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# Groups with Root-System of Type $BC_\ell$

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## 1. Introduction

Let $\mathcal{B}$ be an irreducible, spherical Moufang building of rank $\ell \geq 2$, $\mathcal{A}$ an apartment of $\mathcal{B}$ and $\Phi$ the set of roots (half-apartments) of $\mathcal{A}$ with corresponding root-subgroups $A_r, r \in \Phi$ in the sense of Tits. Then we call $G = \langle A_r \mid r \in \Phi \rangle \leq \mathrm{Aut}(\mathcal{B})$ the group of *Lie-type* $\mathcal{B}$. The notion of a group of Lie-type $\mathcal{B}$ is very general, since it includes:

- simple classical groups over division rings of finite Witt-index $\ell \geq 2$,
- simple algebraic groups over arbitrary fields of relative rank $\ell \geq 2$,
- the finite simple groups of Lie-type of rank $\ell \geq 2$.

The theory of such groups of Lie-type $\mathcal{B}$ was developed in [8], see also [3, I §4 and II §5]. In particular it was shown that one can enlarge $\Phi$ to some possibly nonreduced root-system $\widetilde{\Phi}$ ($\Phi \neq \widetilde{\Phi}$ only if $\Phi$ is of type $B_\ell$ and $\widetilde{\Phi}$ of type $BC_\ell$ or $\Phi$ is of type $I_2(8)$ and $\widetilde{\Phi}$ of type ${}^2F_4$; for the latter see [9, (5.4)]) such that the $A_r, r \in \widetilde{\Phi}$, satisfy:

(1) $X_r = \langle A_r, A_{-r} \rangle$ is a rank one group with unipotent subgroups $A_r$ and $A_{-r}$ for $r \in \widetilde{\Phi}$. (For definition of a rank one group see [3, I].). Further $A_{2r} \leq A_r$ if also $2r \in \widetilde{\Phi}$.

(2) If $r, s \in \widetilde{\Phi}$ with $s \neq -r$ and $-2r$, then

$$[A_r, A_s] \leq \langle A_{\lambda r + \mu s} \mid \lambda r + \mu s \in \widetilde{\Phi} \text{ and } \lambda, \mu \in \mathbb{N} \rangle$$

(We use the convention $\langle \emptyset \rangle = 1$. Hence (2) implies $A_r' = 1$ if $2r \notin \widetilde{\Phi}$ and $A_r' \leq A_{2r} \leq Z(A_r)$ if $2r \in \widetilde{\Phi}$!)

Now it would be desirable to prove the converse. That is to show that, if $G$ is a group generated by nonidentity subgroups $A_r, r \in \widetilde{\Phi}$ and $\widetilde{\Phi}$ as above, satisfying (1) and (2), then

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either $G$ has a proper central factor (also of Lie-type) or $G$ is a perfect central extension of a group of Lie-type $\mathcal{B}$ (with same $\widetilde{\Phi}$). Notice that the first possibility always occurs. If for example $[A_r, A_s] = 1$ for all $r, s$ in (2), then $G$ is a central product of the rank one groups $X_r, r \in \Phi$ (which will be considered as groups of Lie-type of rank one).

Now this problem has been solved already to a large extent. First it was shown in [4], that if always equality holds in (2), then indeed $G$ is a perfect central extension of a group of Lie-type $\mathcal{B}$. Next in [5] the special case, when $\Phi$ has only single bonds (i.e. $\Phi$ of type $A_\ell, D_\ell$ or $E_\ell$) was considered. Finally in [6] we treated the case when $\Phi = \widetilde{\Phi}$ is of type $B_\ell, C_\ell$ or $F_4$ and the "characteristic" is different from 2. (The special case $\Phi = \widetilde{\Phi} = B_2 = C_2$ has been treated in [2]. So apart from the special cases $\Phi = \widetilde{\Phi} = G_2$ and $\Phi = I_2(8)$ and $\widetilde{\Phi} = {}^2F_4$, it just remains to treat the case $\widetilde{\Phi} = BC_\ell$, which corresponds to unitary groups which are not of maximal Witt-Index, of the above problem, which will be the purpose of this paper. (For a survey of these results and also Curtis-Tits type presentations of Lie-type groups see [7].)

To state our Main-theorem we need some notation:
If $\Psi$ is a root-system as above (i.e. $\Psi$ is of type $A_\ell, B_\ell, C_\ell, BC_\ell, D_\ell, E_\ell, G_2, F_4$ or ${}^2F_4$) and $G$ is a group generated by subgroups $A_r \neq 1, r \in \Psi$, satisfying (1) and (2), then we say that $G$ is of *type* $\Psi$, if there exists a surjective homomorphism $\varphi : G \to \overline{G}$, where $\overline{G}$ is a group of Lie-type $\mathcal{B}$, with $\ker \varphi \leq Z(G)$ mapping the $A_r$, $r \in \Psi$ with $r \neq 2s$ for all $s \in \Psi$, onto the root-subgroups corresponding to the roots of some apartment $\mathcal{A}$ of $\mathcal{B}$. (The complication $r \neq 2s$ only plays a role if $\Psi$ is of type $BC_\ell$ or ${}^2F_4$. In the latter cases roots of the form $2s$ are not roots (i. e. halfapartments) of $\mathcal{A}$.) If $\Delta \subseteq \Psi$ we set $G(\Delta) := \langle X_r \mid r \in \Delta \rangle$. If $\Delta$ carries the structure of a root-system (also denoted by $\Delta$), then we say $G(\Delta)$ is of type $\Delta$ if it satisfies the above conditions with respect to $\Delta$. With this notation we have:

**Main-theorem.** *Suppose $\Phi$ is a root-system of type $BC_\ell, \ell \geq 2$ and $G$ is a group generated by subgroups $A_r \neq 1$, $r \in \Phi$, satisfying* (1) *and* (2)*. Then one of the following holds:*

(a) *Always equality holds in* (2)*. In this case $G$ is perfect and of type $BC_\ell$.*

(b) *$A_r = A_{2r}$ for all $r \in \Phi$ with $2r \in \Phi$, $\Phi_0 = \{2r \mid r, 2r \in \Phi\} \cup \{s \mid s \in \Phi, 2s \notin \Phi\}$ is a root system of type $C_\ell$ and equality holds in* (2) *for all $r, s \in \Phi_0$ with $s \neq -r$. In this case $G$ is perfect and of type $C_\ell$.*

(c) *$\Phi = J \dot\cup K$ with $J \neq \emptyset \neq K$ and either $J = \{\pm r\}$ resp. $J = \{\pm r, \pm 2r\}$ or $J$ carries the structure of an irreducible root system $\Psi$ of rank $r \geq 2$. Moreover $G = G(J) * G(K)$ and $G(J)$ is of type $\Psi$ resp. $G(J) = X_r$ is a rank one group.*

(d) *$J' = \{r \in \Phi \mid A_r$ is an elementary abelian 2-group $\} \neq \emptyset$. Let $J = J' \cup \{s \in \Phi \mid 2s \in J'\}$ and $K = \Phi - J$. Then $G = G(J) * G(K)$ and $A_s^2 = \langle a^2 \mid a \in A_s \rangle \leq A_s' \leq A_{2s}$ for all $s \in J - J'$.*

Notice that if $\Phi$ is a root-system of type $BC_\ell$ then in any case $\Phi_0 = \{2r \mid r, 2r \in \Phi\} \cup \{s \mid s \in \Phi, 2s \notin \Phi\}$ is a root-system of type $C_\ell$. Now the proof of the above theorem proceeds by discussing the possibilities obtained for $G(\Phi_0)$ in §3 of [6].

Obviously the case of groups with root-system $\Phi_0$ of type $C_\ell$ is included in our Main-theorem, since one can enlarge $\Phi_0$ to a root-system $\Phi$ of type $BC_\ell$ and then simply sets $A_r = A_{2r}$, if

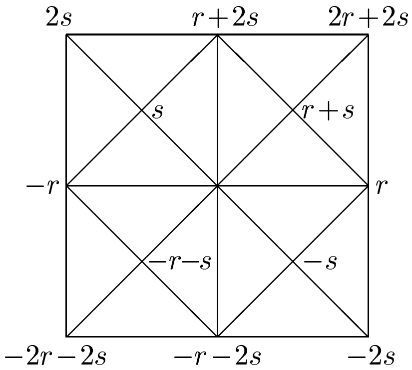
$r, 2r \in \Phi$. (But of course for the proof of the Main-theorem the treatment of groups with root-system of type $C_\ell$ in [6] is used.) The case of groups with root-system of type $B_\ell$ is not included in the statement of the Main-theorem, since we demand $A_{2r} \neq 1$ if $r, 2r \in \Phi$. (If $A_{2r} = 1$ for all $r \in \Phi$ with $2r \in \Phi$, then $G$ has root-system of type $B_\ell$, whence we can apply [6].)

For definition of a root-system of type $BC_\ell$ see [1]. (We will give a short description in the beginning of Section 3.)

