, clearly, $\mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \dots \supseteq \mathcal{A}_{a_c}$.

Remark 1.5. Let $V \subset \mathbb{P}^r$ be a c-partial intersection with support on $\mathcal{A}$ and hyperplanes A_{ij}, with $1 \leq i \leq c$, then

$$V = \bigcup_{1 \leq j \leq a_c} (V_j \cap A_{cj})$$

where V_j are $(c-1)$-partial intersections, with $V_j \supseteq V_{j+1}$. Namely, it is enough to set $V_j = \bigcup_{H \in \mathcal{A}_j} L_H$.

Conversely, if $V_1 \supseteq \dots \supseteq V_s$ are $(c-1)$-partial intersections with support on the $(c-1)$-left segments $\mathcal{A}_1 \supseteq \dots \supseteq \mathcal{A}_s$, respectively, and with respect to $c-1$ families of hyperplanes $\{A_{ij}\}$, $1 \leq i \leq c-1$, if we consider another family of "generic" hyperplanes $\{A_{cj}\}$, $j = 1, \dots, s$, then

$$V = \bigcup_{1 \leq j \leq s} (V_j \cap A_{cj})$$

is a c-partial intersection with support on $\mathcal{A} = \{(H, j) \mid H \in \mathcal{A}_j,\ 1 \leq j \leq s\}$. Note that such an $\mathcal{A}$ is a c-left segment: in fact, if $(H, j) \in \mathcal{A}$ and $(H', j') \leq (H, j)$, we have $j' \leq j \leq s$ and $H' \leq H \in \mathcal{A}_j \subseteq \mathcal{A}_{j'}$, hence $(H', j') \in \mathcal{A}$.

In the sequel we need the following lemma.

Lemma 1.6. *Let* $V_1 \supseteq \dots \supseteq V_s$ *be* $s \geq 2$ $(c-1)$*-codimensional aCM subschemes of* $\mathbb{P}^r$ *and* A_j, *with* $I_{A_j} = (f_j)$, $1 \leq j \leq s$, *be hyperplanes such that* $Y_i = V_i \cap A_i$ *are c-codimensional for each*

i and suppose that Y_i and Y_j have no common components for $i \neq j$. Let $Y = Y_1 \cup \ldots \cup Y_{s-1}$ and $X = Y \cup Y_s$. Then the following sequence of graded R-modules

$$0 \longrightarrow I_{Y_s}(-(s-1)) \xrightarrow{f} I_X \xrightarrow{\varphi} I_Y/(f) \longrightarrow 0$$

is exact, where $f = \prod_{i=1}^{s-1} f_i$ and f_i is the form defining A_i for $i = 1, \ldots, s-1$ and φ is the natural map. Moreover

$$I_X = I_{V_1} + f_1 I_{V_2} + f_1 f_2 I_{V_3} + \cdots + f_1 \ldots f_{s-1} I_{V_s} + (f_1 \ldots f_s).$$

Proof. Observe that the exactness of the above sequence in the middle depends on the fact that f is regular modulo I_{Y_s}, since Y_i and Y_j have no common components for $i \neq j$. So, the only not trivial fact to prove is that the map φ is surjective. For this we use induction on s. For $s = 2$, since V_1 is aCM, $I_Y = I_{V_1} + (f_1)$, therefore every element in $I_Y/(f_1)$ looks like $z + (f_1)$ with $z \in I_{V_1} \subseteq I_{V_2}$. Hence, $z \in I_{Y_1} \cap I_{Y_2} = I_X$. So, φ is surjective and the sequence is exact. Now, from the exactness of this sequence it follows that I_X is generated by I_{V_1} and $f_1 I_{Y_2}$, i.e. $I_X = I_{V_1} + f_1 I_{V_2} + (f_1 f_2)$.

Let us suppose the lemma true for $s-1$. This means, in particular, that $I_Y = I_{V_1} + f_1 I_{V_2} + \cdots + f_1 \ldots f_{s-2} I_{V_{s-1}} + (f_1 \ldots f_{s-1})$; therefore, every element $z \in I_Y/(f_1 \ldots f_{s-1})$ has the form $x + (f_1 \ldots f_{s-1})$ with $x \in I_{V_1} + f_1 I_{V_2} + \cdots + f_1 \ldots f_{s-2} I_{V_{s-1}}$. Hence, $x \in I_{V_s}$ which implies $x \in I_Y \cap I_{Y_s} = I_X$. Again, by the exactness of our sequence we get that I_X is generated by $f_1 \ldots f_{s-1} I_{Y_s}$ and by $I_{V_1} + f_1 I_{V_2} + \cdots + f_1 \ldots f_{s-2} I_{V_{s-1}}$, i.e. $I_X = I_{V_1} + f_1 I_{V_2} + \cdots + f_1 \ldots f_{s-1} I_{V_s} + (f_1 \ldots f_s)$. $\square$

Corollary 1.7. *The ideal I_X associated to any c-partial intersection X is minimally generated by elements which are products of linear forms.*

Proof. Since any c-partial intersection X, by the genericity of the hyperplanes A_{ij}, is in the hypotheses of the previous lemma, we have

$$I_X = I_{V_1} + f_1 I_{V_2} + f_1 f_2 I_{V_3} + \cdots + f_1 \ldots f_{s-1} I_{V_s} + (f_1 \ldots f_s),$$

it is enough to use induction on c. $\square$

Remark 1.8. Note that, by construction of partial intersections, $f_1 \cdot \ldots \cdot f_s$ is a minimal generator for I_X.

Theorem 1.9. *Every c-partial intersection X of $\mathbb{P}^r$ is a reduced aCM subscheme consisting of a union of c-codimensional linear varieties.*

Proof. By definition X is a reduced scheme of codimension c union of linear varieties of the same codimension.

