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Hopf Mappings for Complex Quaternions

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Abstract. The natural mapping of the right quaternion vector space $\mathbb{H}^2$ onto the quaternion projective line (identified with the four-sphere) can be defined for complex quaternions $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ as well. We discuss its exceptional set, the fiber subspaces, and how the linear automorphism groups of two-dimensional quaternion vector spaces and modules induce groups of projective automorphisms of the image quadrics.

1. Notation

Consider the skew field $\mathbb{H}$ of quaternions. We will use the usual notation $a = a_0 + ia_1 + ja_2 + ka_3$ with real numbers a_i and the 'quaternion units' i, j, k. The symbol $\overline{a}$ denotes the quaternion $a_0 - ia_1 - ja_2 - ka_3$ conjugate to a, and the quaternion norm is denoted by $N(a)$. It has the properties that $N(a) = a\overline{a} = \overline{a}a = a_0^2 + a_1^2 + a_2^2 + a_3^2$, and that $N(ab) = N(a)N(b)$. The group of unit quaternions will be identified with the three-sphere $S^3 = \{a \in \mathbb{H} \mid N(a) = 1\}$.

If V is a vector space over the field $\mathbb{K}$, we write $\mathrm{rk}_{\mathbb{K}}$, $\dim_{\mathbb{K}}$, etc., to indicate the field, if ambiguity is possible. Likewise we write $P_{\mathbb{K}}(V)$ for the projective space defined by V. Nonetheless we use 'standard' notation to indicate 'standard' objects: The real projective space $P_{\mathbb{R}}(\mathbb{H}^2)$ is of dimension seven and will be denoted simply by P^7. The subset $\{(q_0, q_1) \mid N(q_0) + N(q_1) = 1\}$ of $\mathbb{H}^2$ is called its unit seven-sphere and is denoted by S^7. P^7 arises from S^7 by identification of antipodal points.

The projective line $P_{\mathbb{H}}(\mathbb{H}^2)$ is the set of one-dimensional right quaternion linear subspaces of $\mathbb{H}^2$, and is denoted by $P^1(\mathbb{H})$, i.e.,

$$P^1(\mathbb{H}) = \{(q_0, q_1)\mathbb{H} \mid (q_0, q_1) \neq (0, 0)\}. \tag{1}$$

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2. The Hopf mapping $P^7 \to S^4$

The topology of $P^1(\mathbb{H})$ as induced by the natural mapping $\mathbb{H}^2 \to P^1(\mathbb{H})$ is homeomorphic to S^4. This homeomorphism is explicitely realized by the stereographic projection: The real vector space $\mathbb{H} \oplus \mathbb{R}$ contains the unit sphere $S^4 = \{\left[\begin{smallmatrix} y \\ \eta \end{smallmatrix}\right] \in \mathbb{H} \oplus \mathbb{R} \mid N(y) + \eta^2 = 1\}$.

The stereographic projection $\sigma : P^1(\mathbb{H}) \to S^4$ is defined by

$$\begin{aligned} \sigma(\begin{bmatrix} 1 \\ y \end{bmatrix}\mathbb{H}) &= \frac{1}{N(y)+1}\begin{bmatrix} 2y \\ N(y)-1 \end{bmatrix}, \\ \sigma(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\mathbb{H}) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned} \tag{2}$$

and

$$\sigma^{-1}\begin{bmatrix} y \\ \eta \end{bmatrix} = \begin{bmatrix} 1-\eta \\ y \end{bmatrix}\mathbb{H}. \tag{3}$$

The *Hopf mapping* is usually defined as the mapping from S^7 onto $P^1(\mathbb{H})$, which maps $\left[\begin{smallmatrix} q_0 \\ q_1 \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} q_0 \\ q_1 \end{smallmatrix}\right]\mathbb{H}$. We will use the fact that both $\left[\begin{smallmatrix} q_0 \\ q_1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} -q_0 \\ -q_1 \end{smallmatrix}\right]$ are mapped to the same projective point, and use the mapping $\left[\begin{smallmatrix} q_0 \\ q_1 \end{smallmatrix}\right]\mathbb{R} \mapsto \left[\begin{smallmatrix} q_0 \\ q_1 \end{smallmatrix}\right]\mathbb{H}$. In order to make the image a sphere embedded into a real vector space, we define the Hopf mapping φ by

$$\varphi : P^7 \to S^4, \quad \varphi(\begin{bmatrix} q_0 \\ q_1 \end{bmatrix}\mathbb{R}) = \sigma^{-1}(\begin{bmatrix} q_0 \\ q_1 \end{bmatrix}\mathbb{H}). \tag{4}$$

The φ-preimage of $\left[\begin{smallmatrix} y \\ \eta \end{smallmatrix}\right] \in S^4$ equals

$$\varphi^{-1}\begin{bmatrix} y \\ \eta \end{bmatrix} = \begin{bmatrix} 1-\eta \\ y \end{bmatrix}\mathbb{H}. \tag{5}$$

The set $\left[\begin{smallmatrix} 1-\eta \\ y \end{smallmatrix}\right]\mathbb{H}$ has an interpretation as a projective point of $P^1(\mathbb{H})$, a one-dimensional right quaternion linear subspace of $\mathbb{H}^2$, as a four-dimensional real linear subspace of $\mathbb{H}^2$, or a three-dimensional subspace of P^7.

Remark. We let $H = \{1, -1\} \subset \mathbb{H}$ and consider the group $G = S^3/H$, which is isomorphic to SO_3. The group G acts on the fibers of φ in the following way: If $g = a \cdot H = \pm a$, then $g(\left[\begin{smallmatrix} p_0 \\ q_1 \end{smallmatrix}\right]\mathbb{R}) = \pm\left[\begin{smallmatrix} p_0 a \\ q_1 a \end{smallmatrix}\right]\mathbb{R} = \left[\begin{smallmatrix} p_0 a \\ q_1 a \end{smallmatrix}\right]\mathbb{R}$. This makes $\varphi : P^7 \to S^4$ a principal SO_3-bundle over the base space S^4 (cf. [3], p. 105).

We embed the affine space $\mathbb{H} \oplus \mathbb{R}$ into the real projective space $P^5 = P_{\mathbb{R}}(\mathbb{R} \oplus \mathbb{H} \oplus \mathbb{R})$. The point $\left[\begin{smallmatrix} y \\ \eta \end{smallmatrix}\right] \in \mathbb{H} \oplus \mathbb{R}$ is identified with the projective point $\begin{bmatrix} 1 \\ y \\ \eta \end{bmatrix}\mathbb{R}$. Consider the mapping

$$\psi : \mathbb{H}^2 \to \mathbb{R} \oplus \mathbb{H} \oplus \mathbb{R}, \quad \psi\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} N(a)+N(b) \\ 2b\overline{a} \\ N(b)-N(a) \end{bmatrix}. \tag{6}$$

Because

$$\frac{1}{N(q_1 q_0^{-1})}\begin{bmatrix} 2q_1 q_0^{-1} \\ N(q_1 q_0^{-1}) - 1 \end{bmatrix} = \frac{1}{N(q_1) + N(q_0)}\begin{bmatrix} 2q_1 \overline{q}_0 \\ N(q_1) - N(q_0) \end{bmatrix}, \tag{7}$$

we have the equivalence

$$\psi\begin{bmatrix} a \\ b \end{bmatrix}\mathbb{R} = \begin{bmatrix} y_0 \\ y \\ y_5 \end{bmatrix}\mathbb{R} \iff \varphi\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{y_0}\begin{bmatrix} y \\ y_5 \end{bmatrix}. \tag{8}$$

If we use coordinates with respect to the bases $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} i \\ 0 \end{bmatrix}$, $\begin{bmatrix} j \\ 0 \end{bmatrix}$, $\begin{bmatrix} k \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ i \end{bmatrix}$, $\begin{bmatrix} 0 \\ j \end{bmatrix}$, $\begin{bmatrix} 0 \\ k \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ i \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ j \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ k \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, ψ reads

$$\psi\begin{bmatrix} x_0 \\ \vdots \\ x_8 \end{bmatrix} = \begin{bmatrix} x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 \\ 2(+x_0x_4 + x_1x_5 + x_2x_6 + x_3x_7) \\ 2(-x_0x_5 + x_1x_4 - x_2x_7 + x_3x_6) \\ 2(-x_0x_6 + x_1x_7 + x_2x_4 - x_3x_5) \\ 2(+x_0x_7 + x_1x_6 - x_2x_5 - x_3x_4) \\ x_4^2 + x_5^2 + x_6^2 + x_7^2 - x_0^2 - x_1^2 - x_2^2 - x_3^2 \end{bmatrix}. \tag{9}$$

3. The projective automorphism group of S^4

The group S^3 acts as a subgroup of SO_8 via right translations $R_a\begin{bmatrix} q_0 \\ q_1 \end{bmatrix} = \begin{bmatrix} q_0 a \\ q_1 a \end{bmatrix}$, and this action leaves the fibers of φ invariant. This right multiplication however is not right $\mathbb{H}$-linear, but semilinear with respect to the inner automorphism $x \mapsto a^{-1}xa$ of $\mathbb{H}$.