## 2. $BC_2$

Let in this section

$\Phi =$




be a root system of type $BC_2$ and

$$\Phi_0 = \{\pm r, \pm 2s, \pm(r+2s), \pm(2r+2s)\} \text{ the subsystem of type } C_2.$$

Let $G = G(\Phi) = \langle A_r \mid r \in \Phi \rangle$ be a group, satisfying (1) and (2) of Section 1 and $G_0 = G(\Phi_0)$. We fix the following notation:

For $\alpha \in \Phi$ let $H_\alpha = N_{X_\alpha}(A_\alpha) \cap N_{X_\alpha}(A_{-\alpha})$ and pick $n_\alpha \in X_\alpha$ with $A_\alpha^{n_\alpha} = A_{-\alpha}$ and $n_\alpha^2 \in H_\alpha$. Then

$$H_\alpha n_\alpha = \{x \in X_\alpha \mid A_\alpha^x = A_{-\alpha}, A_{-\alpha}^x = A_\alpha\}.$$

Further, if $2\alpha \in \Phi$, then $X_{2\alpha} \leq X_\alpha$, $H_{2\alpha} \leq H_\alpha$ and $n_{2\alpha} \in H_\alpha n_\alpha$ all by [3, I §1]. Hence in this situation we may and will pick $n_\alpha$ such that $n_\alpha = n_{2\alpha}$. Let $U_\alpha = \langle A_\beta \mid \beta \in \Phi$ is between $\alpha$ and $-\alpha$ in clockwise sense $\rangle$. (For example $U_{-r} = \langle A_s, A_{r+2s}, A_{r+s} \rangle$ as $A_{2s} \leq A_s$)

If $\alpha \in \Phi_0$ let:

$$V_\alpha := \langle A_\beta \mid \beta \in \Phi_0 \text{ is between } \alpha \text{ and } -\alpha \text{ in clockwise sense } \rangle.$$

Then $V_{-r} = \langle A_{2s}, A_{r+2s}, A_{2r+2s} \rangle \leq U_{-r}$. Notice that by [4, (2.1), (2.2)] we have:

**2.0.** The following hold:

(1) $X_\alpha$ normalizes $U_\alpha$ and $U_{-\alpha}$.

(2) $A_\alpha U_\alpha$ and $A_{-\alpha} U_\alpha$ are nilpotent.

(3) $A_\alpha \cap U_\alpha = 1 = A_{-\alpha} \cap U_\alpha$.

(4) If $\alpha \neq 2\beta$ for all $\beta \in \Phi$, then $\langle U_\alpha, U_{-\alpha} \rangle \trianglelefteq G$ and $G = \langle U_\alpha, U_{-\alpha} \rangle X_\alpha$.

For the convenience of the reader we state the main result of [2] and a corollary obtained in [6, (2.8)] from it.

**2.1 Proposition.** *For $G_0 = G(\Phi_0)$ one of the following holds:*

(1) *All $A_\alpha, \alpha \in \Phi_0$, are elementary abelian 2-groups.*

(2) $G_0 = X_r * X_{2s} * X_{r+2s} * X_{2r+2s}$.

(3) *There exists a long root $\alpha \in \Phi_0$, such that for $\Delta = \Phi_0 - \{\pm\alpha\}$ we have $G_0 = X_\alpha * G(\Delta)$ and $G(\Delta)$ is of type $A_2$. I.e. $\Delta = \{\pm\beta, \pm\gamma, \pm(\beta+\gamma)\}$ and for all $\sigma, \tau \in \Delta$ with $\sigma + \tau \in \Delta$ we have*

$$[A_\sigma, A_\tau] = A_{\sigma+\tau} \text{ and } A_\sigma^{n_\tau} = A_{\sigma+\tau} = A_\tau^{n_\sigma}.$$

(4) *For all $\alpha, \beta \in \Phi_0$ with $\beta \neq -\alpha$ we have*

$$[A_\alpha, A_\beta] = \langle A_{\lambda\alpha+\mu\beta} \mid \lambda\alpha + \mu\beta \in \Phi_0; \lambda, \mu \in \mathbb{N}\rangle.$$

*Moreover $G_0$ is of type $C_2$.*

We now describe the possibilities for $G$ over a series of Lemmata.

**2.2 Lemma.** *Suppose $[A_s, A_{r+s}] = 1$ and possibility (4) of Proposition 2.1 holds for $G_0$. Suppose further that some $A_\alpha, \alpha \in \Phi_0$, is not an elementary abelian 2-group. Then we have $A_\beta = A_{2\beta}$ for all $\beta \in \Phi$ with $2\beta \in \Phi$. Moreover $G = G_0$ is of type $C_2$.*

*Proof.* Since $G_0$ is of type $C_2$ clearly all $A_\alpha, \alpha \in \Phi_0$ are not elementary abelian 2-groups.
Consider the action of $X_s$ on $\widetilde{U}_s = U_s/A_{2r+2s}$. Then $[A_s, \widetilde{A}_r] \leq \widetilde{A}_{r+s}\widetilde{A}_{r+2s} \leq C_{\widetilde{U}_s}(A_s)$. Hence $A_s' \leq C_{A_{2s}}(\widetilde{U}_s) = 1$ by the 3-subgroup lemma and since $G_0$ is of type $C_2$. With the same argument we also obtain $A_{r+s}' = 1$. Hence

$$U_{-r}' \leq A_s'[A_s, A_{r+s}]A_{r+s}' = 1,$$

since $V_{-r} \leq Z(U_{-r})$ by the commutator relations of §1 (2).
Since $A_{-s}' = 1$ we obtain similarly

$$U_{r+2s}' \leq A_{r+s}'[A_{r+s}, A_{-s}]A_{-s}' \leq A_r.$$

But $U_{r+2s}'$ is invariant under $X_{r+2s}$. Hence we obtain

$$U_{r+2s}' \leq C_{A_r}(X_{r+2s}) \leq Z(V_{2s}).$$

But by [2, (3.12)] we have either $[A_r, A_{r+2s}] = 1$ or $C_{A_r}(A_{r+2s}) = 1$. Since the first possibility contradicts our hypothesis that (4) of Proposition 2.1 holds, this shows $U_{r+2s}' = 1$. Now the same arguments imply $U_r' = 1 = U_{-r-2s}'$.
With a repeated application of (3) from Subsection 2.0 we obtain $\widetilde{U}_s = \widetilde{A}_{r+2s} \oplus \widetilde{A}_{r+s} \oplus \widetilde{A}_r$. Moreover, by [2, (3.3)], $C_{A_{2s}}(\widetilde{a}) = 1 = C_{A_{-2s}}(\widetilde{b})$ for all $1 \neq \widetilde{a} \in \widetilde{A}_r, 1 \neq \widetilde{b} \in \widetilde{A}_{r+2s}$. This implies $A_{r+s} = C_{U_s}(X_s)$. Since $[\widetilde{U}_s, A_s, A_s] = 1$, [3, I(2.5)] shows that $X_s$ is a special rank one

group. Now for each $a \in A_s^\#$ pick $b(a) \in A_{-s}^\#$ such that $a^{b(a)} = b(a)^{-a}$ and let by [3, I(5.6)] $X_s(a) = \langle A_s(a), A_{-s}(b(a))\rangle \le X_s$ such that $X_s(a)$ is a perfect central extension of $PSL_2(k)$, $k$ a primefield and $A_s(a) \le A_s$ and $A_{-s}(b(a)) \le A_{-s}$ are unipotent subgroups of $X_s(a)$. Set $n = n(a) = ab(a)^{-1}a$. Then the proof of [3, I(3.5)] shows that $n^2 \in Z(X_s(a))$ and $n^2$ inverts $\widetilde{U}_s/\widetilde{A}_{r+s}$. In particular, since by the hypothesis of Lemma 2.2 $\mathrm{Char}\, k \neq 2$, we have $n^2 \neq 1$. Clearly $n^2 \in H_s$ by [3, I(2.7)]. Hence $[n^2, A_s] \le C_{A_s}(\widetilde{U}_s/\widetilde{A}_{r+s}) = 1$, since we assume that (4) of Proposition 2.1 holds. We obtain $[n^2, X_s] = 1$ and thus $n^2$ normalizes $\widetilde{A}_r = [\widetilde{U}_s, A_{-2s}]$ and $\widetilde{A}_{r+2s} = [\widetilde{U}_s, A_{2s}]$.

This shows that $n^2$ centralizes $\widetilde{A}_{r+s}$ and inverts $\widetilde{A}_{r+2s}\widetilde{A}_r = [\widetilde{U}_s, n^2]$. In particular $\widetilde{A}_{r+2s}\widetilde{A}_r$ is $X_s$-invariant, as $X_s \le C(n^2)$. Hence $X_s \le N(A_{r+2s}A_rA_{2r+2s})$. The same argument also shows that $X_s \le N(A_{-r}A_{-2r-2s}A_{-r-2s})$. Now by Theorem 2 of [4] $G_0$ is quasisimple and by (4) of Proposition 2.1 $\langle V_{2s}, V_{-2s}\rangle \trianglelefteq G_0$. Hence $G_0 = \langle V_{2s}, V_{-2s}\rangle$ is normalized by $X_s$. This implies $X_{2s} \le G_0 \cap X_s \trianglelefteq X_s$.