To show that X is aCM we use induction on c. The property is trivially true for $c = 1$, so we can assume that every $(c-1)$-partial intersection is aCM. Since X is a c-partial intersection we have $X = \bigcup_{1 \leq i \leq s} (V_i \cap A_i)$, where V_i are $(c-1)$-partial intersections, $V_i \supseteq V_{i+1}$ for $i = 1, \ldots, s-1$ and A_i are hyperplanes. Now we use induction on s. For $s = 1$, X is aCM as it is a hyperplane section of an aCM scheme. Suppose that $Y = \bigcup_{1\le i\le s-1} (V_i \cap A_i)$ is aCM and show that $X = Y \cup (V_s \cap A_s)$ is aCM. Applying the previous lemma we get the exact sequence

$$0 \longrightarrow I_{V_s\cap A_s}(-(s-1)) \xrightarrow{f} I_X \longrightarrow I_Y/(f) \longrightarrow 0$$

where $f = \prod_{i=1}^{s-1} f_i$ and f_i is the form defining A_i for $i = 1, \dots, s-1$, from which we see that a resolution of I_X can be obtained as direct sum of the resolutions of $I_{V_s\cap A_s}(-(s-1))$ and $I_Y/(f)$; since both have resolutions of length c the same is true for I_X and we are done. □

2. Hilbert functions for partial intersections

In this section we compute the Hilbert function of a partial intersection V of codimension c in terms of its support $\mathcal{A}$.

In the sequel, if $H = (m_1, \dots, m_c) \in \mathbb{N}^c$, then we will write $v(H) = m_1 + \cdots + m_c$.

Theorem 2.1. *If $V \subset \mathbb{P}^r$ is a partial intersection of codimension c with support on $\mathcal{A}$, then the $(r-c+1)$-st difference of its Hilbert function is*

$$\Delta^{r-c+1} H_V(n) = \Big| \{H \in \mathcal{A} \mid v(H) = n+c\} \Big| .$$

Proof. We work by induction on c. For $c = 1$, $\mathcal{A} = \{1, 2, \dots, a_1\}$, $V = A_{11} \cup \ldots \cup A_{1a_1}$. We have

$$\Delta^r H_V(n) = \begin{cases} 1 & \text{if } 0 \le n \le a_1 - 1 \\ 0 & \text{if } n \ge a_1 \end{cases} .$$

On the other hand, there is only one $H \in \mathcal{A}$ whose $v(H) = n+1$, for $n = 0, 1, \dots, a_1 - 1$. Let us suppose that the conclusion is true for every $(c-1)$-partial intersection. Now let $V = \bigcup_{H\in\mathcal{A}} L_H = \bigcup_{1\le j\le a_c} (V_j \cap A_{cj})$ be a c-partial intersection, where the V_j's are $(c-1)$-partial intersections with support on $\mathcal{A}_j$ and the A_{cj}'s are hyperplanes. By the inductive hypothesis we know that $\Delta^{r+1-(c-1)} H_{V_j}(n) = \Big| \{H \in \mathcal{A}_j \mid v(H) = n+c-1\} \Big|$. On the other hand, since the V_j's are aCM

$$\Delta^{r+1-c} H_{V_j\cap A_{cj}} = \Delta^{r+1-(c-1)} H_{V_j}.$$

Now we will prove, by induction on a_c that

$$\Delta^{r+1-c} H_V(n) = \sum_{j=1}^{a_c} \Delta^{r+1-c} H_{V_j\cap A_{cj}}(n+1-j).$$

Since there is nothing to say for $a_c = 1$ we can assume that for $W = \bigcup_{1\le j\le a_c-1} (V_j \cap A_{cj})$ we have

$$\Delta^{r+1-c} H_W(n) = \sum_{j=1}^{a_c-1} \Delta^{r+1-c} H_{V_j\cap A_{cj}}(n+1-j),$$

and it remains to verify that

$$\Delta^{r+1-c} H_V(n) = \Delta^{r+1-c} H_W(n) + \Delta^{r+1-c} H_{V_{a_c} \cap A_{ca_c}}(n+1-a_c).$$

From the short exact sequence of Lemma 1.6 we get

$$\dim(I_V)_n = \dim(I_{V_{a_c} \cap A_{ca_c}})_{(n+1-a_c)} + \dim(I_W)_n - \dim R(-(a_c-1))_n$$

from which we deduce

$$H_V(n) = H_W(n) + H_{V_{a_c} \cap A_{a_c}}(n+1-a_c).$$

Therefore

$$\Delta^{r+1-c} H_V(n) = \sum_{j=1}^{a_c} \Big| \{H \in \mathcal{A}_j \mid v(H) = n+c-j\} \Big| =$$

$$\Big| \bigcup_{j=1}^{a_c} \{H \in \mathcal{A}_j \mid v(H) = n+c-j\} \Big|,$$

since the previous union runs over disjoint sets. To conclude the proof it is enough to perform a bijection

$$\varphi : \bigcup_{j=1}^{a_c} \{H \in \mathcal{A}_j \mid v(H) = n+c-j\} \to \{H \in \mathcal{A} \mid v(H) = n+c\}.$$

If $K \in \bigcup_{j=1}^{a_c} \{H \in \mathcal{A}_j \mid v(H) = n+c-j\}$, there exists a unique $1 \leq h \leq a_c$ such that $K \in \mathcal{A}_h$ and $v(K) = n+c-h$; then, define

$$\varphi(K) = (K,h) \in \{H \in \mathcal{A} \mid v(H) = n+c\}.$$

One easily shows that such a map is bijective. □

Example 2.2. Let $\mathcal{A}$ be the 3-left segment generated by the following elements of $\mathbb{N}^3$