We ask if the action of SO_5 on S^4 is induced by a subgroup of SO_8, or by a subgroup of $\mathrm{GL}(2,\mathbb{H})$. It will turn out that the answer to an even more general question is affirmative. We use the notation $\mathrm{PGL}(S^4) \le \mathrm{PGL}_5$ for the projective automorphisms of the unit sphere when embedded into P^5. A projective automorphism $x\mathbb{R} \mapsto (A \cdot x)\mathbb{R}$ is determined by the set of scalar multiples $A\mathbb{R}$ of a regular matrix A. It leaves the quadric $x^T \cdot J \cdot x = 0$ invariant, if and only if $A^T J A = \lambda J$ with $\lambda \neq 0$. Thus

$$\mathrm{PGL}(S^4) = \{A\mathbb{R} \mid A^T J A = \lambda J\}, \tag{10}$$

$$J = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \ \lambda \in \mathbb{R} \setminus 0.$$

Lemma 1. $\mathrm{PGL}(S^4)$ *has two connected components. The component containing the identity, which consists of orientation-preserving transformations, is denoted by* $\mathrm{PGL}^+(S^4)$*, and is isomorphic to* $\mathrm{SO}_{5,1}/\{\pm 1\}$.

Proof. If $\lambda < 0$ in equation (10), then the matrix $\overline{A} = A/\sqrt{|\lambda|}$ fulfills $\overline{A}^T J \overline{A} = -J$. This means that the last five column vectors of $\overline{A}$ span a subspace where the scalar product $\langle x, y\rangle = x^T J y$ is negative definite in. This contradicts the inertia theorem, so $\lambda > 0$.

Let $\mathrm{O}_{5,1} = \{A \mid A^T J A = J\}$ and $\mathrm{SO}_{5,1} = \{A \in \mathrm{O}_{5,1} \mid \det(A) > 0\}$. Then for all $A\mathbb{R} \in \mathrm{PGL}(S^4)$, $A/\sqrt{\lambda} \in \mathrm{O}_{5,1}$. Conversely, $A \in \mathrm{O}_{5,1}$ implies that $A\mathbb{R} \in \mathrm{PGL}(S^4)$. This shows the isomorphism $\mathrm{PGL}(S^4) \cong \mathrm{O}_{5,1}/\{\pm 1\}$. The group $\mathrm{SO}_{5,1}$ has two connected components, distinguished by the sign of the upper left 5×5 minor (cf. [1], p. 44). As this minor is of odd order, $\mathrm{SO}_{5,1}/\{\pm 1\}$ is connected, which shows the statement of the theorem. □

Lemma 2. *The Hopf mapping provides an isomorphism*

$$\varphi_\star : \mathrm{GL}(2, \mathbb{H})/\mathbb{R}^\times \cong \mathrm{PGL}^+(S^4), \quad \varphi_\star(L\mathbb{R}) = \varphi L \varphi^{-1}. \tag{11}$$

Here $\mathbb{R}^\times$ denotes the subgroup of homothetical transformations with nonzero real factors.

Proof. We first note that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ q \end{bmatrix} = \begin{bmatrix} 1 \\ d^{-1}(1 + (d^{-1}c - a^{-1}b)(q + a^{-1}b)^{-1}a^{-1}) \end{bmatrix} \mathbb{H}, \tag{12}$$

wherever defined. This shows that $L \in \mathrm{GL}(2, \mathbb{H})$ induces in $P^1(\mathbb{H})$ a composition of transformations of the following types: $q \mapsto q + a$, $q \mapsto qa$, $q \mapsto aq$, and $q \mapsto 1/q$. All of them are Möbius transformations and map the set of lines and circles onto itself. This property is not destroyed by σ, and so $\varphi_\star L = \varphi L \varphi^{-1}$ maps circles to circles. It is well known (cf. [2], p. 992) that then $\varphi_\star L = \kappa | S^4$ with $\kappa \in \mathrm{PGL}(S^4)$. Because $\mathrm{GL}(2, \mathbb{H})$ is connected, $\kappa \in \mathrm{PGL}^+(S^4)$ (the connectedness of $\mathrm{GL}(2, \mathbb{H})$ can be shown in a way completely analogous to the proof of Lemma 13).

Clearly $\varphi_\star : \mathrm{GL}(2, \mathbb{H}) \to \mathrm{PGL}(S^4)$ is a homomorphism with $\ker \varphi_\star$ consisting of the homothetical transformations $\begin{bmatrix} a \\ b \end{bmatrix} \mapsto \lambda \begin{bmatrix} a \\ b \end{bmatrix}$, $\lambda \in \mathbb{R}$. To show that $\varphi_\star$ is onto, we allow a topological argument:

$$\begin{aligned} &\dim_\mathbb{R} \mathrm{GL}(2, \mathbb{H}) = 16, \\ &\dim_\mathbb{R} \mathrm{GL}(2, \mathbb{H})/\mathbb{R}^\times = \dim_\mathbb{R} \mathrm{PGL}^+(S^4) = 15, \end{aligned} \tag{13}$$

and $\mathrm{PGL}^+(S^4)$ is connected. The image of $\varphi_\star$ is an open subgroup of $\mathrm{PGL}^+(S^4)$ because $\varphi_\star$ is of constant rank 15 (its kernel has dimension one), and therefore coincides with $\mathrm{PGL}^+(S^4)$. □

We are going to find explicit $\varphi_\star$-preimages of generators of SO_5. The stabilizer of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in SO_5 is isomorphic to SO_4 and acts on the invariant subspace $\mathbb{H} \oplus 0$. We know that every element of SO_4, when acting in the standard way on the three-sphere $S^3 \subset \mathbb{H}$, is a product of a left and a right multiplication with unit quaternions. Thus all $\gamma \in \mathrm{SO}_5$ with $\gamma \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ have the form

$$\gamma \begin{bmatrix} y \\ \eta \end{bmatrix} = \lambda_a \rho_b^{-1} \begin{bmatrix} y \\ \eta \end{bmatrix} = \begin{bmatrix} ayb^{-1} \\ \eta \end{bmatrix}. \tag{14}$$

If $a, b \in S^3$ and $\varphi \begin{bmatrix} q_0 \\ q_1 \end{bmatrix} = \begin{bmatrix} y \\ \eta \end{bmatrix}$, we have

$$\varphi \begin{bmatrix} bq_0 \\ aq_1 \end{bmatrix} = \sigma^{-1} \begin{bmatrix} 1 \\ aq_1 q_0^{-1} b^{-1} \end{bmatrix} \mathbb{R}$$

$$= \frac{1}{N(bq_0)+N(aq_1)}\begin{bmatrix} 2aq_1\bar q_0\bar b \\ N(aq_1)-N(bq_0)\end{bmatrix}$$
$$= \frac{1}{N(q_0)+N(q_1)}\begin{bmatrix} 2aq_1\bar q_0 b^{-1} \\ N(q_1)-N(q_0)\end{bmatrix} = \begin{bmatrix} ayb^{-1} \\ \eta \end{bmatrix}.$$

by multiplicativity of $N(\cdot)$. Thus the action of SO_4 as stabilizer of $c(0,1)$ in SO_5 is induced by the subgroup of matrices

$$L_{a,b} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in \mathrm{GL}(2,\mathbb{H}), \quad (a,b \in S^3). \tag{15}$$

Consider the subgroups

$$\sigma_{1,t} : \begin{bmatrix} y_1 \\ \vdots \\ y_5 \end{bmatrix} \mapsto \begin{bmatrix} y_1\cos t - y_5 \sin t \\ y_2 \\ y_3 \\ y_4 \\ y_1 \sin t + y_5\cos t \end{bmatrix}, \tag{16}$$

$$\sigma_{a,t} = \lambda_a \sigma_{1,t}\lambda_a^{-1}. \tag{17}$$

The family $\sigma_{1,t}$ of rotations generates SO_5 together with the stabilizer of $\left[\begin{smallmatrix} 0 \\ 1\end{smallmatrix}\right]$. It is directly verified that for all $\left[\begin{smallmatrix} y \\ \eta\end{smallmatrix}\right] \in S^4$ the equation

$$\begin{aligned} &((1-\eta)\sin t + y\cos t)((1-\eta)\cos t - y \sin t)^{-1} \\ = \ &(y_1 \cos 2t - \eta \sin 2t + iy_2 + jy_3 + ky_4) \\ &(1-(\eta\cos 2t + y_1 \sin 2t))^{-1} \end{aligned} \tag{18}$$

holds. This means that

$$S_{1,t} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \implies \varphi S_{1,t}\varphi^{-1} = \sigma_{1,2t} \in \mathrm{SO}_5. \tag{19}$$

It follows that $\sigma_{a,2t}$ is induced by

$$S_{a,t} = \varphi^{-1}\sigma_{a,2t}\varphi = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \bar a \end{bmatrix}. \tag{20}$$

At last we see that

$$I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \implies \iota\begin{bmatrix} y \\ \eta \end{bmatrix} = \varphi I \varphi^{-1}\begin{bmatrix} y \\ \eta \end{bmatrix} = \begin{bmatrix} \bar y \\ -\eta \end{bmatrix}. \tag{21}$$

We define the $\mathbb{R}$-basis $\left[\begin{smallmatrix} 1 \\ 0\end{smallmatrix}\right], \ldots, \left[\begin{smallmatrix} 0 \\ k\end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1\end{smallmatrix}\right], \ldots, \left[\begin{smallmatrix} 0 \\ k\end{smallmatrix}\right]$ of $\mathbb{H}^2$ to be orthonormal. Then the following makes sense:

Lemma 3. *The preimage* $\varphi_\star^{-1}(\mathrm{SO}_5)$ *is contained in* $\mathrm{SO}_8 \cap \mathrm{GL}(2,\mathbb{H})$.