Now, since $G_0$ is of type $C_2$, $P_0 = N_{G_0}(V_{2s}) = V_{2s}X_{2s}H_0$, $H_0 = \langle H_\alpha \mid \alpha \in \Phi_0\rangle$ is a maximal parabolic subgroup of $G_0$ and $X_s \le N(P_0)$. Hence $A_s$ normalizes $V_{2s}X_{2s} = \langle (V_{2s}A_{2s})^{P_0}\rangle$ and also $A_{-s} \le N(V_{2s}X_{2s})$. This implies $X_s \le N(V_{2s}X_{2s})$.

Now

$$X_{2s} \le X_s \cap V_{2s}X_{2s} = X_{2s}(V_{2s} \cap X_s) \trianglelefteq X_s \text{ and } V_{2s} \cap X_s \lhd X_s.$$

Hence by [3, I(1.10)] $V_{2s} \cap X_s \le Z(X_s)$. Thus

$$V_{2s} \cap X_s \le C_{V_{2s}}(X_s) \le A_{2r+2s},$$

since we assume that (4) of Proposition 2.1 holds for $G_0$. Suppose $V_{2s} \cap X_s \neq 1$. Then also $V_{-2s} \cap X_s \neq 1$, since $V_{2s}^{n_s} = V_{-2s}$. Now $[X_s, X_{2r+2s}] = 1$ and thus also $V_{-2s} \cap X_s \le A_{2r+2s}$, which is obviously impossible since $A_{2r+2s} \cap A_{-2r-2s} = 1$ and

$$V_{-2s} \cap X_s = (V_{2s} \cap X_s)^{n_s} \le C_{V_{2s}}(X_s)^{n_s} = C_{V_{-2s}}(X_s) \le A_{-2r-2s}.$$

This shows $V_{2s} \cap X_s = 1$ and thus $X_{2s} \trianglelefteq X_s$. Hence by [3, I(1.10)] $X_s = X_{2s}A_s$. We obtain

$$A_{2s}^{X_s} = A_{2s}^{A_sX_{2s}} = A_{2s}^{X_{2s}} = A_{2s} \cup \{A_{-2s}^{A_{2s}}\}$$

and also $A_{-2s}^{X_s} = A_{2s} \cup \{A_{-2s}^{A_{2s}}\}$. Now pick $a \in A_s - A_{2s}$. Then there exists an $y \in A_{2s}$ with $A_{-2s}^a = A_{-2s}^y$. Hence $ay^{-1} \in N_{A_s}(A_{-2s}) = N_{A_s}(A_{-s}) = 1$ and $a = y \in A_{2s}$.

This shows $A_s = A_{2s}$. Since we have shown that $U'_{r+2s} = U'_r = U'_{-r-2s} = 1$, the same argument implies $A_\alpha = A_{2\alpha}$ for all $\alpha \in \Phi$ with $2\alpha \in \Phi_0$, which proves Lemma 2.2. □

**2.3 Lemma.** *Suppose that all $A_\alpha$, $\alpha \in \Phi_0$, are elementary abelian 2-groups. Then one of the following holds:*

(1) *$X_s \lhd G$ and $G = X_s * C(X_s)$ with $X_\beta \le C(X_s)$ for all $\beta \in \Phi - \{\pm s, \pm 2s\}$.*

(2) *$A_s^2 \le A'_s \le A_{2s}$. In particular $A_s$ is a 2-group.*

*Proof.* Suppose (1) does not hold. Let $\widetilde{U}_s = U_s/A_{2r+2s}$. Then by [4, (2.6)] $[\widetilde{V}_{2s}, X_{2s}] = \widetilde{V}_{2s}$.

Now, because of $[A_{r+s}, A_s, A_s] \leq [A_{r+2s}, A_s] = 1$, we have $[A_{r+s}, A_s^2] \leq A_{r+2s}^2 = 1$. Let $1 \neq v \in A_r$. Then for each $a \in A_s^2$ we have

$$[\widetilde{v}, a^2] = [\widetilde{v}, a]^2 = [\widetilde{v}^2, a] = 1.$$

Hence $(A_s^2)^2$ centralizes $\widetilde{U}_s$. Suppose there exists an element $1 \neq a \in (A_s^2)^2$. Let $Y = \langle a^{X_s} \rangle$. Then by [3, I(2.13)(10) and (1.10)] $X_s = YA_s$ and thus $[\widetilde{U}_s, X_s] \leq [\widetilde{U}_s, A_s] \leq \widetilde{A}_{r+s}\widetilde{A}_{r+2s}$, a contradiction to $\widetilde{V}_s \leq [\widetilde{U}_s, X_{2s}]$.

This shows $(A_s^2)^2 = 1$. Hence $A_s^2$ and $A_s/A_s^2$ are elementary abelian 2-groups and thus (2) holds. □

**2.4 Lemma.** *Suppose* (4) *of Proposition* 2.1 *holds and some* $A_\alpha$, $\alpha \in \Phi_0$, *is not an elementary abelian* 2*-group. Then the following are equivalent:*

(1) $[A_s, A_{r+s}] \neq 1$

(2) $A_s' \neq 1$ *(resp.* $A_{r+s}' \neq 1$*)*

(3) $U_{-r}' = A_{2s}A_{r+2s}A_{2r+2s}$.

*Proof.* Because of $A_{2s}A_{r+2s}A_{2r+2s} \leq Z(U_{-s})$ we have $U_{-r}' = A_s'[A_s, A_{r+s}]A_{r+s}'$.

Suppose that (1) holds. Then $U_{-r}' \neq 1$. Assume $U_{-r}' \leq A_{r+2s}$. Then $[U_{-r}', A_r] \leq [A_{r+2s}, A_r] \cap U_{-r}' \leq A_{2r+2s} \cap U_{-r}' = 1$. By [2, (3.12)] applied to $G_0$ this implies $[A_{r+2s}, A_r] = 1$, a contradiction to our hypothesis that (4) of Proposition 2.1 holds.

This shows that $U_{-r}' \not\leq A_{r+2s}$ and thus, since $U_{-r}'$ is invariant under $X_r$, also $U_{-r}' \not\leq A_{r+2s}A_{2r+2s}$. (Otherwise $U_{-r}' \leq A_{r+s}A_{2r+2s} \cap (A_{r+s}A_{2r+2s})^{n_r} = A_{r+s}A_{2r+2s} \cap A_{r+s}A_{2s} = A_{r+s}$ since $G_0$ is of type $C_2$ and thus $A_\beta^{n_r} = A_{\beta w_r}$ for all $\beta \in \Phi_0$.) Now pick $x \in U_{-r}' - A_{r+2s}A_{2r+2s}$. Then by [4, (2.4)] $[x, A_r]A_{2r+2s} = A_{r+2s}A_{2r+2s}$ and thus $A_{r+2s}A_{2r+2s} \leq U_{-r}'A_{2r+2s}$. Because of

$$A_{2r+2s} = [A_{r+2s}, A_r] \leq [U_{-r}', A_r] \leq U_{-r}'$$

by (4) of Proposition 2.1 we obtain $A_{r+2s}A_{2r+2s} \leq U_{-r}'$. Now applying $n_r$ to this inequality this shows that (3) holds.

If now $1 \neq A_s' \leq U_{-r}' \cap A_{2s}$, then picking $1 \neq x \in A_s'$ it follows as above that (3) holds. Since of course (3) implies (2) and (1) this proves Lemma 2.4. □

**2.5 Corollary.** *Suppose that* (4) *of Proposition* 2.1 *holds and some* $A_\alpha$, $\alpha \in \Phi_0$, *is not an elementary abelian* 2*-group. Then one of the following holds:*

(1) $A_\beta = A_{2\beta}$ *for all* $\beta \in \Phi$ *with* $2\beta \in \Phi$*. Further* $G = G_0$ *is of type* $C_2$*.*

(2) $A_\beta' = A_{2\beta}$ *for all* $\beta \in \Phi$ *with* $2\beta \in \Phi$*. Further* $U_\alpha' = V_\alpha$ *for all short roots* $\alpha \in \Phi_0$*.*

*Proof.* If $[A_{\epsilon s}, A_{\mu(r+s)}] = 1$ for some $\epsilon = \pm 1$ and $\mu = \pm 1$, then Lemma 2.2 shows that (1) holds. So we may assume that $[A_{\epsilon s}, A_{\mu(r+s)}] \neq 1$ for all choices of $\epsilon = \pm 1$ and $\mu = \pm 1$. Hence by Lemma 2.4 we obtain $U_{\epsilon r}' = V_{\epsilon r}$ and $U_{\epsilon(r+2s)}' = V_{\epsilon(r+2s)}$ for $\epsilon = \pm 1$. As $U_{-r}' = A_s'[A_r, A_{r+s}]A_{r+s}'$ again Lemma 2.4 shows that $A_s' = A_{2s}$. With symmetry this shows that (2) holds. □