$$(1,2,3),(2,3,2),(3,4,1),(4,1,3),(4,2,2),(4,3,1),(5,1,2),(5,2,1).$$

Then the elements $H \in \mathcal{A}$ such that
$v(H) = 3$ is $(1,1,1)$;
$v(H) = 4$ are $(1,1,2)$, $(1,2,1)$, $(2,1,1)$;
$v(H) = 5$ are $(1,1,3)$, $(1,2,2)$, $(1,3,1)$, $(2,1,2)$, $(2,2,1)$, $(3,1,1)$;
$v(H) = 6$ are $(1,2,3)$, $(1,3,2)$, $(1,4,1)$, $(2,1,3)$, $(2,2,2)$, $(2,3,1)$, $(3,1,2)$, $(3,2,1)$, $(4,1,1)$;
$v(H) = 7$ are $(2,3,2)$, $(2,4,1)$, $(3,1,3)$, $(3,2,2)$, $(3,3,1)$, $(4,1,2)$, $(4,2,1)$, $(5,1,1)$;
$v(H) = 8$ are $(3,4,1)$, $(4,1,3)$, $(4,2,2)$, $(4,3,1)$, $(5,1,2)$, $(5,2,1)$;
therefore, if $V \subset \mathbb{P}^r$ is any 3-partial intersection with support on $\mathcal{A}$, we have

$$\Delta^{r-2} H_V : \quad 1,3,6,9,8,6,0 \to \quad .$$

Now we want to construct a free resolution of length c (not necessarily minimal) for a c-codimensional partial intersection $V \subset \mathbb{P}^r$.

Construction 2.3. We start by working by induction on c.

If $c = 1$ then $V = \bigcup_{i=1}^n A_i$, where A_i are hyperplanes and the minimal graded free resolution of I_V is

$$0 \longrightarrow R(-n) \xrightarrow{f} I_V \longrightarrow 0$$

where $f = f_1 \cdot \ldots \cdot f_n$ and f_i is the form defining A_i.

Let $V = \bigcup_{1 \le i \le s} (V_i \cap A_i)$, where V_i are $(c-1)$-partial intersections, $V_i \supseteq V_{i+1}$ for $i = 1, \ldots, s-1$ and A_i are hyperplanes. Now we use induction on s. If $s = 1$ then $V = V_1 \cap A_1$; let

$$\mathbb{F}_\bullet \longrightarrow I_{V_1} \longrightarrow 0$$

be a graded free resolution for I_{V_1} of length $c-1$, since V_1 is aCM, using a mapping cone of the map

$$I_{V_1} \xrightarrow{f_1} I_{V_1}$$

we get the following graded free resolution of length c

$$0 \longrightarrow F_{c-1}(-1) \longrightarrow F_{c-2}(-1) \oplus F_{c-1} \longrightarrow \cdots \longrightarrow F_1(-1) \oplus F_2 \longrightarrow F_1 \longrightarrow I_V \longrightarrow 0.$$

Denote by $Y = \bigcup_{1 \le i \le s-1} (V_i \cap A_i)$ and $Y_s = V_s \cap A_s$. By inductive hypothesis we know resolutions of length c for I_Y and I_{Y_s} :

$$0 \longrightarrow \cdots \longrightarrow F_i \xrightarrow{M_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow I_Y \longrightarrow 0$$

$$0 \longrightarrow \cdots \longrightarrow G_i \xrightarrow{N_i} G_{i-1} \longrightarrow \cdots \longrightarrow G_1 \longrightarrow I_{Y_s} \longrightarrow 0$$

Note that $f = f_1 \cdot \ldots \cdot f_{s-1}$ is a minimal generator for I_Y (see Remark 1.8); moreover, using Lemma 1.6 we can assume that the system of generators for I_Y used in the previous resolution consists of f and elements in I_V. Therefore, $F_1 = F_1' \oplus R(-(s-1))$, $M_1 = (M_1' | f)$ with the entries of M_1' in I_V and

$$M_2 = \begin{pmatrix} M_2' \\ Q_2 \end{pmatrix}$$

where M_2' is the matrix defining the map $F_2 \to F_1'$. From the short exact sequence of Lemma 1.6 one gets the following resolution of length c for I_V

$$\cdots \longrightarrow G_i(1-s) \oplus F_i \xrightarrow{P_i} G_{i-1}(1-s) \oplus F_{i-1} \longrightarrow \cdots \xrightarrow{P_2} G_1(1-s) \oplus F_1' \xrightarrow{P_1} I_V$$

where

$$P_1 = \begin{pmatrix} fN_1 & M_1' \end{pmatrix}, \quad P_2 = \begin{pmatrix} N_2 & X_2 \\ 0 & M_2' \end{pmatrix},$$

X_2 is a matrix such that $fN_1X_2 = -M_1'M_2'$ and, in general, for $i > 2$,

$$P_i = \begin{pmatrix} N_i & X_i \\ 0 & M_i \end{pmatrix}$$

with X_i satisfying the equality $N_{i-1}X_i = -X_{i-1}M_i$.

3. Minimal generators and last syzygies for partial intersections

In this section we compute a minimal set of generators of a partial intersection scheme of codimension c of $\mathbb{P}^r$.

Let $V \subset \mathbb{P}^r$ be a c-partial intersection with respect to the hyperplanes $\{A_{ij}\}$, $1 \leq i \leq c$ and $1 \leq j \leq a_i$, and support on the c-left segment $\mathcal{A}$ of size $T = (a_1, \ldots, a_c)$.

Recall that $\mathcal{A}_j = \hat{\alpha}_c(\alpha_c^{-1}(j))$ is the set of $(c-1)$-tuples H such that $(H, j) \in \mathcal{A}$. Denote by V_j the $(c-1)$-partial intersection with support $\mathcal{A}_j$ and relative hyperplanes A_{ij}, for $1 \leq i \leq c-1$. As we saw $V = \bigcup_{i=1}^{c} (V_j \cap A_{cj})$. Now, set $I_{A_{ij}} = (f_{ij})$, where $f_{ij} \in R_1$, for all i, j.