Proof. The right $\mathbb{H}$-linear automorphisms $S_{a,t}$ and $L_{a,b}$ of $\mathbb{H}^2$ are contained in SO_8, if we see them as $\mathbb{R}$-linear automorphisms of $\mathbb{R}^8$. □

Lemma 4. *Consider a point* $\begin{bmatrix} y_0 \\ y \\ y_5 \end{bmatrix}\mathbb{R} \in S^4 \subset (\mathbb{R}\oplus\mathbb{H}\oplus\mathbb{R}) \cong P^5$. *We define the vectors* $v_1, \ldots, w_6 \in \mathbb{H}^2$ *by*

$$\begin{aligned}
w_3 &= \begin{bmatrix} y_0 - y_1 \\ y - y_5 - y_1 \end{bmatrix}, w_4 = \begin{bmatrix} y_0 - y_2 \\ y - iy_5 - iy_2 \end{bmatrix},\\
w_5 &= \begin{bmatrix} y_0 - y_3 \\ y - jy_5 - jy_3 \end{bmatrix},\ w_6 = \begin{bmatrix} y_0 - y_4 \\ y - ky_5 - ky_4 \end{bmatrix},\\
v_1 &= \begin{bmatrix} y_0 - y_5 \\ y \end{bmatrix},\ v_2 = \begin{bmatrix} \overline{y} \\ y_0 + y_5 \end{bmatrix},\ v_3 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot w_3,\\
v_4 &= \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \cdot w_4, v_5 = \begin{bmatrix} 1 & -j \\ -j & 1 \end{bmatrix} \cdot w_5,\ v_6 = \begin{bmatrix} 1 & -k \\ -k & 1 \end{bmatrix} \cdot w_6.
\end{aligned}$$

Then the preimage $\varphi^{-1}\begin{bmatrix} y_0 \\ y \\ y_5 \end{bmatrix}$ *equals* $v_1\mathbb{H} = \ldots = v_6\mathbb{H}$.

Proof. equation (3) shows the result for v_1. The equation $v_2 = I^{-1}\varphi^{-1}\iota(v_1)$ shows the statement for v_2. The rest is a translation of the formulas $w_3 = \varphi^{-1}\sigma_{1,\frac{\pi}{2}}(\frac{1}{y_0}\begin{bmatrix} y \\ y_5 \end{bmatrix})$, $w_4 = \varphi^{-1}\sigma_{i,\frac{\pi}{2}}(\frac{1}{y_0}\begin{bmatrix} y \\ y_5 \end{bmatrix})$, $w_5 = \varphi^{-1}\sigma_{j,\frac{\pi}{2}}(\frac{1}{y_0}\begin{bmatrix} y \\ y_5 \end{bmatrix})$, $w_6 = \varphi^{-1}\sigma_{k,\frac{\pi}{2}}(\frac{1}{y_0}\begin{bmatrix} y \\ y_5 \end{bmatrix})$, and $v_3 = \frac{1}{\sqrt{2}}S^{-1}_{1,\frac{\pi}{4}}(w_3)$, $v_4 = \frac{1}{\sqrt{2}}S^{-1}_{i,\frac{\pi}{4}}(w_4)$, $v_5 = \frac{1}{\sqrt{2}}S^{-1}_{j,\frac{\pi}{4}}(w_5)$, $v_6 = \frac{1}{\sqrt{2}}S^{-1}_{k,\frac{\pi}{4}}(w_6)$. □

4. Complex quaternions

Definition. *The tensor product*

$$\widetilde{\mathbb{H}} = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}, \tag{22}$$

is called the algebra of complex quaternions.

We denote the imaginary unit in $\mathbb{C}$ by the symbol $i_{\mathbb{C}}$. We further naturally extend the definition of conjugate quaternion and norm, using the same formulas as in the real case). The relations $N(a) = a\overline{a} \in 1\otimes\mathbb{C}$, $N(ab) = N(a)N(b)$, $a^{-1} = \frac{1}{N(a)}\overline{a}$ remain true. A polynomial identity in quaternions carries over to a polynomial identity for complex quaternions, as the embedding $\mathbb{H} \to \widetilde{\mathbb{H}}$ defines a homomorphism $\mathbb{H}[x_1, \ldots, x_n] \to \widetilde{\mathbb{H}}[x_1, \ldots, x_n]$ of polynomial rings.

Consider the left and right multiplication operators λ_a and ρ_a, which are defined by $\lambda_a(x) = ax$ and $\rho_a(x) = xa$.

Lemma 5. *The sets* $\widetilde{\mathbb{H}}a$ *and* $a\widetilde{\mathbb{H}}$ *are* $\mathbb{C}$*-linear subspaces of* $\widetilde{\mathbb{H}}$. *Their* $\mathbb{C}$*-dimension equals four if* $N(a) \neq 0$, *and two if* $N(a) = 0$, $a \neq 0$. *In the latter case* $\ker(\rho_a) = \overline{a}\widetilde{\mathbb{H}}$ *and* $\ker(\lambda_a) = \widetilde{\mathbb{H}}\overline{a}$.

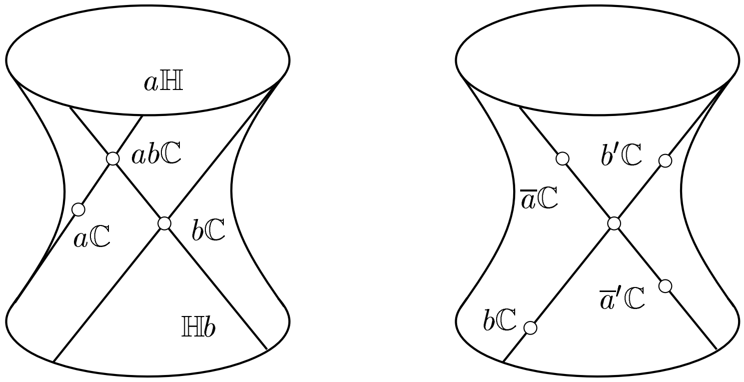

Figure 1. Left: The quadric $N(x) = 0$ in $P_{\mathbb{C}}(\mathbb{H} \otimes \mathbb{C})$. Right: see the proof of Lemma 8.

Proof. The coordinate matrix of λ_a with respect to the basis $(1 \otimes 1 + i \otimes i_{\mathbb{C}}, 1 \otimes 1 - i \otimes i_{\mathbb{C}}, j \otimes 1 + k \otimes i_{\mathbb{C}}, j \otimes 1 - k \otimes i_{\mathbb{C}})$ is given by $\overline{L}_a = \begin{bmatrix} B_1 & -B_2 \\ B_2 & B_1 \end{bmatrix}$, with

$$B_1 = \begin{bmatrix} a_0 - i_{\mathbb{C}} a_1 & 0 \\ 0 & a_0 + i_{\mathbb{C}} a_1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & a_2 - i_{\mathbb{C}} a_3 \\ a_2 + i_{\mathbb{C}} a_3 & 0 \end{bmatrix}. \tag{23}$$

If one of $a_0 \pm i_{\mathbb{C}} a_1$, $a_2 \pm i_{\mathbb{C}} a_3$ is nonzero (which is the case if $a \neq 0$), then obviously $\mathrm{rk}(\lambda_a) \geq 2$, and $\dim_{\mathbb{C}}(\ker \lambda_a) \leq 2$. The determinant of λ_a equals $N(a)^2$, so $\dim_{\mathbb{C}}(a\widetilde{\mathbb{H}}) = 4$ if $N(a) \neq 0$. The same results hold for $a\widetilde{\mathbb{H}}$, because $\overline{xa} = \bar{a}\bar{x}$ for all x, a.

If $N(a) = 0$, then $\widetilde{\mathbb{H}}a \subset \ker \lambda_{\bar{a}}$. Because $\dim(\widetilde{\mathbb{H}}a) = \mathrm{rk}(\lambda_a) \geq 2$, we have actually $\dim(\widetilde{\mathbb{H}}a) = 2$ and $\widetilde{\mathbb{H}}a = \ker \lambda_{\bar{a}}$. The argument for $a\widetilde{\mathbb{H}}$ is similar. □

Consider the three-dimensional complex projective space $P_{\mathbb{C}}(\widetilde{\mathbb{H}})$, consisting of elements $a\mathbb{C}$ with $a \in \widetilde{\mathbb{H}}$. The equation $N(a) = 0$ defines a quadric, i.e., a nonsingular quadratic variety. It carries two families of projective subspaces of $\mathbb{C}$-dimension one (its *generator lines*), which have the property that (i) all points of the quadric are incident with exactly one line of each family (ii) any two lines of different families intersect in one point.