Now we are able to show:

**2.6 Proposition.** *One of the following holds:*

(1) *There exists an $\alpha \in \Phi$ with $2\alpha \in \Phi_0$ such that $X_\alpha \trianglelefteq G$ and $X_\beta \leq C(X_\alpha)$ for all $\beta \in \Phi - \{\pm\alpha, \pm 2\alpha\}$.*
(2) *All $A_\alpha, \alpha \in \Phi_0$, are elementary abelian 2-groups and $A_\beta^2 \leq A'_\beta \leq A_{2\beta}$ for all $\beta \in \Phi - \Phi_0$.*
(3) *$A_\beta = A_{2\beta}$ for all $\beta \in \Phi - \Phi_0$. Moreover (4) of Proposition 2.1 holds and $G = G_0$ is of type $C_2$.*
(4) *For all $\alpha, \beta \in \Phi$ with $\beta \neq -\alpha$, $-2\alpha$ we have*

$$(*) \qquad [A_\alpha, A_\beta] = \langle A_{i\alpha + j\beta} \mid i\alpha + j\beta \in \Phi; i, j \in \mathbb{N}\rangle.$$

*Moreover $G$ is of type $BC_2$.*

*Proof.* $G_0$ satisfies one of the cases of Proposition 2.1. If (1) of Proposition 2.1 holds, then by Lemma 2.3 and symmetry either (1) or (2) of Proposition 2.6 holds. So we may assume that some $A_\beta, \beta \in \Phi_0$ is not an elementary abelian 2-group. If now $X_{2\alpha} \triangleleft G_0$ for some $\alpha \in \Phi - \Phi_0$, then by [4, (2.6)] $X_\alpha \triangleleft G$ and also (1) holds. So we may by Proposition 2.1 assume that $G_0$ satisfies (4) of Proposition 2.1. Hence the hypothesis of Corollary 2.5 is satisfied. Now case (1) of Corollary 2.5 is case (3) of Proposition 2.6. Thus we may, to prove Proposition 2.6, assume that we are in case (2) of Corollary 2.5 and then show that (4)(∗) holds. (If this is the case then $G$ is of type $BC_2$ by Theorem 2 of [4] as mentioned in the introduction.)
Now by case (2) of Corollary 2.5 it just remains to show that

$$[A_r, A_s] = A_{r+s}A_{r+2s}A_{2r+2s}$$

(and the symmetric equations, applying symmetries of $\Phi$) hold. For this consider the action of $X_r$ on $\overline{U}_{-r} = U_{-r}/V_{-r}$. By (3) of Subsection 2.0 we have $A_{r+s} \cap A_sA_{r+2s} = 1$. This implies $A_{r+s} \cap A_sA_{r+2s}A_{2r+2s} = A_{2r+2s}$. Whence multiplying this equation by $V_{-r}$ we obtain:

$$A_{r+s}V_{-r} \cap A_sV_{-r} = V_{-r},$$

since $V_{-r} \leq A_sA_{r+2s}A_{2r+2s}$. This shows $\overline{U}_{-r} = \overline{A}_{r+s} \oplus \overline{A}_s$.
Now $[A_s, A_{-2r-2s}] = 1$ and thus also $[A_s^{n_r}, A_{-2s}] = 1$. ($G_0$ is of type $C_2$) Since by [3, I(2.13)(10)] $\langle x, A_{-2s}\rangle U_s/U_s$ is not nilpotent for each $1 \neq x \in A_sU_s - U_s$ we obtain

$$A_s^{n_r} \leq U_{-r} \cap U_s = U_{-r} \cap A_rA_{r+s}A_{r+2s} = (U_{-r} \cap A_r)A_{r+s}A_{r+2s} = A_{r+s}A_{r+2s},$$

by (3) of Subsection 2.0 and since $A_s^{n_r} \leq U_{-r} \leq A_sU_s$. Hence $\overline{A}_s^{n_r} \leq \overline{A}_{r+s}$ and, by the same argument, $\overline{A}_{r+s}^{n_r} \leq \overline{A}_s$ for all $n_r \in X_r$ interchanging $A_r$ and $A_{-r}$. Applying $n_r^{-1}$ we obtain $\overline{A}_s^{n_r} = \overline{A}_{r+s}$ and $\overline{A}_{r+s}^{n_r} = \overline{A}_s$.
On the other hand, clearly $[\overline{A}_s, A_r] \leq \overline{A}_{r+s}$ and $\overline{A}_s[\overline{A}_s, A_r]$ is $X_r$ invariant. Hence $\overline{A}_{r+s} \leq \overline{A}_s[\overline{A}_s, A_r]$ and thus $[\overline{A}_s, A_r] = \overline{A}_{r+s}$.
We have shown $[A_s, A_r]A_{r+2s}A_{2r+2s} = A_{r+s}A_{r+2s}$. Since

$$A_{r+2s} = [A_s, A_{r+s}] = [A_r, A_s, A_s] \leq [A_r, A_s]$$

and

$$A_{2r+2s} = A'_{r+s} = (A_{r+s}A_{r+2s})' = ([A_s, A_r]A_{2r+2s})' = [A_s, A_r]',$$

it follows that $[A_s, A_r] = A_{r+s}A_{r+2s}$. □

**2.7 Corollary.** *Suppose case* (1) *of Proposition* (2.6) *holds and no* $A_r, r \in \Phi_0$, *is an elementary abelian* 2*-group. Let* $\Psi = \Phi - \{\pm\alpha, \pm 2\alpha\}$. *Then we get the following possibilities for* $G(\Psi)$.

(1) $G(\Psi)$ *is a central product of rank one groups.*

(2) *If without loss* $\alpha = s$, *then* $A_{r+s} = A_{2r+2s}$ *and* $G(\Psi) = G(\Psi \cap \Phi_0)$ *is of type* $A_2$.

*Proof.* Assume without loss $\alpha = s$. Then by Proposition 2.1 we have the following possibilities for $G_0$:

(a) $G_0 = X_{2s} * X_{r+2s} * X_{2r+2s} * X_r$.

(b) Let $\Psi_0 = \Psi \cap \Phi_0$. Then $G_0 = X_{2s} * G(\Psi_0)$ and $G(\Psi_0)$ is of type $A_2$.

We will show that in case (a) (1) and in case (b) (2) holds.

In case (a) we have $X_{r+s} = \langle X_{2r+2s}, A_{r+s}\rangle = \langle X_{2r+2s}, A_{-r-s}\rangle \leq C(A_{r+2s}) \cap C(A_{-r-2s}) = C(X_{r+2s})$. Similarly $X_{r+s} \leq C(X_r)$. Thus we obtain:

$$G = X_s * \langle X_\beta \mid \beta \in \Psi\rangle = X_s * (X_{r+2s} * X_{r+s} * X_r).$$

In case (b) it suffices to show that $A_{r+s} = A_{2r+2s}$. Now we have

$$[A_{r+s}, A_{-r-2s}] \leq A_{-s}A_r \cap C(X_s) = A_r.$$

But since $G(\Psi_0)$ is of type $A_2$

$$[a, A_{2r+2s}] = [A_{-r-2s}, b] = A_r \text{ for all } a \in A^\#_{-r-2s} \text{ and } b \in A^\#_{2r+2s}.$$

Suppose $\overline{b} \in A_{r+s} - A_{2r+2s}$ and $a \in A^\#_{-r-2s}$. Then there exists $b \in A_{2r+2s}$ with $[a, \overline{b}] = [a, b^{-1}]$, whence $[a, b\overline{b}] = 1$. This implies $X_{r+2s} = \langle a, A_{r+2s}\rangle \leq C(b\overline{b})$. Since this holds for arbitrary $\overline{b} \in A_{r+s} - A_{2r+2s}$ it shows:

$$(*) \qquad A_{r+s} = A_{2r+2s}C_{A_{r+s}}(X_{r+2s}).$$

The same argument implies $A_{-r-s} = A_{-2r-2s}C_{A_{-r-s}}(X_{r+2s})$.