Finally, to every $H = (m_1, \ldots, m_c) \in \overline{\mathcal{S}}_T$ we associate the following form

$$P_H = \prod_{i=1}^{c} \prod_{j=1}^{m_i - 1} f_{ij}.$$

Theorem 3.1. *Let $V \subset \mathbb{P}^r$ be a partial intersection of codimension c with support $\mathcal{A}$. Then a minimal set of generators for I_V is*

$$\{P_H \mid H \in F(\mathcal{A})\}.$$

Proof. We use induction on c. For $c = 1$, $\mathcal{A} = \{1, 2, \ldots, a_1\}$, $F(\mathcal{A}) = \{a_1 + 1\}$, the hyperplanes are A_{1j}, $j = 1, \ldots, a_1$, $V = \bigcup_{j=1}^{a_1} A_{1j}$, $P_{a_1+1} = \prod_{j=1}^{a_1} f_{1j}$, hence $I_V = (P_{a_1+1})$. Let us suppose the theorem true for partial intersections of codimension $c-1$. Using Lemma 1.6 we have

$$I_V = I_{V_1} + f_{c1} I_{V_2} + \cdots + f_{c1} \ldots f_{c,a_c-1} I_{V_{a_c}} + (f_{c1} \ldots f_{ca_c}).$$

By the inductive hypothesis I_{V_j} is minimally generated by $G_j = \{P_H \mid H \in F(\mathcal{A}_j)\}$, $1 \leq j \leq a_c$. Set

$$G = G_1 \cup f_{c1} G_2 \cup \ldots \cup \prod_{j=1}^{a_c - 1} f_{cj} G_{a_c} \cup \left\{ \prod_{j=1}^{a_c} f_{cj} \right\}.$$

Now, if we set $F = \{P_H \mid H \in F(\mathcal{A})\}$, we prove that $(G) = (F)$.

Indeed, consider first $\prod_{j=1}^{a_c} f_{cj} \in G$; since $H = (1, \ldots, 1, a_c + 1) \in F(\mathcal{A})$ and, trivially, $P_H = \prod_{j=1}^{a_c} f_{cj}$, we have $\prod_{j=1}^{a_c} f_{cj} \in F$. Take now $1 \leq j \leq a_c$, an element $H \in F(\mathcal{A}_j)$ and the form $g = P_H \prod_{h=1}^{j-1} f_{ch}$. We first show that there is an integer t, $1 \leq t \leq j$, such that $(H, t) \in F(\mathcal{A})$. For this, define $D = \{n \in \mathbb{N} \mid (H, n) \in \mathcal{A}\}$. We want to show that

$$t = \begin{cases} \max D + 1 & \text{if } D \neq \emptyset \\ 1 & \text{if } D = \emptyset \end{cases}$$

is the required integer, i.e. $(H, t) \in F(\mathcal{A})$. Note that, $t < j$ as $(H, j) \notin \mathcal{A}$. By definition of t, clearly $(H, t) \notin \mathcal{A}$, so it remains to prove that $\mathcal{S}_{(H,t)} \subseteq \mathcal{A}$. Take $K \in \mathcal{S}_{(H,t)}$, write $K = (K', u)$, with $K' \in \mathbb{N}^{c-1}$; since $K < (H, t)$ we have either $K' < H$ and $u \leq t$ or $K' = H$ and $u < t$. In the first case, $K' \in \mathcal{S}_H$ and since $H \in F(\mathcal{A}_j)$ we have $K' \in \mathcal{A}_j$, therefore $(K', j) \in \mathcal{A}$; on the other hand $u \leq t < j$ from which we get $K = (K', u) \in \mathcal{A}$. In the second case, by definition of t, $K = (K', u) \in \mathcal{A}$. Consider the form $P_{(H,t)} = P_H \prod_{h=1}^{t-1} f_{ch}$; then $g = P_{(H,t)} \prod_{h=t}^{j-1} f_{ch}$, therefore $g \in (F)$. Conversely, consider an element P_H with $H \in F(\mathcal{A})$; write $H = (H', u)$ with $H' \in \mathbb{N}^{c-1}$. We want to show that $H' \in F(\mathcal{A}_u)$. Namely, $H' \notin \mathcal{A}_u$ since $(H', u) \notin \mathcal{A}$; take now $K < H'$, this implies $(K, u) < (H', u)$, i.e. $(K, u) \in \mathcal{S}_H \subseteq \mathcal{A}$, therefore $K \in \mathcal{A}_u$. Then, $P_H = P_{H'} \prod\limits_{h=1}^{u-1} f_{ch} \in G$.

It remains to prove that the set F of generators of I_V is minimal. To do this, let $H = (m_1, m_2, \dots, m_c) \in F(\mathcal{A})$ and consider the linear variety L whose defining ideal is $I_L = (f_{1m_1}, f_{2m_2}, \dots, f_{cm_c})$, where we set $f_{ia_i+1} = 0$, for $1 \leq i \leq c$. Observe that if $P_H \in I_L$ some linear factor of P_H should stay in I_L but this cannot happen by the genericity of the hyperplanes A_{ij}. On the other hand, for every $K = (n_1, \dots, n_c) \in F(\mathcal{A})$ with $K \neq H$, P_K belongs to I_L : in fact, since $K \not\leq H$ there is an integer t such that $n_t > m_t$ which implies that $m_t \leq a_t$, therefore $f_{tm_t} \neq 0$ and it divides P_K, i.e. $P_K \in I_L$. Since this was proved for every $H \in F(\mathcal{A})$ we get that the set F of generators of I_V is minimal. □

The previous theorem, in particular, shows that the first graded Betti numbers of partial intersections are determined by its support.

Corollary 3.2. *Let V be as above then its first graded Betti numbers depend only on $\mathcal{A}$ and they are the following integers*

$$d_H = v(H) - c \quad \forall\ H \in F(\mathcal{A}).$$

Now we compute the last graded Betti numbers of a c-codimensional partial intersection in terms of its support.

