Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 44 (2003), No. 1, 203-212.

Special Curves and Ruled Surfaces

Dedicated to Professor Koichi Ogiue on his sixtieth birthday

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Abstract. We study cylindrical helices and Bertrand curves as curves on ruled surfaces. Some results in this paper clarify that the cylindrical helix is related to Gaussian curvature and the Bertrand curve is related to mean curvature of ruled surfaces. All arguments in this paper are elementary and classical. There are some articles which investigate curves on ruled surfaces [1, 7]. However, the main results are not obtained in these articles and do shade new light on an old subject.
MSC2000: 58C27, 53A25, 53A05
Keywords: cylindrical helix, Bertrand curve, ruled surfaces

1. Introduction

In [5] we have studied singularities of the rectifying developable (surface) of a space curve. We observed that the rectifying developable along a curve γ is non-singular if and only if γ is a cylindrical helix. In this case the rectifying developable is a cylindrical surface. The notion of cylindrical helices is a generalization of the notion of circular helices. On the other hand, the notion of Bertrand curves is another generalization of the notion of circular helices. These two curves have been classically studied as special curves in Euclidean space.

In this paper we study these curves from the view point of geometry of curves on ruled surfaces. The principal normal surface of a space curve γ is defined to be a ruled surface along $\boldsymbol{\gamma}$ whose rulings are given by the principal normals of $\boldsymbol{\gamma}$. Principal normal surfaces are naturally related to Bertrand curves by definition. In §2 we review basic notions and properties of space curves and ruled surfaces. In §3 we study cylindrical helices and Bertrand curves as curves on ruled surfaces. We prove that a ruled surface is the rectifying developable of $\boldsymbol{\gamma}$ if and only if $\boldsymbol{\gamma}$ is a geodesic of the ruled surface which is transversal to rulings and Gaussian curvature vanishes along $\boldsymbol{\gamma}$ (Theorem 3.4). As a corollary of Theorem 3.4, we give a characterization of a cylindrical surface as a developable surface by the existence of a geodesic which is a cylindrical helix with non-zero curvature (cf. Corollary 3.5). We also prove that a ruled surface is the principal normal surface of a space curve $\boldsymbol{\gamma}$ if and only if $\boldsymbol{\gamma}$ is an asymptotic curve of the ruled surface which is transversal to rulings and mean curvature vanishes along $\boldsymbol{\gamma}$ (Theorem 3.10). As a corollary of Theorem 3.10, we show that if there exist two disjoint asymptotic curves on a ruled surface both of which are transversal to rulings and mean curvature of the ruled surface vanishes along these curves, then these curves are Bertrand curves (cf. Proposition 3.12). We also show that if there exist three disjoint Bertrand curves on a ruled surface, then the ruled surface is a helicoid (cf. Proposition 3.11). In [6] we have constructed many examples of cylindrical helices and Bertrand curves. Moreover, we have shown that all cylindrical helices and Bertrand curves can be constructed by using the method in [6].

This is one of the papers of the authors joint project entitled "Geometry of ruled surfaces and line congruences".

All manifolds and maps considered here are of class C^∞ unless otherwise stated.

2. Basic notions and properties

We now review some basic concepts on classical differential geometry of space curves and ruled surfaces in Euclidean space. For any two vectors $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$, we denote $\boldsymbol{x} \cdot \boldsymbol{y}$ as the standard inner product. Let $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^3$ be a curve with $\dot{\boldsymbol{\gamma}}(t) \neq 0$, where $\dot{\boldsymbol{\gamma}}(t) = d\boldsymbol{\gamma}/dt(t)$. We also denote the norm of $\boldsymbol{x}$ by $\|\boldsymbol{x}\|$. The *arc-length parameter* s of a curve $\boldsymbol{\gamma}$ is determined such that $\|\boldsymbol{\gamma}'(s)\| = 1$, where $\boldsymbol{\gamma}'(s) = d\boldsymbol{\gamma}/ds(s)$. Let us denote $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ and we call $\boldsymbol{t}(s)$ a *unit tangent vector* of $\boldsymbol{\gamma}$ at s. We define the *curvature* of $\boldsymbol{\gamma}$ by $\kappa(s) = \sqrt{\|\boldsymbol{\gamma}''(s)\|}$. If $\kappa(s) \neq 0$, then the *unit principal normal vector* $\boldsymbol{n}(s)$ of the curve $\boldsymbol{\gamma}$ at s is given by $\boldsymbol{\gamma}''(s) = \kappa(s)\boldsymbol{n}(s)$. The unit vector $\boldsymbol{b}(s) = \boldsymbol{t}(s) \times \boldsymbol{n}(s)$ is called the *unit binormal vector* of $\boldsymbol{\gamma}$ at s. Then we have the Frenet-Serret formulae:

$$\begin{cases} \boldsymbol{t}'(s) &= \kappa(s)\boldsymbol{n}(s) \\ \boldsymbol{n}'(s) &= -\kappa(s)\boldsymbol{t}(s) + \tau(s)\boldsymbol{b}(s) \\ \boldsymbol{b}'(s) &= -\tau(s)\boldsymbol{n}(s), \end{cases}$$

where $\tau(s)$ is the torsion of the curve $\boldsymbol{\gamma}$ at s. For any unit speed curve $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^3$, we call $D(s) = \tau(s)\boldsymbol{t}(s) + \kappa(s)\boldsymbol{b}(s)$ *the Darboux vector field* of $\boldsymbol{\gamma}$ (cf. [8], Section 5.2). We define a vector field $\widetilde{D}(s) = (\tau/\kappa)(s)\boldsymbol{t}(s) + \boldsymbol{b}(s)$ along $\boldsymbol{\gamma}$ under the condition that $\kappa(s) \neq 0$ and we call it *the modified Darboux vector field* of $\boldsymbol{\gamma}$.

A curve $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^3$ with $\kappa(s) \neq 0$ is called a *cylindrical helix* if the tangent lines of $\boldsymbol{\gamma}$ make a constant angle with a fixed direction. It has been known that the curve $\boldsymbol{\gamma}(s)$ is a cylindrical helix if and only if $(\tau/\kappa)(s)$ is constant. We call a curve a *circular helix* if both of $\kappa(s) \neq 0$ and $\tau(s)$ are constant. On the other hand, a curve $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^3$ with $\kappa(s) \neq 0$ is called a *Bertrand curve* if there exists a curve $\bar{\boldsymbol{\gamma}} : I \longrightarrow \mathbb{R}^3$ such that the principal normal lines of $\boldsymbol{\gamma}$ and $\bar{\boldsymbol{\gamma}}$ at $s \in I$ are equal. In this case $\bar{\boldsymbol{\gamma}}$ is called a *Bertrand mate* of $\boldsymbol{\gamma}$. Any plane curve $\boldsymbol{\gamma}$ is a Bertrand curve whose Bertrand mates are parallel curves of $\boldsymbol{\gamma}$. Bertrand curves have the following fundamental properties [3, 4, 9, 10].

Proposition 2.1. *Let $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^3$ be a space curve.*

(1) *Suppose that $\tau(s) \neq 0$. Then $\boldsymbol{\gamma}$ is a Bertrand curve if and only if there exist nonzero real numbers A, B such that $A\kappa(s) + B\tau(s) = 1$ for any $s \in I$. It follows from this fact that a circular helix is a Bertrand curve.*

(2) *Let $\boldsymbol{\gamma}$ be a Bertrand curve with the Bertrand mate $\bar{\boldsymbol{\gamma}}$. Then $\tau(s)\bar{\tau}(s)$ is non-negative constant, where $\bar{\tau}(s)$ is the torsion of $\bar{\boldsymbol{\gamma}}$.*

We have the following corollary of the proposition.

Corollary 2.2. *Let $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^3$ be a space curve with $\kappa(s) \neq 0$ and $\tau(s) \neq 0$. Then $\boldsymbol{\gamma}$ is a Bertrand curve if and only if there exists a real number $A \neq 0$ such that*

$$A(\tau'(s)\kappa(s) - \kappa'(s)\tau(s)) - \tau'(s) = 0.$$

In this case the Bertrand mate of $\boldsymbol{\gamma}$ is given by $\bar{\boldsymbol{\gamma}}(s) = \boldsymbol{\gamma}(s) + A\boldsymbol{n}(s)$.