Now suppose that $A_{r+s} \neq A_{2r+2s}$. Then we obtain:

$$\begin{aligned} X_{r+s} &= \langle C_{A_{-r-s}}(X_{r+2s}), A_{r+s}\rangle \leq C(A_{r+2s}) \\ &= \langle C_{A_{r+s}}(X_{r+2s}), A_{-r-s}\rangle \leq C(A_{-r-2s}). \end{aligned}$$

Hence $X_{r+s} \leq C(X_{r+2s})$, a contradiction to $[A_{2r+2s}, A_{-r-2s}] = A_r$ since $G(\Psi_0)$ is of type $A_2$. □

## 3. $BC_\ell$, $\ell \geq 3$

In this section we assume that $\Phi$ is a root-system of type $BC_\ell$, $\ell \geq 3$. For the convenience of the reader we give a short description of $\Phi$. Let $(e_i, i = 1, \ldots, \ell)$ be an orthonormal basis of $\mathbb{R}^\ell$. Then the roots of $\Phi$ are

$$\pm e_i, \pm 2e_i, \pm e_i \pm e_j \text{ with } i < j \text{ and } 1 \leq i, j \leq \ell.$$

Then $\Phi_0 = \{\pm 2e_i, \pm e_i \pm e_j\}$ is a root subsystem of type $C_\ell$ and $\Phi = \Phi_0 \cup \{r \in \Phi \mid 2r \in \Phi_0\}$. Let $G = \langle A_r \mid r \in \Phi\rangle$ be a group satisfying the hypothesis of the Main-theorem. Then $G_0 = \langle A_r \mid r \in \Phi_0\rangle$ is a group satisfying the hypothesis of the Main-theorem of [6] for a root system $\Phi_0$ of type $C_\ell$. In particular the results of Section 3 of [6] hold for $G_0$ on which our proof is based. (Notice that $\Psi = \{\pm e_i, \pm e_i \pm e_j\}$ is a root system of type $B_\ell$, but it is not a root subsystem of $\Phi$, since $2e_i = e_i + e_i \notin \Psi$, although $2e_i \in \Phi$. Hence we cannot apply [6] for $\langle A_r \mid r \in \Psi\rangle$) In addition we will assume in this section that no $A_r$, $r \in \Phi$ is an elementary abelian 2-group. (We will see in the next section that case (d) of the Main-theorem holds, if some $A_r$ is an elementary abelian 2-group.)

For the rest of the section we fix the following notation:

$X_r := \langle A_r, A_{-r}\rangle$ for $r \in \Phi$. Then, as $A_{2r} \leq A_r$ if $2r \in \Phi$, we have $X_{2r} = \langle A_{2r}, A_{-2r}\rangle \leq X_r$.

Let $H_r := N_{X_r}(A_r) \cap N_{X_r}(A_{-r})$. Then by [3, I(1.4)] $H_{2r} \leq H_r$ if $2r \in \Phi$. If $r, s \in \Phi$ then $\langle r, s\rangle$ is the root subsystem of $\Phi$ spanned by $r$ and $s$. Fix an element $n_r \in X_r$ with $A_r^{n_r} = A_{-r}, A_{-r}^{n_r} = A_r$. Then, again by [3, I(1.4)] we may and will choose $n_r$ such that $n_r = n_{2r}$ if $2r \in \Phi$. If $\Delta$ is a subset of $\Phi$ let $G(\Delta) := \langle X_r \mid r \in \Delta\rangle$. Then we have:

**3.1 Lemma.** *The following hold for all $r, s \in \Phi$ with $s \neq \lambda r$:*

(1) $H_r \leq N(A_s)$

(2) $[H_r, H_s] \leq H_r \cap H_s$

(3) $H_s^{n_r} \leq H_s H_r$

(4) $A_s^{n_r} = A_{ir+js}$ *for some pair $i, j \in \mathbb{N} \cup \{0\}$ with $ir + js \in \Phi$.*

*Proof.* If $\langle r, s\rangle$ is a subsystem of type $A_1 \times A_1, A_2$ or $B_2 = C_2$ Lemma 3.1 is a consequence of (2.5)–(2.9) of [6]. So we may assume that $\langle r, s\rangle$ is of type $BC_2$. Hence one of the cases of Proposition 2.6 holds for $G(\langle r, s\rangle)$. If now $G(\langle r, s\rangle)$ is of type $BC_2$ then it follows from Theorem 2 of [6] that $A_\alpha^{\overline{n}_\beta} = A_{\alpha^{w_\beta}}$ for all $\alpha, \beta \in \langle r, s\rangle$ and all $\overline{n}_\beta \in H_\beta n_\beta$, since $H_\beta n_\beta$ is the set of all elements of $X_\beta$ interchanging $A_\beta$ and $A_{-\beta}$. As $H_\beta = \langle n_\beta \overline{n}_\beta \mid \overline{n}_\beta \in H_\beta n_\beta\rangle$, this implies $H_\beta \leq N(A_\alpha)$ for all $\alpha, \beta \in \langle r, s\rangle$. Hence (1) and (2) hold. Since $H_\beta$ also normalizes $\langle H_\alpha n_\alpha\rangle = H_\alpha\langle n_\alpha\rangle$ we also obtain $[H_\beta, n_\alpha] \leq H_\alpha$, which proves (3).

So we may assume that $G(\langle r, s\rangle)$ is not of type $BC_2$. If $A_\alpha = A_{2\alpha}$ for all $\alpha \in \langle r, s\rangle$ with $2\alpha \in \langle r, s\rangle$, then by Proposition 2.6 $G(\langle r, s\rangle)$ is of type $C_2$ and whence Lemma 3.1 holds by (2.7)–(2.9) of [6]. So we may assume that $X_\alpha \lhd G(\langle r, s\rangle)$ for some $\alpha \in \langle r, s\rangle$ with $2\alpha \in \langle r, s\rangle$. Let $\Delta = \langle r, s\rangle - \{\pm\alpha, \pm 2\alpha\}$. Then by Corollary 2.7 either $G(\langle r, s\rangle)$ is a central product of rank one groups or $G(\langle r, s\rangle) = X_\alpha * G(\Delta)$ and $G(\Delta)$ is of type $A_2$.

In the first case obviously Lemma 3.1 holds. In the second case it easily follows from [6, (2.6)]. $\square$

In the next lemma we will see that Lemma 3.1 remains nearly true for $s = 2r$.

**3.2 Lemma.** *Let $r, 2r \in \Phi$. Then either $X_r \trianglelefteq G$ and $X_\alpha \leq C(X_r)$ for all $\alpha \in \Phi - \{\pm r, \pm 2r\}$ or we have:*

(1) $H_r \leq N(A_{2r})$ (3) $A_{2r}^{n_r} = A_{-2r}$ *and* $H_{2r}^{n_r} = H_{2r}$

(2) $H_r \leq N(H_{2r})$ (4) $A_r' = A_{2r}$ *or* $A_r = A_{2r}$

*Proof.* To prove Lemma 3.2 we may assume $A_r \neq A_{2r}$. Let $s \in \Phi$ with $2s \notin \Phi$ and $s \neq \lambda r, \lambda \in \mathbb{Z}$. Then $\langle r, s\rangle$ is either of type $A_1 \times A_1$ or of type $BC_2$. In the second case we may apply Proposition 2.6 to $\langle r, s\rangle$. Thus either $A_r' = A_{2r}$ or there exists an $\alpha \in \langle r, s\rangle$ with $2\alpha \in \Phi$ such that $X_\alpha \trianglelefteq G(\langle r, s\rangle)$. If now $X_\alpha \neq X_r$, then by Corollary 2.7 either $[X_r, X_s] = 1$ or $A_r = A_{2r}$, which we assume is not the case. Hence we obtain that either $[X_r, X_s] = 1$ or $A_r' = A_{2r}$. But since clearly Lemma 3.2 holds in the second case since $H_r = H_{-r}$ and thus $H_r \leq N(X_{2r})$, we may assume that $[X_r, X_s] = 1$ for all $s \in \Phi$ with $2s \notin \Phi$.

Next suppose $s, 2s \in \Phi$ with $r, s$ linearly independent. Then by the description of the root system of type $BC_\ell$, $\langle r, s\rangle$ is of type $BC_2$. But then we obtain again from Proposition 2.6 and Corollary 2.7 that either $A_r = A_{2r}, A_r' = A_{2r}$ or $[X_r, X_s] = 1$. This shows that either (1)–(4) hold or $[X_r, X_s] = 1$ for all $s \in \Phi - \{\pm r, \pm 2r\}$. □

**3.3 Notation.** We assume from now on for the rest of this section that no $X_r$ with $r, 2r \in \Phi$ is normal in $G$, since in case $X_r \triangleleft G$ case (c) of the Main-theorem holds. Thus from now on we know that always (1)–(4) of Lemma 3.2 are satisfied, which in turn implies that (1)–(4) of Lemma 3.1 hold for all $r, s \in \Phi$. Now set

$$H := \Pi H_r, r \in \Phi \text{ and } N := \langle H, n_r \mid r \in \Phi\rangle.$$

Then by (3) of Lemma 3.1 $H \triangleleft N$. Let $\overline{N} = N/H$ and $\overline{n}_r$ be the image of $n_r$. Then by (1) and (4) of Lemma 3.1 the $\overline{n}_r$ act on $\{A_s \mid s \in \Phi\}$ and thus they act on $\Phi$ by

$$A_{s^{\overline{n}_r}} := A_s^{\overline{n}_r}.$$

Finally let $W = W(\Phi) = \langle w_r \mid r \in \Phi\rangle$ be the Weyl-group of $\Phi$.