Let V be a c-codimensional partial intersection with support $\mathcal{A}$ of size $T = (a_1, \dots, a_c)$ and respect to the families of hyperplanes $\{A_{ij}\}$ whose defining forms are f_{ij}. Consider the complete intersection Z where

$$I_Z = \left(\prod_{j=1}^{a_1} f_{1j}, \dots, \prod_{j=1}^{a_c} f_{cj} \right).$$

Let V^* be the scheme linked to V in the complete intersection Z.

Proposition 3.3. *V^* is a partial intersection of codimension c with support $\mathcal{A}^*$.*

Proof. Observe first that $V^* = \bigcup_{H \in \overline{S}_T \setminus \mathcal{A}} L_H$. Now consider the following families of hyperplanes $B_{ij} = A_{i,a_i+1-j}$. With respect to these families we see that $V^* = \bigcup_{H \in C_T(\overline{S}_T \setminus \mathcal{A})} L'_H = \bigcup_{H \in \mathcal{A}^*} L'_H$. □

Theorem 3.4. *Let $V \subset \mathbb{P}^r$ be a partial intersection of codimension c with support $\mathcal{A}$. Then the last graded Betti numbers of V are*

$$s_H = v(H) \quad \forall\ H \in G(\mathcal{A}).$$

Proof. By liaison theory we know that the degrees of a set of generators for the last syzygies of I_V can be computed in terms of the degrees of a set of generators for I_{V^*}, precisely, $s_H = v(T) - v(H) + c$, for all $H \in F(\mathcal{A}^*)$. By item 2. of Proposition 1.3 we have $F(\mathcal{A}^*) = C_T(G(\mathcal{A})) \cup \{T_1^*, \dots, T_c^*\}$. On the other hand, one of these syzygies is not minimal if and only if it comes from a generator of I_{V^*} which is also a generator for I_Z. But I_{V^*} and I_Z have a common generator only for those $i \in \{1, \dots, c\}$ such that $T_i = T_i^*$. Then, using item 3. of the mentioned proposition we have that the minimal syzygies are exactly those coming from $C_T(G(\mathcal{A}))$. Therefore, the degrees of a minimal set of generators for the last syzygies are $v(T) - v(K) + c$, for $K \in C_T(G(\mathcal{A}))$; now, write $K = T + I - H$ with $H \in G(\mathcal{A})$ and we obtain that

$$s_H = v(T) - v(K) + c = v(T) - (v(T) + v(I) - v(H)) + c = v(H). \qquad \square$$

From the previous results it follows easily

Corollary 3.5. *For a c-partial intersection X with support on a c-left segment $\mathcal{A}$ the following are equivalent*

1. *X is a complete intersection;*
2. *X is arithmetically Gorenstein;*
3. *$\mathcal{A}$ is principal.*

Remark 3.6. All the graded Betti numbers for a 3-codimensional partial intersection V can be computed easily in terms of $\mathcal{A}$. Precisely, from the previous results we know the generators' and last syzygies' degrees and the Hilbert function; therefore, using the relationship $\alpha_{2j} = \Delta^{r+1} H_V(j) + \alpha_{1j} + \alpha_{3j}$ we can compute the second syzygies' degrees.

Example 3.7. Since we are now able to compute all the graded Betti numbers for any partial intersection $V \subset \mathbb{P}^r$ of codimension 3 we will apply the results to the Example 2.2. The support $\mathcal{A}$ of V was minimally generated by

$$G(\mathcal{A}) = \{(1,2,3), (2,3,2), (3,4,1), (4,1,3), \\ (4,2,2), (4,3,1), (5,1,2), (5,2,1)\};$$

thus, one can compute first $\mathcal{A}^*$ getting

$$G(\mathcal{A}^*) = \{(1,4,1), (4,3,1), (5,2,1), (1,3,2), \\ (3,2,2), (5,1,2), (1,2,3), (2,1,3)\};$$

then

$$F(\mathcal{A}) = \{(5,1,3), (2,2,3), (1,3,3), (5,2,2), (3,3,2), \\ (1,4,2), (5,3,1), (4,4,1), (6,1,1), (1,5,1), (1,1,4)\}.$$

Applying the previous results a minimal free resolution of I_V looks like

$$0 \longrightarrow R(-6) \oplus R(-7) \oplus R(-8)^6 \longrightarrow R(-5)^5 \oplus R(-6)^2 \oplus R(-7)^{11} \longrightarrow \\ \longrightarrow R(-3) \oplus R(-4)^4 \oplus R(-5)^2 \oplus R(-6)^4 \longrightarrow I_V \longrightarrow 0.$$

We conclude this section emphasizing that these partial intersections can be used to provide schemes with fixed generators' degrees or syzygies' degrees (for instance level schemes or schemes with an assigned Cohen Macaulay type).

4. Linear decompositions of Hilbert functions

The goal of this section is to construct partial intersections with a given Hilbert function H. For this we will give the notion of linear decomposition of an O-sequence and we will explain the connection with the c-left segments.

Decompositions of Hilbert functions were used by Geramita, Harima and Shin. Nevertheless our decompositions will generalize those in [8] and will have unimportant overlap with those in [10].

If $\varphi\colon \mathbb{N}_0 \to \mathbb{N}$ is an O-sequence we call the integer $\varphi(1)$ the *embedding dimension* of φ and the set of integers n such that $\varphi(n) \neq 0$ the *support* of φ; furthermore if φ and ψ are O-sequences such that $\varphi(n) \geq \psi(n)$ for every $n \in \mathbb{N}_0$, we will write simply $\varphi \geq \psi$.

Definition 4.1. *Let φ be an O-sequence with finite support, $\varphi(1) = c \geq 2$. A linear decomposition of φ is a succession of O-sequences of embedding dimension $< c$, $(\varphi_1, \varphi_2, \dots, \varphi_d)$ such that*

1) $\varphi_1 \geq \varphi_2 \geq \dots \geq \varphi_d$;

2) $\varphi(n) = \sum_{j=1}^{d} \varphi_j(n+1-j)$

for every $n \in \mathbb{N}_0$ (we use the convention $\varphi(n) = 0$ for all $n < 0$).