Lemma 6. *If $N(a) = 0$, then the sets $a\widetilde{\mathbb{H}}$ and $\widetilde{\mathbb{H}}a$ coincide with the two generator lines of the quadric $N(x) = 0$ incident with $a\mathbb{C}$. One family of generators consists of the sets $a\widetilde{\mathbb{H}}$, the other one of the sets $\widetilde{\mathbb{H}}a$.*

Proof. By Lemma 5, the subspaces $a\widetilde{\mathbb{H}}$ and $\widetilde{\mathbb{H}}a$ are of $\mathbb{C}$-projective dimension one. Obviously they are contained in the quadric $N(x) = 0$. As all points of this quadric are incident with exactly one generator line, this description exhausts all generators.

If $ab \neq 0$, the generators $a\widetilde{\mathbb{H}}$ and $\widetilde{\mathbb{H}}b$ intersect in $ab\mathbb{C}$, which shows that they belong to different families of generators (cf. Fig. 1). □

5. Extension of the Hopf mapping to $\mathbb{H} \otimes \mathbb{C}$.

We consider the affine space $\widetilde{\mathbb{H}} \oplus \mathbb{C}$, and embed it into the projective space

$$P^5(\mathbb{C}) \cong P_{\mathbb{C}}(\mathbb{C} \oplus \widetilde{\mathbb{H}} \oplus \mathbb{C}) \tag{24}$$

via

$$(a, \lambda) \mapsto (1, a, \lambda)\mathbb{C}.$$

The projective extension $S^4(\mathbb{C})$ of the complex unit sphere has the equation

$$S^4(\mathbb{C}) : y_0^2 = N(y) + y_5^2. \tag{25}$$

Further we consider the mapping

$$\widetilde{\psi} : \widetilde{\mathbb{H}} \oplus \widetilde{\mathbb{H}} \to \mathbb{C} \oplus \widetilde{\mathbb{H}} \oplus \mathbb{C}, \quad \begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} N(a) + N(b) \\ 2b\overline{a} \\ N(b) - N(a) \end{bmatrix}. \tag{26}$$

There is a corresponding mapping

$$\widetilde{\varphi} : P^7(\mathbb{C}) \to P^5(\mathbb{C}), \quad \widetilde{\varphi}(\begin{bmatrix} a \\ b \end{bmatrix}\mathbb{C}) = \widetilde{\psi}\begin{bmatrix} a \\ b \end{bmatrix}\mathbb{C}, \tag{27}$$

which is undefined for those $\begin{bmatrix} a \\ b \end{bmatrix}$ with $\widetilde{\psi}\begin{bmatrix} a \\ b \end{bmatrix} = 0$.

Lemma 7. *$\widetilde{\psi}$ is zero (and $\widetilde{\varphi}$ is undefined) precisely for the elements of the set*

$$\psi^{-1}(0) = \{\begin{bmatrix} ca \\ da \end{bmatrix} \mid N(a) = 0\} = \{\begin{bmatrix} a \\ da \end{bmatrix} \mid N(a) = 0\}. \tag{28}$$

Proof. Obviously $\psi\begin{bmatrix} ca \\ da \end{bmatrix} = \begin{bmatrix} (N(c) + N(d))N(a) \\ cN(a)d \\ (N(d) - N(a))N(a) \end{bmatrix} = 0$ for all c, d if $N(a) = 0$, which shows the '$\supseteq$' part of the statement. We show the reverse inclusion:

If $N(a) + N(b) = N(a) - N(b) = 0$ then $N(a) = N(b) = 0$. If $N(a) = 0$, Lemma 6 shows that the set of x such that $x\bar{a} = 0$ equals $\widetilde{\mathbb{H}}a$. This implies $\psi^{-1}(0) = \{\begin{bmatrix} a \\ da \end{bmatrix} \mid N(a) = 0\}$. This set is contained in $\{\begin{bmatrix} ca \\ da \end{bmatrix} \mid N(a) = 0\}$, and we are done. $\square$

We show a lemma whose proof uses geometry to show an algebraic relation:

Lemma 8. *If a, b, a', b' are complex quaternions of zero norm, then the equation $b'\overline{a}'\mathbb{C} = b\overline{a}\mathbb{C}$ is equivalent to $a' = ax$, $b' = by$.*

Proof. Consider the quadric $N(x) = 0$ in $\widetilde{\mathbb{H}}$ and its two families of generator lines. Generators of the same family do not intersect unless they are equal, and generators of different families intersect in precisely one point. $b'\overline{a}'\mathbb{C} = b\overline{a}\mathbb{C}$ means that the generators $b'\widetilde{\mathbb{H}}$ and $b\widetilde{\mathbb{H}}$ intersect, and therefore are equal. Likewise the generators $\widetilde{\mathbb{H}}\overline{a}'$ and $\widetilde{\mathbb{H}}\bar{a}$ are equal. Thus $b' = bx$ and $\overline{a}' = \bar{y}\bar{a}$ (see Fig. 1).

Conversely, $a' = ax$, $b' = by$ imply these equalities of generator lines. Generators of different families intersect in precisely one point, which must be $b'\overline{a}\mathbb{C} = b\overline{a}'\mathbb{C}$. $\square$

Theorem 1. *The image of* $\widetilde{\varphi} = S^4(\mathbb{C})$. *If* $v_1, \ldots, v_6$ *are defined as in Lemma 4 with scalars* $1\otimes 1$, $i\otimes 1$, $j\otimes 1$ *and* $k\otimes 1$ *instead of* $1, i, j$, *and* k, *then the preimage* $\widetilde{\psi}^{-1}\begin{bmatrix} y_0 \\ y \\ y_5 \end{bmatrix}\mathbb{C}$ *of a point of* $S^4(\mathbb{C})$ *contains the vectors* $v_1, \ldots, v_6$. *There is an* r *such that* $\widetilde{\varphi}(v_r\mathbb{C}) = \begin{bmatrix} y_0 \\ y \\ y_5 \end{bmatrix}\mathbb{C}$.

Proof. It follows directly from Lemma 4 that for all r the vectors $\widetilde{\psi}(v_r)$ and $\begin{bmatrix} y_0 \\ y \\ y_5 \end{bmatrix}$ are $\mathbb{C}$-linearly dependent. We show that $\widetilde{\varphi}$ is onto, i.e., there is an r such that $\widetilde{\psi}(v_r)$ is nonzero.

If $\widetilde{\psi}(v_1) = 0$, then $(y_0 - y_5)^2 = 0$, which means $y_0 - y_5 = 0$. If $\widetilde{\psi}(v_2) = 0$, then necessarily $y_0 + y_5 = 0$. This shows that $\begin{bmatrix} y_0 \\ y \\ y_5 \end{bmatrix}$ is a $\widetilde{\psi}$-image if either y_0 or y_5 are nonzero.

Now consider the case $y_0 = y_5 = 0$. If $\begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} \in S^4(\mathbb{C})$, then $y\overline{y} = 0$. If $y_1 \neq 0$, consider $w_3 = \begin{bmatrix} -y_1 \\ y - y_1 \end{bmatrix}$ and $v_3 = \begin{bmatrix} -2y_1 + y \\ y \end{bmatrix}$. Then $\widetilde{\psi}(v_3) = \begin{bmatrix} 0 \\ y(-2y_1 + \overline{y}) \\ 0 \end{bmatrix}\mathbb{C} = \begin{bmatrix} 0 \\ -2y_1 y \\ 0 \end{bmatrix}\mathbb{C} = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix}\mathbb{C}$. If $y_2 \neq 0$, we use v_4 instead of v_3, and analogously for the cases $y_3 \neq 0$ and $y_4 \neq 0$. $\square$

Lemma 9. *The complete* $\widetilde{\varphi}$*-preimage of a point* $\begin{bmatrix} y_0 \\ y \\ y_5 \end{bmatrix}\mathbb{C} \in P^5(\mathbb{C})$ *consists of* $\widetilde{\psi}^{-1}(0)$ *and a three-dimensional projective subspace* $U \leq P^7(\mathbb{C})$. *Assume that* $\widetilde{\psi}(a, b) = \begin{bmatrix} y_0 \\ y \\ y_5 \end{bmatrix}\mathbb{C}$. *If* $y_0 = y_5 = 0$, *then* $U = \begin{bmatrix} a \\ 0 \end{bmatrix}\widetilde{\mathbb{H}} + \begin{bmatrix} 0 \\ b \end{bmatrix}\widetilde{\mathbb{H}}$. *Otherwise,* $U = \begin{bmatrix} a \\ b \end{bmatrix}\widetilde{\mathbb{H}}$.

Proof. We let $\widetilde{\psi}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_0 \\ y \\ y_5 \end{bmatrix} \in \mathbb{C} \oplus \widetilde{\mathbb{H}} \oplus \mathbb{C}$. In the proof we consider only points where $\widetilde{\varphi}$ is defined, i.e., $\widetilde{\psi}$ is nonzero.