We show next:

**3.4 Lemma.** *$\{\overline{n}_r \mid r \in \Phi\}$ is a set of $\{3,4\}$ transpositions of $\overline{N}$. Moreover for $r, s \in \Phi$ with $s \neq \lambda r$ and $R = G(\langle r, s\rangle)$ one of the cases (1)–(4) of [6, (2.11)] holds or we have up to symmetry between $r$ and $s$:*

(5) *$\langle r, s\rangle$ is of type $BC_2$ and one of the following holds:*

(i) *$2r \in \Phi$, $A_r = A_{2r}$, $R$ is of type $B_2$ and $\overline{n}_r^{\overline{n}_s} = \overline{n}_{r\pm s}$.*

(ii) *$R$ is of type $BC_2$ and*

$$\overline{n}_r^{\overline{n}_s} = \begin{cases} \overline{n}_{r\pm s} & \text{if } 2r \in \Phi, 2s \notin \Phi \\ \overline{n}_{r\pm 2s} & \text{if } 2r \notin \Phi, 2s \in \Phi \end{cases}$$

(iii) *$R$ is a central product of the $X_\alpha$, $\alpha \in \langle r, s\rangle$.*

(iv) *There exists an* $\alpha \in \langle r, s\rangle$ *with* $2\alpha \in \langle r, s\rangle$ *such that for* $\Delta = \langle r, s\rangle - \{\pm\alpha, \pm 2\alpha\}$ *we have* $R = X_\alpha * G(\Delta)$ *and* $G(\Delta)$ *is of type* $A_2$. *Moreover, if* $\pm r \neq \alpha \neq \pm s$, *then* $\overline{n}_r^{\overline{n}_s} = \overline{n}_s^{\overline{n}_r} = \overline{n}_{r\pm 2s}$ *resp.* $\overline{n}_{2r\pm s}$ *if* $2s \in \Phi$ *resp.* $2r \in \Phi$.

*Proof.* We first show that one of the cases (1)–(4) of [6,(2.11)] or case (5) of Lemma 3.4 holds. If $\langle r, s\rangle$ is not of type $BC_2$ this follows from [6,(2.11)]. So assume $\langle r, s\rangle$ is of type $BC_2$. Then it follows from Proposition 2.6 and Corollary 2.7 that $R$ satisfies one of the cases (5)(i)–(iv). If $R$ is of type $B_2$ or $BC_2$ then $\overline{n}_r^{\overline{n}_s} = n_{r^{w_s}}$, whence (i) or (ii) holds. Finally, if $R$ satisfies (iv) then by Corollary 2.7 $A_\beta = A_{2\beta}$ for $\beta \in \Delta$ with $2\beta \in \Phi$. Hence if $2s \in \Phi$ then $\overline{n}_r^{\overline{n}_s} = \overline{n}_{r\pm 2s}$. Now $\overline{D} = \{\overline{n}_r \mid r \in \Phi\} = \{\overline{n}_s \mid s \in \Phi_0\}$ since $\overline{n}_r = \overline{n}_{2r}$ if $r, 2r \in \Phi$. Hence it follows already from [6,(2.11)] that $\overline{D}$ is a set of $\{3,4\}$ transpositions of $\overline{N}$. (By Lemma 3.1 and Lemma 3.2 we have $H_0 = \Pi_{s\in\Phi_0} H_s \leq H$ and $H_0 \lhd N$) □

As in Section 3 of [6] we choose now a root subsystem $\Psi_1$ of $\Phi_0$ of type $A_k$ consisting only of short roots of $\Phi_0$ with $k \leq \ell - 1$, satisfying:

(1) For all $r, s \in \Psi_1$ with $r + s \in \Phi$ we have $[A_r, A_s] = A_{r+s}$.

(2) $O_2(W_1) \not\leq O_2(W)$, where $W_1 = \langle w_r \mid r \in \Psi_1\rangle$ (i.e. we cannot have $W_1 \simeq \Sigma_4$ and $O_2(W_1) \leq O_2(W)$)

(3) If $\Psi_0$ is a root subsystem of type $A_{\ell-1}$ containing $\Psi_1$ with $O_2(\langle w_r \mid r \in \Psi_0\rangle) \not\leq O_2(W)$, then $[X_r, X_s] = 1$ for all $r \in \Psi_1$ and $s \in \Psi_0 - \Psi_1$.

(4) $k$ is maximal with (1)–(3).

(The existence of $\Psi_1$ was discussed at the beginning of Section 3 of [6].)

Let $\Lambda = \Phi - \Phi_0$. Then $\Lambda_0 = \{2\alpha \mid \alpha \in \Lambda\}$ is the set of long roots of $\Phi_0$. Set $\Psi = \Phi - (\Lambda \cup \Lambda_0)$. As in [6] we first treat the case $k = \ell - 1$.

**3.5 Theorem.** *Suppose* $k = \ell - 1$. *Then one of the following holds:*

I $o(\overline{n}_r\overline{n}_\alpha) = 2$ *or* 4 *for all* $r \in \Psi_1$ *and* $\alpha \in \Lambda$ *and one of the following holds:*

(a) $o(\overline{n}_r\overline{n}_\alpha) = 4$ *for some* $r \in \Psi_1$ *and* $\alpha \in \Lambda$. *In this case we get the possibilities:*

(i) *G is of type* $BC_\ell$ *or*

(ii) $A_\alpha = A_{2\alpha}$ *for all* $\alpha \in \Lambda$ *and G is of type* $C_\ell$.

(b) $o(\overline{n}_r\overline{n}_\alpha) = 2$ *for all* $r \in \Psi_1$ *and* $\alpha \in \Lambda$. *In this case* $G = G(\Psi) * G(\Lambda)$ *and one of the following holds:*

(i) $G(\Psi)$ *is of type* $D_\ell$ *(i.e.* $\Psi$ *carries the structure of a root-system of type* $D_\ell$*) or*

(ii) $G(\Psi) = G(\Psi_1) * C_{G(\Psi)}(G(\Psi_1))$ *with* $X_s \leq C(G(\Psi_1))$ *for all* $s \in \Psi - \Psi_1$ *and* $G(\Psi_1)$ *is of type* $A_{\ell-1}$.

II *There exists an* $r \in \Psi_1$ *and* $\alpha \in \Lambda$ *with* $o(\overline{n}_r\overline{n}_\alpha) = 3$. *In this case* $\pm\alpha$ *are the only roots in* $\Lambda$ *with* $o(\overline{n}_r\overline{n}_\alpha) = 3$ *and the following holds:*

(i) $\overline{N}_0 = \langle \overline{n}_\alpha, \overline{n}_t \mid t \in \Psi_1\rangle \simeq \Sigma_{\ell+1}$, $\Delta = \{(\pm 2\alpha)^{\overline{N}_0}\}$ *carries the structure of a root-system of type* $A_\ell$ *and* $A_\alpha = A_{2\alpha}$. *Moreover* $G(\Delta)$ *is of type* $A_\ell$.

(ii) $\pm 2\alpha$ *are the only long roots of* $\Phi_0$ *in* $\Delta$.

(iii) $G = G(\Delta) * C(G(\Delta))$ *with* $X_s \leq C(G(\Delta))$ *for all* $s \in \Phi - (\Delta \cup \{\pm\alpha\})$.

*Proof.* We may apply Theorem (3.1) of [6] to $G(\Phi_0)$. Suppose first $o(\overline{n}_r\overline{n}_\alpha) = 2$ for all $r \in \Psi_1$ and $\alpha \in \Lambda$. Then case I(b) holds for $G(\Phi_0)$. Hence it remains to show that for all $r \in \Psi$ resp. $\Psi_1$ and all $\alpha \in \Lambda$ we have $[X_r, X_\alpha] = 1$.

If now $\langle r, \alpha\rangle$ is of type $A_1 \times A_1$ this follows from condition (2) of the introduction. Hence we may assume $\langle r, \alpha\rangle$ is of type $BC_2$. Since $[X_r, X_{2\alpha}] = 1$ by assumption (i.e. $G(\Phi_0)$ satisfies I(b)), case (1) of Proposition 2.6 holds for $G(\langle r, \alpha\rangle)$. Hence by Corollary 2.7 $[X_r, X_\alpha] = 1$, which shows that I(b) holds for $G$.