A linear decomposition of φ, $(\varphi_1, \varphi_2, \dots, \varphi_d)$ is said principal if $d = \min\left\{n \in \mathbb{N} \mid \varphi(n) < \binom{c+n-1}{n}\right\}$.

All the O-sequences φ, $\varphi(1) = c \geq 2$, with finite support, have principal linear decompositions. For instance now we build, recursively, the *maximal decomposition* $(\varphi_1, \varphi_2, \dots, \varphi_d)$ of φ (the proofs can be deduced by [9], Theorem 2.3), where

$$d = \min\left\{n \in \mathbb{N} \;\middle|\; \varphi(n) < \binom{c+n-1}{n}\right\}.$$

Note that every linear decomposition $(\psi_1, \dots, \psi_e)$ of the O-sequence φ has $e \geq d$.

If $\varphi_1' = \varphi$, φ_1 is defined by

$$\varphi_1(0) = 1;\ \varphi_1(1) = c - 1;\ \varphi_1(n) = \min\left\{\varphi_1(n-1)^{<n-1>}, \varphi_1'(n)\right\}.$$

Let us suppose that we built the O-sequences φ_{j-1}' and φ_{j-1}, then we define for $2 \leq j \leq d$

$$\varphi_j'(n) = \varphi_{j-1}'(n+1) - \varphi_{j-1}(n+1),$$

and for $2 \leq j \leq d-1$

$$\varphi_j(0) = 1;\ \varphi_j(1) = c - 1;\ \varphi_j(n) = \min\left\{\varphi_j(n-1)^{<n-1>}, \varphi_j'(n)\right\};$$

note that

$$\min\left\{n \in \mathbb{N} \;\middle|\; \varphi_j'(n) < \binom{c+n-1}{n}\right\} = d - j + 1;$$

finally we set $\varphi_d = \varphi_d'$.

The linear decomposition of φ that we obtain in this way is maximal in the sense that every linear decomposition $(\psi_1, \dots, \psi_e)$ must satisfy $\psi_i \leq \varphi_i$ for $i = 1, \dots, d$.

Remark 4.2. Note that if φ and ψ are O-sequences, with finite support, $\varphi(1) = \psi(1)$, such that $\varphi \geq \psi$ and $(\varphi_1, \varphi_2, \dots, \varphi_d)$, $(\psi_1, \psi_2, \dots, \psi_e)$ are their maximal linear decompositions, then $d \geq e$ and $\varphi_j \geq \psi_j$ for $1 \leq j \leq e$.

If $\mathcal{A}$ is a c-left segment, the function $\varphi\colon \mathbb{N}_0 \to \mathbb{N}$, defined by

$$\varphi(n) = \Big| \{H \in \mathcal{A} \mid v(H) = n + c\} \Big|$$

is an O-sequence, with finite support, $\varphi(1) \leq c$: in fact if $V \subset \mathbb{P}^r$, $r \geq c$, is a partial intersection with support on $\mathcal{A}$, as we have seen in Theorem 2.1, $\varphi = \Delta^{r-c+1} H_V$. We will call φ the *O-sequence associated* to the c-left segment $\mathcal{A}$.

If $\mathcal{A}$ is a degenerate c-left segment and φ is its associated O-sequence, then $\varphi(1) = c' < c$. If $\mathcal{A}$ is degenerate, let $J \subseteq \{1, 2, \dots, c\}$, $|J| = c - c'$, such that $\alpha_j(H) = 1$ for every $H \in \mathcal{A}$ and $j \in J$. Denote $\hat{\alpha}_J\colon \mathbb{N}^c \to \mathbb{N}^{c'}$ the map which kills the indices belonging to J. Then $\mathcal{A}' = \hat{\alpha}_J(\mathcal{A})$ is a not degenerate c'-left segment such that its associated O-sequence is again φ; consequently, we can limit ourselves to consider only not degenerate c-left segments.

Every not degenerate c-left segment $\mathcal{A}$ provides linear decompositions of its associated O-sequence φ. Now we will construct a particular linear decomposition of φ, $\sigma(\mathcal{A})$, depending on $\mathcal{A}$.

If $d = \max\{\alpha_c(H) \mid H \in \mathcal{A}\}$, $\mathcal{A}_j = \hat{\alpha}_c(\alpha_c^{-1}(j))$, for $1 \leq j \leq d$ and φ_j is the O-sequence associated to $\mathcal{A}_j$, then $\sigma(\mathcal{A}) = (\varphi_1, \varphi_2, \dots, \varphi_d)$ is a linear decomposition of φ. In fact, since $\mathcal{A}$ is not degenerate we have that $\varphi_j(1) < \varphi(1)$ for $1 \leq j \leq d$; since $\mathcal{A}_j \supseteq \mathcal{A}_{j+1}$, $\varphi_j \geq \varphi_{j+1}$ for $1 \leq j \leq d-1$; finally $\varphi(n) = \sum\limits_{j=1}^{d} \varphi_j(n + 1 - j)$ for every $n \in \mathbb{N}_0$ as it was shown in the proof of Theorem 2.1.

Now we prove the following

Proposition 4.3. *If φ is an O-sequence with finite support, $\varphi(1) = c$, then there exists a not degenerate c-left segment $\mathcal{A}^\varphi$ whose associated O-sequence is just φ. Moreover if φ and ψ are O-sequences, with finite support, such that $\varphi \geq \psi$ then $\mathcal{A}^\varphi \supseteq \mathcal{A}^\psi$.*

Proof. We work by induction on c.