Proof. By the proposition, $\boldsymbol{\gamma}$ is a Bertrand curve if and only if there exist real numbers $A \neq 0$ and B such that $A\kappa(s) + B\tau(s) = 1$. This is equivalent to the condition that there exists a real number $A \neq 0$ such that $(1 - A\kappa(s))/\tau(s)$ is constant. Differentiating both sides of the last equality, we have $A(\tau'(s)\kappa(s) - \kappa'(s)\tau(s)) = \tau'(s)$.

The converse assertion is also true. $\square$

On the other hand, *a ruled surface* in $\mathbb{R}^3$ is (locally) the map $F_{(\gamma,\delta)} : I \times \mathbb{R} \to \mathbb{R}^3$ defined by $F_{(\gamma,\delta)}(t, u) = \boldsymbol{\gamma}(t) + u\boldsymbol{\delta}(t)$, where $\boldsymbol{\gamma} : I \to \mathbb{R}^3$, $\boldsymbol{\delta} : I \to \mathbb{R}^3 \setminus \{\mathbf{0}\}$ are smooth mappings and I is an open interval. We call $\boldsymbol{\gamma}$ *the base curve* and $\boldsymbol{\delta}$ *the director curve.* The straight lines $u \mapsto \boldsymbol{\gamma}(t) + u\boldsymbol{\delta}(t)$ are called *rulings.* We can calculate that

$$\frac{\partial F_{(\gamma,\delta)}}{\partial t}(t, u) \times \frac{\partial F_{(\gamma,\delta)}}{\partial u}(t, u) = \boldsymbol{\gamma}'(t) \times \boldsymbol{\delta}(t) + u\boldsymbol{\delta}'(t) \times \boldsymbol{\delta}(t).$$

Therefore (t_0, u_0) is a singular point of $F_{(\gamma,\delta)}$ if and only if $\boldsymbol{\gamma}'(t_0) \times \boldsymbol{\delta}(t_0) + u_0\boldsymbol{\delta}'(t_0) \times \boldsymbol{\delta}(t_0) = \mathbf{0}$.

We say that the ruled surface $F_{(\gamma,\delta)}$ is a *cylindrical surface* if $\boldsymbol{\delta}(t) \times \boldsymbol{\delta}'(t) \equiv \mathbf{0}$. Thus, we say that the ruled surface $F_{(\gamma,\delta)}$ is *non-cylindrical* if $\boldsymbol{\delta}(t) \times \boldsymbol{\delta}'(t) \neq \mathbf{0}$. We now consider a curve $\boldsymbol{\sigma}(t)$ on the ruled surface $F_{(\gamma,\delta)}$ with the property that $\boldsymbol{\sigma}'(t) \cdot \widetilde{\boldsymbol{\delta}}'(t) = 0$. We call such a curve a *line of striction*. If $F_{(\gamma,\delta)}$ is non-cylindrical, it has been known that there exists the line of striction uniquely.

In this paper we consider the following two special ruled surfaces associated to a space curve $\boldsymbol{\gamma}$ with $\kappa(s) \neq 0$ which are respectively related to cylindrical helices and Bertrand curves. The ruled surface $F_{(\gamma,\widetilde{D})}(s, u) = \boldsymbol{\gamma}(s) + u\widetilde{D}(s)$ is called the *rectifying developable of* $\boldsymbol{\gamma}$.

We also define the ruled surface $F_{(\gamma,n)}(s,u) = \boldsymbol{\gamma}(s)+u\boldsymbol{n}(s)$ which is called *the principal normal surface of* $\boldsymbol{\gamma}$.