Next assume $o(\overline{n}_r\overline{n}_\alpha) = 4$ for some $r \in \Psi_1$ and $\alpha \in \Lambda$. Since $\overline{n}_\alpha = \overline{n}_{2\alpha}$ Theorem (3.1) of [6] implies that $G(\Phi_0)$ is of type $C_\ell$ and

$$(*) \qquad [A_\beta, A_\gamma] = \langle A_{i\beta+j\gamma} \mid i, j \in \mathbb{N}, i\beta + j\gamma \in \Phi_0\rangle \text{ for all } \beta, \gamma \in \Phi_0 \text{ with } \beta \neq -\gamma.$$

We must show that in this case either $G = G(\Phi_0)$ or $(*)$ holds for all $\beta, \gamma \in \Phi$ with $\beta \neq -\gamma, -2\gamma$, since in the latter case by Theorem 2 of [4] $G$ is of type $BC_\ell$. For this pick such a pair $\beta, \gamma$ with $\{\beta, \gamma\} \not\subseteq \Phi_0$. Then, without loss, $\gamma \in \Lambda$. If $\beta = \gamma$, then by (4) of Lemma 3.2 either $[A_\beta, A_\beta] = A_{2\beta}$ and $(*)$ holds or $A_\beta = A_{2\beta}$. Now in the second case we obtain $A_\delta = A_{2\delta}$ for all $\delta \in \Lambda$, since, as $G(\Phi_0)$ is of type $C_\ell$, $\overline{N}$ acts transitively on $\Lambda_0$. Hence $G = G(\Phi_0)$ is of type $C_\ell$.

Thus we may assume $\beta \notin \langle\gamma\rangle$. If also $\beta \in \Lambda$, then $\beta + \gamma \in \Phi$ und $\langle\beta, \gamma\rangle$ is of type $BC_2$. Now, since $G(\Phi_0)$ is of type $C_\ell$, $G(\Phi_0 \cap \langle\beta, \gamma\rangle)$ must be of type $C_2$. Hence either case (3) or (4) of Corollary 2.7 holds for $G(\langle\alpha, \beta\rangle)$. In case (3) we get $A_\delta = A_{2\delta}$ for all $\delta \in \Lambda$ as shown and thus $G = G(\Phi_0)$. Thus we may assume that (4) of Corollary 2.7 holds and whence $(*)$ is satisfied for the pair $\beta, \gamma$.

So we may assume $\beta \in \Psi$. (If $\beta \in \Lambda_0$, then $[A_\gamma, A_\beta] = 1$ by condition (2) of Section 1 and $(*)$ holds for the pair $\gamma, \beta$) If $\langle\beta, \gamma\rangle$ is of type $A_1 \times A_1$ clearly $(*)$ holds. Thus we may assume that $\langle\beta, \gamma\rangle$ is of type $BC_2$. Then again, since we may assume $A_\beta \neq A_{2\beta}$ and since $G(\Phi_0 \cap \langle\beta, \gamma\rangle)$ is of type $C_2$, we are in case (4) of Proposition 2.6. Hence $(*)$ holds for the pair $\beta, \gamma$.

We have shown that in case $o(\overline{n}_r\overline{n}_\alpha) = 4$ either $G = G(\Phi_0)$ or $(*)$ holds for all pairs $\beta, \gamma \in \Phi$ with $\gamma \neq -\beta, -2\beta$. Hence in this case I(a) holds.

Finally assume $o(\overline{n}_r\overline{n}_\alpha) = 3$ for some $r \in \Psi_1$ and $\alpha \in \Lambda$. Since $\overline{n}_\alpha = \overline{n}_{2\alpha}$ in this case (3.1) II of [6] holds. Hence $\overline{N}_0 \simeq \Sigma_{\ell+1}$, $\Delta$ carries the structure of a root-system of type $A_\ell$ and $G(\Delta)$ is of type $A_\ell$. Moreover II(ii) of Theorem 3.5 holds. In particular $G(\Delta) = \langle X_t \mid t \in \Psi \cap \Delta\rangle$.

It remains to show that $X_s \leq C(G(\Delta))$ for all $s \in \Phi - \Delta$. If $s \in \Lambda - \Delta$, then $2s \in \Lambda_0$ and $\{\pm 2s\} \neq \{\pm 2\alpha\}$. Hence $X_{2s} \leq C(G(\Delta))$. Let $t \in \Psi \cap \Delta$. Then either $\langle t, s\rangle$ is of type $A_1 \times A_1$ or of type $BC_2$. But in the second case (1) of Proposition 2.6 holds, since $[X_{2s}, X_t] = 1$. Hence in any case $[X_s, X_t] = 1$ for all $t \in \Psi \cap \Delta$ and thus $X_s \leq C(G(\Delta))$. If now $s \in \Psi - \Delta$, then $s \in \Phi_0 - \Delta$ and thus $X_s \leq C(G(\Delta))$ by [6, (3.1)]. Hence $X_s \leq C(G(\Delta))$ for all $s \in \Phi - \Delta$ and thus case II of Theorem 3.5 is satisfied. □

Assume now $k < \ell - 1$. Then we construct a root-subsystem $\Phi_1$ of type $BC_{k+1}$ containing $\Psi_1$ such that for $G(\Phi_1)$ one of the cases of Theorem 3.5 is satisfied.

Let $\Lambda_0 = \Lambda_1 \dot{\cup} \Lambda_2$ such that $W(\Psi) = \langle w_\alpha \mid \alpha \in \Psi_1 \rangle$ acts naturally on $\Lambda_1$ and fixes all roots in $\Lambda_2$. Then $|\{w_r \mid r \in \Lambda_1\}| = k+1$ and $|M_1| = 2^{k+1}$ for $M_1 = \langle w_r \mid r \in \Lambda_1 \rangle$. Let $M_2 = \langle w_s \mid s \in \Lambda_2 \rangle$. Then $O_2(W) = M_1 \times M_2$ and $M_1 W(\Psi_1) \simeq W(C_{k+1})$. Let $\Phi^1 = \{\alpha \in \Phi_0 \mid w_\alpha \in M_1 W(\Psi_1)\}$. Then $\Phi^1$ is a root subsystem of type $C_{k+1}$ of $\Phi^0$. Let finally $\Phi_1 = \Phi^1 \cup \Lambda^1$, where $\Lambda^1 = \{\beta \in \Lambda \mid 2\beta \in \Lambda_1\}$. Then we get:

**3.6 Lemma.** *The following hold:*

(1) *$\Phi_1$ is a root subsystem of type $BC_{k+1}$ of $\Phi$ containing $\Psi_1$.*

(2) *$G(\Phi_1)$ satisfies one of the cases of Theorem* 3.5.

*Proof.* Without loss we may choose the enumeration of the orthonormal basis of $\mathbb{R}^\ell$ such that $\Psi_1 = \{e_i - e_j \mid i \neq j, i, j \leq k+1\}$. Then $\Lambda_1 = \{\pm 2e_i \mid i \leq k+1\}$ and $\Lambda^1 = \{\pm e_i \mid i \leq k+1\}$. Hence $\Phi_1 = \{\pm e_i, \pm 2e_i, \pm e_i \pm e_j \mid i \neq j, i, j \leq k+1\}$ is a root-system of type $BC_{k+1}$ by the description of a root system of type $BC_\ell$. Now since $\Phi_1$ is a root-subsystem it is clear that $G(\Phi_1)$ satisfies the hypothesis of Theorem 3.5 with respect to the root system $\Phi_1$. □

**3.7 Proposition.** *$G = G(\Phi_1) * C(G(\Phi_1))$ with $X_s \leq C(G(\Phi_1))$ for all $s \in \Phi - \Phi_1$.*

*Proof.* Suppose first $s \in \Phi_0 - \Phi_1$. Then by (3.3) and (3.4) of [6] $X_s \leq C(G(\Phi^1))$. Let $t \in \Phi_1 - \Phi^1$. Then $t \in \Lambda^1$ and $2t \in \Lambda_1$. Hence $[X_s, X_{2t}] = 1$. Now $\langle s, t \rangle$ is either of type $A_1 \times A_1$ or of type $BC_2$ and, to show $[X_s, X_t] = 1$, we may assume that we are in the second case. Then also $s \notin \Lambda_0$, since $\langle s, t \rangle$ is not of type $A_1 \times A_1$ and we may assume that $A_t \neq A_{2t}$. Hence possibility (1) of Proposition 2.6 holds for $G(\langle s, t \rangle)$ and thus $[X_s, X_t] = 1$.

We have shown $X_s \leq C(G(\Phi_1))$ for all $s \in \Phi_0 - \Phi_1$. Finally assume $s \in \Phi - (\Phi_1 \cup \Phi_0)$. Then $s \in \Lambda$ and $2s \in \Lambda_2$. Suppose $r \in \Phi_1$ with $[X_s, X_r] \neq 1$. Then, since $\langle s, r \rangle$ is of type $A_1 \times A_1$ for all $r \in \Psi \cap \Phi_1$ and for all $r \in \Lambda_1$, we obtain $r \in \Lambda^1$. Now, using the description of $\Phi$ in the beginning of this section and the description of $\Phi_1$ in Lemma 3.6 we obtain $s = e_m$ with $k+1 \leq m \leq \ell$ and $r = e_i$ with $1 \leq i \leq k+1$. Hence $r + s = e_i + e_m \in \Phi_0 - \Phi_1$ and thus $[X_{r+s}, X_r] = 1$ as shown above. But by Proposition 2.6 and Corollary 2.7 $G(\langle r, s \rangle)$ is of type $BC_2$ since $[X_s, X_r] \neq 1$. (If $G(\langle r, s \rangle)$ is of type $C_2$, then $X_s = X_{2s}$ and thus $[X_s, X_r] = 1$) But then $[A_{r+s}, A_{-r}] \geq A_r$ by $(*)$ in (4) of Proposition 2.6, a contradiction to $[X_{r+s}, X_r] = 1$. This finally shows $[X_s, X_r] = 1$ for all $s \in \Phi - \Phi_1$ and all $r \in \Phi_1$, which proves Proposition 3.7. □

## 4. Proof of the Main-theorem

In this section we assume that the hypothesis of the Main-theorem holds. We carry on with the notation introduced in Section 2 and 3. We first show how case (d) of the Main-theorem can be split of.