If $c = 1$, $\varphi(1) = 1$, so there is an integer u such that $\varphi(n) = 1$ for $0 \leq n \leq u - 1$, and $\varphi(n) = 0$ for $n \geq u$. Then $\mathcal{A}^\varphi = \{1, 2, \dots, u\}$ is an 1-left segment whose associated O-sequence is φ. Moreover if φ and ψ are O-sequences, $\varphi(1) = \psi(1) = 1$, with finite support, such that $\varphi \geq \psi$ then it is trivial that $\mathcal{A}^\varphi \supseteq \mathcal{A}^\psi$.

Suppose the proposition is true for O-sequences with finite support whose embedding dimension is less than c, and let φ be an O-sequence, with finite support, such that $\varphi(1) = c$. Let $(\varphi_1, \varphi_2, \dots, \varphi_d)$ be its maximal decomposition. Recall that $\varphi_j(1) \leq c - 1$ for $1 \leq j \leq d$. Let $\mathcal{A}_j = \mathcal{A}^{\varphi_j}$ for $1 \leq j \leq d - 1$; set $c_d = \varphi_d(1)$ and $\overline{\beta}_d\colon \mathbb{N}^{c_d} \to \mathbb{N}^{c-1}$ the map defined by $\overline{\beta}_j(H) = (H, 1, \dots, 1)$ let $\mathcal{A}_d = \overline{\beta}_d(\mathcal{A}^{\varphi_d})$. Since $\varphi_1 \geq \varphi_2 \geq \dots \geq \varphi_d$, by the inductive hypothesis we have that $\mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \dots \supseteq \mathcal{A}_d$.

If j is an integer let $\overline{\alpha}_j\colon \mathbb{N}^{c-1} \to \mathbb{N}^c$ be the map defined by $\overline{\alpha}_j(H) = (H, j)$, for every $H \in \mathbb{N}^{c-1}$. Now we set $\mathcal{A} = \mathcal{A}^\varphi = \bigcup\limits_{j=1}^{d} \overline{\alpha}_j(\mathcal{A}_j)$.

$\mathcal{A}$ is a c-left segment. Namely, if $H \in \mathcal{A}$ and $K \in S_H$, we can write $H = (H', t)$, $K = (K', u)$ where $H', K' \in \mathbb{N}^{c-1}$, $K' \leq H'$ and $u \leq t$. But $H \in \mathcal{A}$, so $H' \in \mathcal{A}_t$; since $\mathcal{A}_t$ is a $(c-1)$-left segment $K' \in \mathcal{A}_t \subseteq \mathcal{A}_u$, i.e. $K = (K', u) \in \overline{\alpha}_u(\mathcal{A}_u) \subseteq \mathcal{A}$.

Now we prove that the O-sequence associated to $\mathcal{A}$ is just φ. In fact

$$\varphi(n) = \sum_{j=1}^{d} \varphi_j(n+1-j) = \sum_{j=1}^{d} \left| \{H \in \mathcal{A}_j \,|\, v(H) = n+1-j+c-1\} \right|$$

$$= \sum_{j=1}^{d} \left| \{H \in \overline{\alpha}_j(\mathcal{A}_j) \,|\, v(H) = n+c\} \right| = \left| \{H \in \mathcal{A} \,|\, v(H) = n+c\} \right|,$$

since $\overline{\alpha}_j(\mathcal{A}_j) \cap \overline{\alpha}_h(\mathcal{A}_h) = \emptyset$, for every $j \neq h$.

Finally we prove that if $\varphi \geq \psi$ are O-sequences with finite support, $\varphi(1) = \psi(1) = c$, then $\mathcal{A}^{\varphi} \supseteq \mathcal{A}^{\psi}$. Let $(\varphi_1, \varphi_2, \ldots, \varphi_d)$, $(\psi_1, \psi_2, \ldots, \psi_e)$ be the maximal decompositions of φ and ψ respectively; we know that $d \geq e$ and $\varphi_j \geq \psi_j$ for $1 \leq j \leq e$. Given $H \in \mathcal{A}^{\psi}$, there is an index t, $1 \leq t \leq e$, such that $H \in \overline{\alpha}_t((\mathcal{A}^{\psi})_t)$, so we can write $H = (H', t)$, $H' \in (\mathcal{A}^{\psi})_t$. Since $\varphi_t \geq \psi_t$ and $\varphi_t(1) = \psi_t(1) \leq c-1$ we have that $(\mathcal{A}^{\psi})_t \subseteq (\mathcal{A}^{\varphi})_t$, which implies $H' \in (\mathcal{A}^{\varphi})_t$, i.e. $H \in \mathcal{A}^{\varphi}$. □

Our next result will permit to construct for every Hilbert function H (admissible for aCM schemes) many partial intersections $V \subset \mathbb{P}^r$ (with different supports and different graded Betti numbers) with $H_V = H$.

Theorem 4.4. *If φ is an O-sequence with finite support, $\varphi(1) = c \geq 2$, and σ is a linear decomposition of φ, then there exists a not degenerate c-left segment $\mathcal{A}$ such that $\sigma(\mathcal{A}) = \sigma$. In particular, the O-sequence associated to $\mathcal{A}$ is φ.*

Proof. Let $\sigma = (\varphi_1, \varphi_2, \ldots, \varphi_d)$. It is enough to prove that for $1 \leq j \leq d$ there exists a $(c-1)$-left segment $\mathcal{A}_j$ whose associated O-sequence is φ_j and such that $\mathcal{A}_j \supseteq \mathcal{A}_{j+1}$ for $1 \leq j \leq d-1$. If this happens we can set $\mathcal{A} = \bigcup_{j=1}^{d} \overline{\alpha}_j(\mathcal{A}_j)$. Now we describe a way to build such $\mathcal{A}_j$'s.