There are the equivalences

$$\begin{aligned}
&(i) && N(a) \neq 0, \quad N(b) \neq 0 &&\iff && y_0 \neq \pm y_5, \\
&(ii) && N(a) = 0, \quad N(b) \neq 0 &&\iff && y_0 = y_5 \neq 0, \\
&(iii) && N(a) \neq 0, \quad N(b) = 0 &&\iff && y_0 = -y_5 \neq 0, \\
&(iv) && N(a) = 0, \quad N(b) = 0 &&\iff && y_0 = y_5 = 0,
\end{aligned} \tag{29}$$

and it is obviously sufficient to treat cases (i)–(iv) separately. If $N(a) \neq 0$, there exists a^{-1} and $\widetilde{\psi}\begin{bmatrix} a \\ b \end{bmatrix} = \widetilde{\psi}\begin{bmatrix} 1 \\ ba^{-1} \end{bmatrix}$. The mapping $\widetilde{\psi}$ is injective for vectors $\begin{bmatrix} 1 \\ x \end{bmatrix}$. If $N(a') \neq 0$ and

$\widetilde{\psi}\begin{bmatrix} a' \\ b' \end{bmatrix}\mathbb{C} = \widetilde{\psi}\begin{bmatrix} a \\ b \end{bmatrix}\mathbb{C}$, we conclude $b'a'^{-1} = ba^{-1}$ which means $\begin{bmatrix} b' \\ a' \end{bmatrix} = \begin{bmatrix} ac \\ bc \end{bmatrix}$. This shows the result for cases (i) and (iii).

An analogous argument based on $N(b) \neq 0$ shows the result for cases (i) (again) and (ii). As to case (iv), we do the following: If $N(a) = N(b) = 0$, then $\widetilde{\psi}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ b\overline{a} \\ 0 \end{bmatrix}$. We look for $\begin{bmatrix} a' \\ b' \end{bmatrix}$ such that $\widetilde{\psi}\begin{bmatrix} a' \\ b' \end{bmatrix}\mathbb{C} = \widetilde{\psi}\begin{bmatrix} a \\ b \end{bmatrix}\mathbb{C}$. By Lemma 8, this is equivalent to $b' = bx$, $\overline{a}' = \bar{a}\bar{y}$, or $\begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix}$. □

Definition. *The three-dimensional subspaces U mentioned in Lemma 9 are called the fiber subspaces of $\widetilde{\psi}$.*

Obviously the sets $U \setminus \widetilde{\psi}^{-1}(0)$ form a partition of $(\widetilde{\mathbb{H}} \oplus \widetilde{\mathbb{H}}) \setminus \widetilde{\psi}^{-1}(0)$.

Remark. Recall that the fiber subspaces of the *complex* Hopf mapping $\varphi : P^3 \to S^2$, $\varphi_{\mathbb{C}}\begin{bmatrix} q_0 \\ q_1 \end{bmatrix}\mathbb{R} = \begin{bmatrix} q_0 \\ q_1 \end{bmatrix}\mathbb{C}$ are straight lines contained in an elliptic linear congruence. After a complex extension, these lines meet the exceptional set of the Hopf mapping, which is a union of two lines. The situation here is similar. A fiber subspace intersects $\widetilde{\psi}^{-1}(0)$, because $\widetilde{\psi}\begin{bmatrix} ac \\ bc \end{bmatrix} = 0$ if $N(c) = 0$.

Definition. *We define two points of $S^4(\mathbb{C})$ to be parallel, if their span is contained in $S^4(\mathbb{C})$.*

Lemma 10. *The fiber subspaces of non-parallel points are skew. The fiber subspaces of a plane of parallel points intersect in a common line, which is contained in the set $\psi^{-1}(0)$.*

Proof. Assume that the fiber subspaces of $\psi\begin{bmatrix} a \\ b \end{bmatrix}$ and $\psi\begin{bmatrix} a' \\ b' \end{bmatrix}$ have a point in common, and that both $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} a' \\ b' \end{bmatrix}$ belong to classes (i)–(iii) of the proof of Lemma 9. Then $\begin{bmatrix} ac \\ bc \end{bmatrix} = \begin{bmatrix} a'c' \\ b'c' \end{bmatrix}$ with $N(c) = 0$, $\begin{bmatrix} ac \\ bc \end{bmatrix} \neq 0$. Further $\begin{bmatrix} ack \\ bck \end{bmatrix} = \begin{bmatrix} a'c'k \\ b'c'k \end{bmatrix}$ for all $k \in \mathbb{H}$, so these fiber subspaces have a line in common.

The generator lines $\widetilde{\mathbb{H}}c$ and $\widetilde{\mathbb{H}}c'$ of the quadric $N(x) = 0$ intersect in the point $ac = a'c'$, so they are equal and $c = c'k$, $N(k) \neq 0$. Thus there is the following chain of equivalences: $\begin{bmatrix} ac \\ bc \end{bmatrix} = \begin{bmatrix} a'c' \\ b'c' \end{bmatrix} \iff a - a'k,\ b - b'k \in \ker(R_c) = \operatorname{im}(\lambda_{\overline{c}}) \iff \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a' \\ b' \end{bmatrix}k + \begin{bmatrix} m \\ n \end{bmatrix}\overline{c}$ with $k, m, n \in \widetilde{\mathbb{H}}$, $N(k) \neq 0 \iff \begin{bmatrix} a \\ b \end{bmatrix}\widetilde{\mathbb{H}} = (\begin{bmatrix} a' \\ b' \end{bmatrix} + \begin{bmatrix} m \\ n \end{bmatrix}\overline{c})\widetilde{\mathbb{H}}$ with $m, n \in \widetilde{\mathbb{H}}$. Direct computation shows that

$$\psi\begin{bmatrix} a' + \lambda m\overline{c} \\ b' + \mu n\overline{c} \end{bmatrix} = \psi\begin{bmatrix} a' \\ b' \end{bmatrix} + 2\lambda\begin{bmatrix} (a'c\overline{m})_0 \\ b'c\overline{m} \\ -(a'c\overline{m})_0 \end{bmatrix} + 2\mu\begin{bmatrix} (b'c\overline{n})_0 \\ n\overline{c}\overline{a}' \\ (b'c\overline{n})_0 \end{bmatrix}, \tag{30}$$

where $(\cdot)_0$ denotes the first component with respect to the standard $\mathbb{C}$-basis $(1, i, j, k) \otimes 1$. It is easily verified that this is indeed a parametrization of a plane, unless $ac = bc = 0$. The remaining cases are similar, but to avoid computations we could also use the fact that

$\mathrm{GL}(2,\widetilde{\mathbb{H}})$ leaves the set $\psi^{-1}(0)$ invariant and apply one of the mappings $S_{a,t}$ of the proof of Lemma 9 such that neither $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ nor $\left[\begin{smallmatrix} a' \\ b' \end{smallmatrix}\right]$ belong to case (iv). □

Lemma 11. *For all planar sections c of $S^4(\mathbb{C})$ which are conics, there is a line $\widetilde{c}$ with $\widetilde{\varphi}(\widetilde{c}) = c$. The same is true for all lines of $S^4(\mathbb{C})$.*

Proof. If the conic c is a planar section of $S^4(\mathbb{C})$, it does not contain parallel points, so the fiber subspaces of c's points do not intersect. We choose three points $P_1, P_2, P_3 \in c$. Consider their fiber subspaces U_1, U_2, U_3. For all $\left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right]\mathbb{C} \in U_1$, there is a unique line $L\left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right]$ which meets U_i in a point $\left[\begin{smallmatrix} a_i \\ b_i \end{smallmatrix}\right]\mathbb{C}$. The correspondences $\left[\begin{smallmatrix} a_i \\ b_i \end{smallmatrix}\right] \leftrightarrow \left[\begin{smallmatrix} a_j \\ b_j \end{smallmatrix}\right]$ are linear and one-to-one, so we can avoid $\left[\begin{smallmatrix} a_i \\ b_i \end{smallmatrix}\right] \in \widetilde{\psi}^{-1}(0)$. We let let $\widetilde{c} = L\left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right]$. The image $\widetilde{\varphi}(\widetilde{c})$ is linear or quadratic and contains P_1, P_2, P_3, which shows that actually $\widetilde{\psi}(\widetilde{c}) = c$.

If P_2, P_3 are parallel, but P_1 is not parallel to P_2, P_3, this procedure must fail because there is no line or conic in $S^4(\mathbb{C})$ which connects P_1, P_2, P_3, and so U_2 and U_3 must have a point in common. As Lemma 10 shows, there is a line $\widetilde{c} \subset \widetilde{\mathbb{H}}^2$ such that $\widetilde{\varphi}(\widetilde{c}) = P_2 \vee P_3$. □

It is not difficult to generalize Lemma 2 to the case of $\widetilde{\mathbb{H}}$. The group $\mathrm{PGL}(S^4(\mathbb{C}))$ of projective automorphisms of $S^4(\mathbb{C})$ equals the set $\{A\mathbb{C} \mid A \in \mathbb{C}^{6\times 6},\ A^T A = \lambda \cdot E_6,\ \lambda \in \mathbb{C}\}$. Because $\bar{A} = \pm(1/\sqrt{\lambda})A$ has the property that $\overline{A}^T A = E_6$, we have $\mathrm{PGL}(S^4(\mathbb{C})) = \mathrm{O}_6(\mathbb{C})/\{\pm 1\}$. The two connected components of $\mathrm{O}_6(\mathbb{C})$ are distinguished by the determinant. Further, $\det(A/\pm\sqrt{\lambda}) = \det(A)/\lambda^3$. Thus we have

Lemma 12. $\mathrm{PGL}(S^4(\mathbb{C})) = \{A\mathbb{C} \mid A^T A = E_6, \lambda \neq 0\}$ *has two connected components distinguished by the value of* $\det(A)/\lambda^3$.