**4.1 Lemma.** *Suppose $J' = \{r \in \Phi_0 \mid A_r$ is an elementary abelian 2-group$\} \neq \emptyset$. Let $K' = \Phi_0 - J'$,*

$$J = \{s \in \Phi \mid 2s \in J'\} \cup J' \text{ and } K = \{s \in \Phi \mid 2s \in K'\} \cup K'.$$

*Then the following hold:*

(1) $\Phi = J\dot\cup K$ *and* $G = G(J) * G(K)$.

(2) *Let* $s \in J - J'$. *Then either* $X_s \lhd G$ *or* $A_s^2 \le A_s' \le A_{2s}$.

*Proof.* By choice of $J'$ and $K'$ we have $\Phi_0 = J'\dot\cup K'$. Let $s \in \Phi - \Phi_0$. Then $2s \in \Phi_0$ and thus $s \in J \cup K$. If now $s \in J \cap K$, then $s \in \Phi - \Phi_0$ and thus $2s \in J' \cap K'$, which is impossible. This shows $\Phi = J\dot\cup K$.

Now by [6, (2.12)] $G_0 = G(J') * G(K')$. Let $s \in J - J'$ and claim $X_s \le C(G(K'))$. For this pick $r \in K'$. Then $\langle s, r\rangle$ is of type $A_1 \times A_1$ or $BC_2$ and, to show $[X_s, X_r] = 1$, we may assume that we are in the second case. Hence we may apply Proposition 2.6 and Corollary 2.7 to $G(\langle s, r\rangle) = Y$. If now $r = 2\alpha$, $\alpha \in K - K'$, then $[X_{2s}, X_r] = 1 = [X_s, X_r]$ by condition (2) of the Main-Theorem. So we may assume that $r$ is not of this form, i.e. $r$ is a short root of $\Phi_0$. Now clearly case (3) and (4) of Proposition 2.6 can not hold, since $[X_{2s}, X_r] = 1$.

Suppose (1) 0f proposition 2.6 holds. Then, if $X_s \lhd Y$ clearly $[X_s, X_r] = 1$. So we have $X_\alpha \lhd Y$ for some $\alpha \ne s$ with $2\alpha \in \Phi_0$. But then by Corollary 2.7 either $Y$ is a central product of rank one groups or $X_s = X_{2s}$. Hence in any case $[X_s, X_r] = 1$. Finally, if (2) of Proposition 2.6 holds, then $A_r$ is also an elementary abelian 2-group, a contradiction to $r \in K'$.

This shows $[X_s, X_r] = 1$ and thus $X_s \le C(G(K'))$ for each $s \in J - J'$. The same argument shows $X_\alpha \le C(G(J'))$ for each $\alpha \in K - K'$. Thus, to prove (1), it remains to show $[X_s, X_\alpha] = 1$ for each $s \in J - J'$, $\alpha \in K - K'$. Suppose this is not the case for some such pair $s, \alpha$. Then $\langle s, \alpha\rangle$ is of type $BC_2$ and $[X_s, X_{2\alpha}] = 1 = [X_{2s}, X_\alpha]$. Let again $Y = G(\langle s, \alpha\rangle)$. Then we may apply Proposition 2.6 to $Y$. Clearly (3) of Proposition 2.6 does not hold for $Y$. If now $Y$ is of type $BC_2$, then $s + \alpha \in \Phi_0$ and $[A_{-s}, A_{s+\alpha}] \ne 1 \ne [A_{-\alpha}, A_{s+\alpha}]$, a contradiction to $\Phi_0 = J' \cup K'$ and $X_s \le C(G(K'))$, $X_\alpha \le C(G(J'))$. This shows that (4) of Proposition 2.6 does not hold. Clearly (2) of proposition 2.6 does not hold, since $2\alpha \in K'$. So case (1) of proposition 2.6 remains. But in this case either $X_s \lhd Y$ or $X_\alpha \lhd Y$ and whence in any case $[X_s, X_\alpha] = 1$. This proves (1).

To prove (2) pick $s \in J - J'$ and assume $X_s$ is not normal in $G$. Then there exists by (1) an $r \in J$ with $[X_s, X_r] \ne 1$. Hence $\langle s, r\rangle$ is of type $BC_2$ and we may apply Proposition 2.6 to $Y = G(\langle s, r\rangle)$. Now in case (2) or (3) of Proposition 2.6 clearly (2) holds. So we may assume that(1) or (4) of Proposition 2.6 are satisfied. If now $Y$ is of type $BC_2$, then as $A_{2s}$ and $A_r$ or $A_{2r}$ are elementary abelian, Proposition 2.1 shows that the hypothesis of Lemma 2.3 is satisfied for $\langle s, r\rangle$ and $Y$. Hence $A_s^2 \le A_s' \le A_{2s}$ since $[X_s, X_r] \ne 1$. So we may finally assume that (1) of Proposition 2.6 holds for $Y$. But then, since $X_s$ is not normal in $Y$, Corollary 2.7 implies $A_s = A_{2s}$, whence $A_s^2 = 1 = A_s'$ and (2) holds. □

**4.2 Proof of the Main-theorem.** Let $\Phi$ be a root system of type $BC_\ell$ and $G = \langle A_r \mid r \in \Phi\rangle$ be a group satisfying (1) and (2) of Section 1. We show that $G$ satisfies one of the cases (a)–(d) of the Main-Theorem.

If $\ell = 2$ then this follows from Proposition 2.6. So we may assume $\ell \ge 3$. Suppose next that some $A_r$, $r \in \Phi_0$ is an elementary abelian 2-group. Then by Lemma 4.1 $G = G(J)*G(K)$ and, to show that case (d) of the Main-theorem is satisfied, it remains to show that $A_s^2 \le A_s' \le A_{2s}$ for each $s \in J$ with $2s \in J'$. But if this is not the case for some such $s$ then by (2) of Lemma 4.1 $X_s \lhd G$ and $X_\alpha \le C(X_s)$ for all $\alpha \in \Phi - \{\pm s, \pm 2s\}$ and thus case (c) of the Main-theorem holds with $J = \{\pm s, \pm 2s\}$.

Hence we may assume that no $A_r$, $r \in \Phi_0$ and whence no $A_r, r \in \Phi$ is an elementary abelian 2-group. Hence the hypothesis of §3 is satisfied. Let now $\Psi_1$ be a root subsystem of type $A_k$, $k \leq \ell - 1$, satisfying conditions (1)–(4) next to the proof of Lemma 3.4. If now $k = \ell - 1$ then Theorem 3.5 shows that one of the cases (a)–(c) of the Main-theorem holds. Hence we may assume $k < \ell - 1$. But in this case it follows from Proposition 3.6 and Proposition 3.7 that the Main-theorem holds. □

## References


[1] Bourbaki, N.: *Groupes et algebres de Lie, Chapitres 4,5 et 6.* Elements de Mathematique, Paris 1981. Zbl 0483.22001

[2] Müller, C.: *On the Steinberg-presentation for Lie-type-groups of type $C_2$.* J. of Algebra **252** (2002), 150–160. Zbl pre01836701

[3] Timmesfeld, F. G.: *Abstract Root Subgroups and simple groups of Lie-type.* Monographs in Mathematics **95**, Birkhäuser Verlag 2001. Zbl 0984.20019

[4] Timmesfeld, F. G.: *On the Steinberg-Presentation of Lie-type groups.* To appear in Forum Math. (2002).

[5] Timmesfeld, F. G.: *A remark on Presentations of certain Chevalley groups.* To appear in Archiv der Mathematik (2002).

[6] Timmesfeld, F. G.: *Groups with a central factor of Lie-type.* To appear in J. of Algebra.

[7] Timmesfeld, F. G.: *Structure and Presentations of Lie-type-groups.* To appear in the Proceedings of the LMS-conference on groups, geometry and combinatorics, Durham 2001.

[8] Timmesfeld, F. G.: *Structure and Presentations of Lie-type groups.* Proc. London Math. Soc. (3) **81** (2000), 428–484. Zbl pre01696288

[9] Van Maldeghem, H.: *Generalized polygons.* Monographs in Mathematics **93**, Birkhäuser, Basel 1998. Zbl 0914.51005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