Let $c_j = \varphi_j(1)$ and $\overline{\beta}_j \colon \mathbb{N}^{c_j} \to \mathbb{N}^{c-1}$ be the map defined by $\overline{\beta}_j(H) = (H, 1, \ldots, 1)$ and call $\mathcal{A}_j = \overline{\beta}_j(\mathcal{A}^{\varphi_j})$. Since $c_1 \geq c_2 \geq \ldots \geq c_d$, these $\mathcal{A}'_j s$ have the requested property. □

Remark 4.5. Partial intersections generalize the notion of k-configuration in [9], where the authors use n-type vectors to show that every k-configuration has the extremal resolution (relative to the Hilbert function) described by Bigatti, Hulett and Pardue ([2], [12], [15]). Partial intersections allow us to build not only extremal resolutions but a great deal of other resolutions which agree with a fixed Hilbert function H.

At this point the following question arises in a natural way: let W be a c-codimensional aCM scheme; does there exist a partial intersection V with the same graded Betti numbers of W ?

For $c = 1, 2$ the answer is positive (see [14] for $c = 2$). For $c \geq 3$ the answer is negative. Indeed, let W be an arithmetically Gorenstein scheme. If $\mathcal{A}$ is the left segment support of a partial intersection V with same graded Betti numbers of W, then, using Theorem 3.4, we have $\mathcal{A} = \overline{S}_T$ for some c-tuple T, so V is a complete intersection. Since for every $c \geq 3$ there are arithmetically Gorenstein schemes which are not complete intersections we are done.

Example 4.6. Consider the following O-sequence

$$\varphi = (1, 3, 5, 2, 0 \rightarrow).$$

For every $r \geq 3$ there are aCM schemes $X \subset \mathbb{P}^r$ such that $\Delta^{r-2} H_X = \varphi$. Looking at $\Delta^3 \varphi = (1, 0, -1, -5, 6, 1, -2, 0 \rightarrow)$, since $-\alpha_{1j} + \alpha_{2j} - \alpha_{3j} = \Delta^3 \varphi(j)$, one sees that any such a scheme should have one generator in degree 2, 5 generators in degree 3, no generator in degree > 4, and by the mentioned result of Bigatti, Hulett, Pardue, at most 2 generators in degree 4.

The O-sequence φ has three principal linear decompositions:

$$\begin{aligned}
\sigma_1 &= ((1, 2, 3, 2, 0 \rightarrow), (1, 2, 0 \rightarrow)) \\
\sigma_2 &= ((1, 2, 3, 1, 0 \rightarrow), (1, 2, 1, 0 \rightarrow)) \\
\sigma_3 &= ((1, 2, 3, 0 \rightarrow), (1, 2, 2, 0 \rightarrow))
\end{aligned}$$

σ_1 is the maximal decomposition. The 3-left segment

$$< (1, 4, 1), (2, 3, 1), (3, 1, 1), (1, 2, 2), (2, 1, 2) >$$

associated to σ_1 provides the extremal resolution:

$$0 \longrightarrow R(-5)^3 \oplus R(-6)^2 \longrightarrow R(-4)^8 \oplus R(-5)^4 \longrightarrow$$
$$\longrightarrow R(-2) \oplus R(-3)^5 \oplus R(-4)^2 \longrightarrow I_X \longrightarrow 0.$$

From this resolution we have that any scheme X with $\Delta^{r-2} H_X = \varphi$ can have 2 or 1 or no generator in degree 4. The 3-left segment

$$< (2, 3, 1), (3, 2, 1), (1, 2, 2), (2, 1, 2) >$$

again associated to σ_1, produces the graded minimal free resolution:

$$0 \longrightarrow R(-5)^2 \oplus R(-6)^2 \longrightarrow R(-4)^7 \oplus R(-5)^3 \longrightarrow$$
$$\longrightarrow R(-2) \oplus R(-3)^5 \oplus R(-4) \longrightarrow I_X \longrightarrow 0.$$

Finally, the 3-left segment

$$< (3, 1, 1), (1, 3, 2), (2, 2, 2) >$$

associated to σ_3 provides the graded minimal free resolution:

$$0 \longrightarrow R(-5) \oplus R(-6)^2 \longrightarrow R(-4)^6 \oplus R(-5)^2 \longrightarrow$$
$$\longrightarrow R(-2) \oplus R(-3)^5 \longrightarrow I_X \longrightarrow 0.$$

In conclusion all the possible degrees for a minimal set of generators of a 3-codimensional aCM scheme $X \subset \mathbb{P}^r$ such that $\Delta^{r-2} H_X = \varphi$ can be reached using partial intersections.

We do not know if the same is possible for every 3-codimensional O-sequence.

The referee suggested us the following counterexample to the above question in codimension 4.

Take the O-sequence $\varphi = (1, 4, 5, 0 \rightarrow)$. The generic set X of 10 points in $\mathbb{P}^4$ has the defining ideal I_X generated by 5 quadratic forms. Nevertheless, there is no 4-partial intersection Y consisting of 10 points with $\Delta H_Y = \varphi$ and with I_Y generated by just 5 forms in degree 2. In fact, if we denote f_{ij}, $i = 1, 2, 3, 4$, the forms defining the 4 families of hyperplanes which give such an Y with $\Delta H_Y = \varphi$, in I_Y there are 5 forms of degree 2. But the only possible quadratic forms in I_Y can be

$$f_{11}f_{12}, f_{11}f_{21}, f_{11}f_{31}, f_{11}f_{41}, f_{21}f_{22}, f_{21}f_{31}, f_{21}f_{41}, f_{31}f_{32}, f_{31}f_{41}, f_{41}f_{42}.$$

Now, if we take any 5 of the above forms we see that at least 2 of them should have a common factor. This implies that I_Y has at least a first syzygy of degree 3. Now, from $\Delta^4\varphi = (1, 0, -5, 0, 15, -16, 5, 0 \rightarrow)$ using again the relation $-\alpha_{1j} + \alpha_{2j} - \alpha_{3j} + \alpha_{4j} = \Delta^4\varphi(j)$, since $\Delta^4\varphi(3) = -\alpha_{13} + \alpha_{23} = 0$, we get that I_Y needs at least a generator of degree 3.

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