The component containing the identity is denoted by $\mathrm{PGL}^+(S^4(\mathbb{C}))$. Obviously

$$\mathrm{PGL}^+(S^4(\mathbb{C})) \cong \mathrm{SO}_6(\mathbb{C})/\{\pm 1\}.$$

Lemma 13. *The group $\mathrm{GL}(2,\widetilde{\mathbb{H}})$ of invertible right $\widetilde{\mathbb{H}}$-linear endomorphisms of the right $\widetilde{\mathbb{H}}$-module $\widetilde{\mathbb{H}}^2$ operates transitively on $\widetilde{\mathbb{H}}^2 \setminus \psi^{-1}(0)$ and is connected.*

Proof. We use the fact that the complement of the quadric $N(x) = 0$ in $\widetilde{\mathbb{H}}$ is arcwise connected, as it is of real codimension two.

The symbol $G_{a,b}^{c,d}$ denotes the set of elements $L \in \mathrm{GL}(2,\widetilde{\mathbb{H}})$ such that $L\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] = \left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$. Obviously the stabilizer $G_{0,1}^{0,1}$ of $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ consists of the matrices $\begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix}$ with $N(d) \neq 0$, and is connected. We show that for $N(a) \neq 0$ there is a path L_t in $\mathrm{GL}(2,\widetilde{\mathbb{H}})$ beginning in $G_{1,0}^{1,0}$ and ending in $G_{1,0}^{a,b}$: If $a(t)$ is a path with $a(0) = 1$, $a(1) = a$, then we let

$$L_t = \begin{bmatrix} a(t) & 0 \\ t \cdot b & 1 \end{bmatrix}. \tag{31}$$

If $N(b) \neq 0$, we analogously find a path L'_t which begins in $G^{0,1}_{0,1}$ and ends in $G^{a,b}_{0,1}$.

If both norms $N(a)$ and $N(b)$ are zero, consider the paths $S_{c,t}$ as defined by equation (20). Obviously $S_{c,0} = \mathrm{id}$ for all c, and the proof of Theorem 1 implies that we can choose c such that $S_{c,\pi/4}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a' \\ b' \end{bmatrix}$ with either $N(a') \neq 0$ or $N(b') \neq 0$. This shows that either there is a path in $\mathrm{GL}(2, \widetilde{\mathbb{H}})$ beginning in $G^{1,0}_{1,0}$ and ending in $G^{a,b}_{1,0}$, or the same for $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ instead of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

As $G^{1,0}_{0,1}$ is nonempty, this shows that $\mathrm{GL}(2, \widetilde{\mathbb{H}})$ acts transitively. If $L \in G^{c,d}_{0,1}$, $L' \in G^{a,b}_{0,1}$, then $G^{0,1}_{0,1} = L^{-1} \cdot G^{c,d}_{a,b} \cdot L'$, so all sets $G^{c,d}_{a,b}$ are homeomorphic. One of them has already been shown to be connected, so all of them are contained in the same and only arc component of $\mathrm{GL}(2, \widetilde{\mathbb{H}})$. □

Theorem 2. *The Hopf mapping $\widetilde{\varphi}$ provides an isomorphism of the groups* $\mathrm{GL}(2, \widetilde{\mathbb{H}})/\mathbb{C}^\times \cong \mathrm{PGL}^+(S^4(\mathbb{C})) \cong \mathrm{SO}_6(\mathbb{C})/\{\pm 1\}$.

Proof. We first show that all $L \in \mathrm{GL}(2, \widetilde{\mathbb{H}})$ induce a projective automorphism of $S^4(\mathbb{C})$. To avoid computations, we appeal to a more general theorem by showing that L induces an automorphism of a circle geometry in the sense of [2]:

The set $\mathcal{C}$ of *proper circles* are those planar sections of $S^4(\mathbb{C})$ which are either conics or pairs of lines. An automorphism of $\mathcal{C}$ is a bijection of $S^4(\mathbb{C})$ which maps $\mathcal{C}$ to $\mathcal{C}$. Lemma 10 characterizes conics and lines as the $\widetilde{\varphi}$-images of certain lines, and it distinguishes between them in a $\mathrm{GL}(2, \widetilde{\mathbb{H}})$-invariant way. Thus L maps $\widetilde{\varphi}^{-1}(\mathcal{C})$ onto itself. We can apply Th. 4.2.3 of [2], p. 992, to conclude that L induces a projective automorphism $\widetilde{\varphi}_\star(L) = \widetilde{\varphi} L \widetilde{\varphi}^{-1} \in \mathrm{PGL}^+(S^4(\mathbb{C}))$.

To show that $\varphi_\star$ is is onto, we note that its kernel is the subgroup of complex homothetical transformations, so $\dim_{\mathbb{R}} \ker(\varphi_\star) = 2$. From the dimensions $\dim_{\mathbb{R}}(\mathrm{GL}(2, \widetilde{\mathbb{H}})) = 32$ and $\dim_{\mathbb{R}}(\mathrm{PGL}^+(S^4(\mathbb{C}))) = 30$ we conclude that $\varphi_\star(\mathrm{GL}(2, \widetilde{\mathbb{H}}))$ is an open subgroup of $\mathrm{PGL}^+(S^4(\mathbb{C}))$. Because the latter is connected, $\varphi_\star$ is onto. □

6. A Hopf mapping onto the Klein quadric

We consider the real subspace

$$\widehat{\mathbb{H}} = [1 \otimes 1, i \otimes 1, j \otimes i_{\mathbb{C}}, k \otimes i_{\mathbb{C}}]_{\mathbb{R}} \tag{32}$$

of $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} = \widetilde{\mathbb{H}}$. It is easily verified that $\widehat{\mathbb{H}}$ is a subring of $\widetilde{\mathbb{H}}$ and an $\mathbb{R}$-subalgebra. Further we consider the real vector space

$$\mathbb{R} \oplus \widehat{\mathbb{H}} \oplus \mathbb{R} = \{\begin{bmatrix} r_0 \\ r \\ r_5 \end{bmatrix} \mid r_0, r_5 \in \mathbb{R},\ r \in \widehat{\mathbb{H}}\}. \tag{33}$$

The set $Q_{4,2} = S^4(\mathbb{C}) \cap \mathbb{R} \oplus \widehat{\mathbb{H}} \oplus \mathbb{R}$ then has the equation

$$r_0^2 + r_3^2 + r_4^2 = r_1^2 + r_2^2 + r_5^2, \tag{34}$$

which describes a real regular quadric of index two. There are the projective spaces $\widehat{P}^7 = P_{\mathbb{R}}(\widehat{\mathbb{H}} \oplus \widehat{\mathbb{H}})$, $\widehat{P}^5 = P_{\mathbb{R}}(\mathbb{R} \oplus \widehat{\mathbb{H}} \oplus \mathbb{R})$, and the Hopf mapping

$$\widehat{\varphi} : \widehat{P}^7 \to \widehat{P}^5, \quad \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]\mathbb{R} \mapsto \begin{bmatrix} N(a) + N(b) \\ b\overline{a} \\ N(b) - N(a) \end{bmatrix}\mathbb{R}. \tag{35}$$

Its representation $\widehat{\psi}$ in homogeneous coordinates with respect to the bases

$$\begin{bmatrix} 1 \otimes 1 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} k \otimes i_{\mathbb{C}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \otimes 1 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ k \otimes i_{\mathbb{C}} \end{bmatrix} \tag{36}$$

and

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \otimes 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ i \otimes 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ j \otimes i_{\mathbb{C}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{37}$$

is given by

$$\widehat{\psi} \begin{bmatrix} x_0 \\ \vdots \\ x_7 \end{bmatrix} = \begin{bmatrix} x_0^2 + x_1^2 - x_2^2 - x_3^2 + x_4^2 + x_5^2 - x_6^2 - x_7^2 \\ 2(+x_0x_4 + x_1x_5 - x_2x_6 - x_3x_7) \\ 2(-x_0x_5 + x_1x_4 - x_2x_7 - x_3x_6) \\ 2(-x_0x_6 + x_1x_7 + x_2x_4 - x_3x_5) \\ 2(+x_0x_7 + x_1x_6 - x_2x_5 - x_3x_4) \\ x_4^2 + x_5^2 - x_6^2 - x_7^2 - x_0^2 - x_1^2 + x_2^2 + x_3^2 \end{bmatrix} \tag{38}$$

The very definitions of $\widehat{\psi}$ and $\widehat{\varphi}$ imply that

$$\begin{aligned} &\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \widehat{\mathbb{H}}^2, \quad \widehat{\varphi}(\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]\mathbb{R}) = \begin{bmatrix} y_0 \\ y \\ y_5 \end{bmatrix}\mathbb{R}, \quad \widetilde{\varphi}(\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]\mathbb{C}) = \begin{bmatrix} y_0' \\ y' \\ y_5' \end{bmatrix}\mathbb{C} \\ &\Longrightarrow \begin{bmatrix} y_0 \\ y \\ y_5 \end{bmatrix}\mathbb{C} = \begin{bmatrix} y_0' \\ y' \\ y_5' \end{bmatrix}\mathbb{C}. \end{aligned} \tag{39}$$

This means that the mapping $\widehat{\varphi}$ becomes the restriction of $\widetilde{\varphi}$ to $\widehat{P}^7 \subset P^7(\mathbb{C})$, if we embed $Q_{4,2}$ into complex projective space.

The group $\mathrm{PGL}(Q_{4,2})$ of projective automorphisms of $Q_{4,2}$ is the set

$$\{A\mathbb{R} \mid A^T J A = \lambda J\} \tag{40}$$

with

$$J = \begin{bmatrix} 1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & -1 \end{bmatrix}. \tag{41}$$

Lemma 14. $\mathrm{PGL}(Q_{4,2})$ *has four connected components. With the notations of equation* (40), *they are distinguished by* $\mathrm{sgn}(\lambda)$ *and* $\mathrm{sgn}(\det(A))$.

Proof. Both $\operatorname{sgn}(\lambda)$ and $\operatorname{sgn}(\det(A))$ are homomorphisms of $\mathrm{PGL}(Q_{4,2})$. It is easy to find examples for all four possible cases, so $\mathrm{PGL}(Q_{4,2})$ has at least four connected components: They are distinguished by whether or not they reverse orientation, and whether they interchange the two families of generator planes or not. If $\det(A) > 0$, $\lambda > 0$, then $A/\sqrt{\lambda}$ preserves the bilinear form α, so after a permutation of coordinates, $A/\sqrt{\lambda} \in \mathrm{SO}_{3,3}$. The group $\mathrm{SO}_{3,3}$ has two connected components, distinguished by the sign of the upper left minor of order three (cf. [1], p. 44). The factor $\mathrm{SO}_{3,3}/\{\pm 1\}$ however is connected, as this minor is of odd order. □

The subgroup of $\mathrm{PGL}(Q_{4,2})$ consisting of orientation-preserving transformations which do not interchange the two families of generator planes will be denoted by $\mathrm{PGL}^+(Q_{4,2})$. It is isomorphic to $\mathrm{SO}_{3,3}/\{\pm 1\}$ and is connected. The following results are closely related to the respective results for $\mathbb{H} \otimes \mathbb{C}$:

Theorem 3. *The image of $\widehat{\varphi} = Q_{4,2}$. If $v_1, \dots, v_4$ are defined as in Lemma 4 with scalars $1 \otimes 1$ and $i \otimes 1$ instead of 1 and i, then the preimage $\widehat{\psi}^{-1}\begin{bmatrix} y_0 \\ y \\ y_5 \end{bmatrix}\mathbb{R}$ of a point of $Q_{4,2}$ contains the vectors $v_1, \dots, v_4$. There is an r such that $\widehat{\varphi}(v_r\mathbb{R}) = \begin{bmatrix} y_0 \\ y \\ y_5 \end{bmatrix}\mathbb{R}$ $(r = 1, \dots, 4)$.*

Proof. Clearly $v_1, \dots, v_4 \in \widehat{\mathbb{H}}^2$ if $y_0, y_5 \in \mathbb{R}$ and $y \in \widehat{\mathbb{H}}$. As the proof of Theorem 1 shows, it is sufficient to consider $v_1, \dots, v_4$ if one of $y_0, y_5, y_1, y_2 \neq 0$. This is always the case, because $\begin{bmatrix} y_0 \\ y \\ y_5 \end{bmatrix} \in Q_{4,2}$ and $y_0 = y_5 = y_1 = y_2 = 0$ implies that $y_3 = y_4 = 0$. □

Lemma 15. *The complete $\widehat{\varphi}$-preimage of a point $\begin{bmatrix} y_0 \\ y \\ y_5 \end{bmatrix}\mathbb{R} \in Q_{4,2}$ consists of $\widehat{\psi}^{-1}(0)$ and a three-dimensional projective subspace $U \leq \widehat{P}^5$. Assume that $\widehat{\psi}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_0 \\ y \\ y_5 \end{bmatrix}\mathbb{R}$. If $y_0 = y_5 = 0$, then $U = \begin{bmatrix} a \\ 0 \end{bmatrix}\widehat{\mathbb{H}} + \begin{bmatrix} 0 \\ b \end{bmatrix}\widehat{\mathbb{H}}$, and $U = \begin{bmatrix} a \\ b \end{bmatrix}\widehat{\mathbb{H}}$ otherwise.*

Proof. This follows immediately from Lemma 9 by intersection of the $\widetilde{\psi}$-preimage with the subspace $\widehat{\mathbb{H}}^2 \leq_{\mathbb{R}} \widetilde{\mathbb{H}}^2$. □

Definition. *The three-dimensional projective subspaces mentioned in this lemma are called the fiber subspace of $\widehat{\varphi}$. Again we call points of $Q_{4,2}$ parallel if their span is contained in $Q_{4,2}$.*

The following is an immediate consequence of the respective result for the complex case $\widetilde{\mathbb{H}}$.

Lemma 16. *The fiber subspaces of non-parallel points are skew. The fiber subspaces of a plane of parallel points intersect in a projective line. For all planar sections c of $Q_{4,2}$ which are conics, there is a line $\widehat{c} \subset \widehat{P}^7$, such that $\widehat{\varphi}(\widehat{c}) = c$. The same is true for all lines of $Q_{4,2}$.*

The connected component of the identity in $\mathrm{GL}(2, \widehat{\mathbb{H}})$ is denoted by $\mathrm{GL}^+(2, \widehat{\mathbb{H}})$. Analogously to the case of $\mathrm{GL}(2, \widetilde{\mathbb{H}})$ we have

Lemma 17. *The Hopf mapping $\widehat{\varphi}$ provides an isomorphism*

$$\mathrm{GL}^+(2,\widehat{\mathbb{H}})/\mathbb{R}^\times \cong \mathrm{PGL}^+(Q_{4,2}).$$

We can show a bit more by using Theorem 2: If $\widehat{\kappa} \in \mathrm{PGL}(Q_{4,2})$, then there is a unique projective automorphism $\widetilde{\kappa}$ automorphic for $S^4(\mathbb{C})$ which extends $\widehat{\kappa}$ after embedding the real projective space $\widehat{P}^5$ into $P^5(\mathbb{C})$. The coordinate matrices of $\widehat{\kappa}$ and $\widetilde{\kappa}$ with respect to the same basis are the same. This is not a basis where the complex unit sphere has the equation $v^TEv = 1$, however. It has the same equation as $Q_{4,2}$, namely $v^TJv = 1$. We therefore consider the matrix group

$$\overline{\mathrm{PGL}}(S^4(\mathbb{C})) = \{A\mathbb{C} \mid A^TJA = \lambda J,\ \lambda \neq 0\}. \tag{42}$$

It is of course isomorphic to $\mathrm{PGL}(S^4(\mathbb{C}))$, and the equations $\det(A)/\lambda^3 = \pm 1$ define its two connected components. The component containing the identity is denoted by $\overline{\mathrm{PGL}}^+(S^4(\mathbb{C}))$. If $\widehat{\kappa}$ is in the subset '$\det(A)\lambda > 0$' of $\mathrm{PGL}(Q_{4,2})$ (which consists of two connected components), then $\widetilde{\kappa}$ is contained in $\overline{\mathrm{PGL}}^+(S^4(\mathbb{C}))$, and vice versa.

Theorem 4. *The Hopf mapping $\widehat{\varphi}$ provides an isomorphism*

$$\mathrm{GL}(2,\widehat{\mathbb{H}})/\mathbb{R}^\times \cong \{A\mathbb{R} \mid A^TA = \lambda J,\ \lambda\det(A) > 0\} \tag{43}$$

The right hand group consists of two of the four connected components of $\mathrm{PGL}(Q_{4,2})$.

Proof. An element $\widetilde{L} \in \mathrm{GL}(2,\widetilde{\mathbb{H}})$ is uniquely determined by its values on the basis $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$, so for each $\widehat{L} \in \mathrm{GL}(2,\widehat{\mathbb{H}})$ there is a unique $\widetilde{L} \in \mathrm{GL}(2,\widetilde{\mathbb{H}})$ with $\widetilde{L}|\widehat{\mathbb{H}}^2 = \widehat{L}$. Thus the $\mathbb{C}$-coordinate matrix of $\widetilde{L}$ with respect to the basis (36) has real entries. Conversely, if $\widetilde{L} \in \mathrm{GL}(2,\widetilde{\mathbb{H}})$, and its (8×8) $\mathbb{C}$-coordinate matrix with respect to the basis (36) is real, then $\widetilde{L}$ actually is contained in $\mathrm{GL}(2,\widehat{\mathbb{H}})$.

First, we do the following: For the sake of brevity, we write G for the right hand group in equation (43). As $\widetilde{L}$ induces (via $\widetilde{\varphi}$) a projective automorphism in $\overline{\mathrm{PGL}}^+(S^4(\mathbb{C}))$ which leaves $Q_{4,2}$ invariant, $\widehat{L}$ induces (via $\widehat{\varphi}$) a projective automorphism of $Q_{4,2}$. This shows that $\widehat{\varphi}(\mathrm{GL}(2,\widehat{\mathbb{H}})) \subset G$.

Second, we show the reverse inclusion: If $\widehat{\kappa} \in G$, then there is a unique projective automorphism $\widetilde{\kappa} \in \overline{\mathrm{PGL}}^+(S^4(\mathbb{C}))$ extending $\widehat{\kappa}$. By Theorem 2, there exists $\widetilde{L} \in \mathrm{GL}(2,\widetilde{\mathbb{H}})$ with $\widetilde{\varphi}_\star(\widetilde{L}) = \widetilde{\kappa}$. Whenever $\begin{bmatrix}a\\b\end{bmatrix} \in \widehat{\mathbb{H}}^2$, and $\widetilde{L}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}a'\\b'\end{bmatrix}$, then there are $\begin{bmatrix}a''\\b''\end{bmatrix} \in \widehat{\mathbb{H}}^2$ such that $\begin{bmatrix}a'\\b'\end{bmatrix}\widetilde{\mathbb{H}} = \begin{bmatrix}a''\\b''\end{bmatrix}\widetilde{\mathbb{H}}$.

As a $\mathbb{C}$-linear mapping, $\widetilde{L}$ permutes the set of subspaces of complex dimension 1 belonging to to $\widehat{\mathbb{H}}^2$, which are spanned by vectors with real coefficients. As the coordinate matrix of $\widetilde{L}$ as a $\mathbb{C}$-linear mapping can be recovered up to a complex factor from the images of nine subspaces, eight of which are $\mathbb{C}$-independent, there is a complex multiple $\alpha\widetilde{L}$ ($\alpha \in \mathbb{C}$) which has a *real* coordinate matrix as a $\mathbb{C}$-linear mapping with respect to the basis (36). This shows that $\alpha\widetilde{L} \in \mathrm{GL}(2,\widehat{\mathbb{H}})$, and obviously $\alpha\widetilde{L}$, like $\widetilde{L}$, induces (via $\widehat{\varphi}$) the original projective automorphism of $\widehat{\kappa} \in Q_{4,2}$. Thus, $\widehat{\varphi}(\mathrm{GL}(2,\widehat{\mathbb{H}})) \supset G$. From the first part of the proof we know that this inclusion is actually an equality.

Third, we compute the kernel of the mapping $\widehat{\varphi}_\star(L) = \widehat{\varphi}L\widehat{\varphi}^{-1}$. The condition $(L\begin{bmatrix} q_0 \\ q_1 \end{bmatrix})\widehat{\mathbb{H}} = \begin{bmatrix} q_0 \\ q_1 \end{bmatrix}\widehat{\mathbb{H}}$ for all $\begin{bmatrix} q_0 \\ q_1 \end{bmatrix}$ quickly leads to $L\begin{bmatrix} q_0 \\ q_1 \end{bmatrix} = \beta\begin{bmatrix} q_0 \\ q_1 \end{bmatrix}$ with $\beta \in \mathbb{R}$. This shows the statement of the theorem. □

Lemma 18. *The set $\widehat{\mathbb{H}}^2 \setminus \widehat{\psi}^{-1}(0)$ is connected.*

Proof. The preimage of 0 is of dimension less than seven. □

Lemma 19. *The right translation R_x for $x \in \widehat{\mathbb{H}}$ is an $\mathbb{R}$-linear endomorphism of the real vector space $\widehat{\mathbb{H}}$. We consider the linear mapping $R = [R_x\, R_y] : \widehat{\mathbb{H}}^2 \to \widehat{\mathbb{H}}$. As to the rank of R, there are the following three cases:*

(i) $\psi\begin{bmatrix} x \\ y \end{bmatrix} \neq 0 : \mathrm{rk}(R) = 4.$

(ii) $\psi\begin{bmatrix} x \\ y \end{bmatrix} = 0,\ (x,y) \neq (0,0) : \mathrm{rk}(R) = 2.$

(iii) $x = y = 0 : \mathrm{rk}(R) = 0.$

Proof. $\mathrm{rk}(R_x)$ equals four, or two, or zero, if $N(x) \neq 0$, or $N(x) = 0$, $x \neq 0$, or $x = 0$, respectively. Thus, $\mathrm{rk}(R)$ equals four if $N(x) \neq 0$ or $N(y) \neq 0$. In this case also $\psi\begin{bmatrix} x \\ y \end{bmatrix} \neq 0$.

If $N(x) = N(y) = 0$, but $x, y \neq 0$, then $\mathrm{rk}(R)$ depends on the position of the planes $R_x\widehat{\mathbb{H}}$ and $R_y\widehat{\mathbb{H}}$. In projective space, these are two generator lines of the quadric $N(\cdot) = 0$, which are skew (i.e., $\mathrm{rk}(R) = 4$) if and only if $x \notin \mathbb{R}_y\widehat{\mathbb{H}}$. This is equivalent to $x\overline{y} \neq 0$. We see that then $\mathrm{rk}(R) = 4$ if and only if $\psi\begin{bmatrix} x \\ y \end{bmatrix} \neq 0$.

If $x = 0$, then $\mathrm{rk}(R) = \mathrm{rk}(R_y)$, and if $y = 0$, then $\mathrm{rk}(R) = \mathrm{rk}(R_x)$. □

We temporarily denote the connected component of the identity in $\mathrm{GL}(2, \widehat{\mathbb{H}})$ with G_0, and $\widehat{\mathbb{H}}^2 \setminus \widehat{\psi}^{-1}(0)$ with the letter X. Then there is the following lemma:

Lemma 20. *G_0 operates transitively on X.*

Proof. The orbit of $\begin{bmatrix} x \\ y \end{bmatrix}$ is the image of the mapping $G_0 \to X$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. This mapping is the restriction of an $\mathbb{R}$-linear mapping to G_0. In terms of right translations R_x and R_y, we may write the first coordinate of the image in the form $[R_x, R_y]\begin{bmatrix} a \\ b \end{bmatrix}$, with the block matrix $R = [R_x, R_y]$ already mentioned in Lemma 19. An analogous statement holds for the second coordinate. Lemma 19 shows that $\mathrm{rk}(R) = 4$, so the rank of the original mapping equals eight. This implies that $G_0\begin{bmatrix} x \\ y \end{bmatrix}$ is open in X.

If two orbits $G_0\begin{bmatrix} x \\ y \end{bmatrix}$ and $G_0\begin{bmatrix} x' \\ y' \end{bmatrix}$ intersect, then they coincide, because G_0 is a group. One orbit is the complement of all the others, whose union is open. So all orbits are closed. As X is connected it follows that there is only one orbit. □

Theorem 5. *The group* $\mathrm{GL}(2, \widehat{\mathbb{H}})$ *has two connected components.*

Proof. $\mathrm{GL}(2, \widehat{\mathbb{H}})$ has at least two components, because factorization with respect to $\mathbb{R}^\times$ produces a group with two connected components. To show the reverse inequality, we use the notation of the previous lemmas. The stabilizer g' of $\begin{bmatrix}1\\0\end{bmatrix}$ in $\mathrm{GL}(2, \widehat{\mathbb{H}})$ consists of the matrices

$$\left\{\begin{bmatrix}1 & b\\ 0 & d\end{bmatrix} \mid b, d \in \widehat{\mathbb{H}},\ N(d) \neq 0\right\}. \tag{44}$$

and has the two connected components $N(d) > 0$ and $N(d) < 0$. As G_0 operates transitively in X, all stabilizers are homeomorphic. If $g \in \mathrm{GL}(2, \widehat{\mathbb{H}})$, choose a $h \in G_0$ with $h\begin{bmatrix}1\\0\end{bmatrix} = g\begin{bmatrix}1\\0\end{bmatrix}$. Further, choose path h_t in G_0 with $h_0 = \mathrm{id}$ and $h_1 = h$. Then $h_t^{-1}g$ is a path which connects g with $h^{-1}g \in G'$. It follows that $\mathrm{GL}(2, \widehat{\mathbb{H}})$ has not more connected components than G', i.e., two. □

The quadric $Q_{4,2}$ is nothing but the Klein quadric or the Grassmann manifold $G_{3,1} = P_{\mathbb{R}}(\Lambda^2\mathbb{R}^4)$, which is a model for the lines of projective three-space: A line which spans the points $a\mathbb{R}$, $b\mathbb{R}$ with $a, b \in \mathbb{R}^4$ has the homogeneous Plücker coordinates

$$x_{ij} = \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}, \tag{45}$$

which depend only on the span of $a\mathbb{R}$, $b\mathbb{R}$, and fulfill the relations

$$x_{ii} = 0, \quad x_{ij} = -x_{ji}, \quad x_{01}x_{23} + x_{02}x_{31} + x_{13}x_{12} = 0. \tag{46}$$

The substitution

$$\begin{aligned} y_0 &= x_{01} + x_{23}, & y_3 &= x_{02} + x_{31}, & y_4 &= x_{03} + x_{12}, \\ y_1 &= x_{01} - x_{23}, & y_2 &= x_{02} - x_{31}, & y_5 &= x_{03} - x_{12} \end{aligned} \tag{47}$$

yields a projective isomorphism $G_{3,1} \cong Q_{4,2}$. The projective automorphisms of the Grassmannian are in one-to-one correspondence with the projective automorphisms of the dual pair $(P^3, P^{3\star})$, consisting of the four following connected components: As P^3 is orientable, there are orientation-preserving and orientation-reversing projective automorphisms. Further, there are projective automorphisms which map P^3 to P^3 and such ones which interchange P^3 and its dual $P^{3\star}$.

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