We now consider the rectifying developable of a unit speed space curve $\boldsymbol{\gamma}(s)$ with $\kappa(s) \neq 0$. We can calculate that $\widetilde{D}'(s) = (\tau/\kappa)'(s)\boldsymbol{t}(s)$. Therefore (s_0, u_0) is a singular point of $F_{(\gamma,\widetilde{D})}$ if and only if $(\tau/\kappa)'(s_0) \neq 0$ and $u_0 = -1/(\tau/\kappa)'(s_0)$. On the other hand, we have the following proposition:

Proposition 2.3. *For a unit speed curve $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^3$ with $\kappa(s) \neq 0$, the following are equivalent.*

(1) *The rectifying developable $F_{(\gamma,\widetilde{D})} : I \times \mathbb{R} \longrightarrow \mathbb{R}^3$ of $\boldsymbol{\gamma}$ is a non-singular surface.*

(2) *$\boldsymbol{\gamma}$ is a cylindrical helix.*

(3) *The rectifying developable $F_{(\gamma,\widetilde{D})}$ of $\boldsymbol{\gamma}$ is a cylindrical surface.*

Proof. By the previous calculation, $F_{(\gamma,\widetilde{D})}$ is non-singular at any point in $I \times \mathbb{R}$ if and only if $(\tau/\kappa)'(s) \equiv 0$. This means that $\boldsymbol{\gamma}$ is a cylindrical helix.

On the other hand, we have calculated that $\widetilde{D}'(s) = (\tau/\kappa)'(s)\boldsymbol{t}(s)$. The rectifying developable $F_{(\gamma,\widetilde{D})}$ is cylindrical if and only if $\widetilde{D}'(s) \equiv 0$, so that the condition (2) is equivalent to the condition (3). □

We also consider the principal normal surface $F_{(\gamma,n)}(s,u)$ of a unit speed curve $\boldsymbol{\gamma}(s)$ with $\kappa(s) \neq 0$. We start to consider the singular point of $F_{(\gamma,n)}(s,u)$. By the Frenet-Serret formulae, we can show that $\boldsymbol{\gamma}'(s)\times\boldsymbol{n}(s)+u\boldsymbol{n}'(s)\times\boldsymbol{n}(s) = (1-u\kappa(s))\boldsymbol{b}(s)-\tau(s)u\boldsymbol{t}(s)$. Therefore (s_0,u_0) is a singular point of $F_{(\gamma,n)}$ if and only if $\tau(s_0) = 0$ and $u_0 = 1/\kappa(s_0)$.

The principal normal surface $F_{(\gamma,n)}$ is non-singular under the assumption that $\tau(s) \neq 0$. For example, the principal normal surface of a circular helix is the *helicoid*. For a Bertrand curve, we have the following proposition.

Proposition 2.4. *Let $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^3$ be a Bertrand curve. The principal normal surface $F_{(\gamma,n)}$ has a singular point if and only if $\boldsymbol{\gamma}$ is a plane curve. In this case the image of $F_{(\gamma,n)}$ is a plane in $\mathbb{R}^3$.*

Proof. By the assertion (2) of Proposition 2.1, if there exists a point $s_0 \in I$ such that $\tau(s_0) = 0$, then $\boldsymbol{\gamma}$ is a plane curve. On the other hand, the singular point of $F_{(\gamma,n)}$ corresponds to the point $s_0 \in I$ with $\tau(s_0) = 0$. The last assertion of the proposition is clear by definition. □

3. Curves on ruled surfaces

In this section we study cylindrical helices and Bertrand curves from the view point of the theory of curves on ruled surfaces. In the previous sections, we have remarked that the rectifying developable of a cylindrical helix is a cylindrical surface and the principal normal surface of a Bertrand curve is non-singular if the Bertrand curve is a space curve. In particular the rectifying developable is a circular cylinder and the principal normal surface is a helicoid if the curve is a circular helix. It has been classically known that the circular cylinder is a non-singular developable surface and the curved minimal surfaces are the helicoids. Here, we say that a ruled surface $F_{(\gamma,\delta)}$ is a *developable surface* if Gaussian curvature of the regular part of $F_{(\gamma,\delta)}$ vanishes. By these facts, we now pay attention to Gaussian curvature and mean curvature of ruled surfaces. Let $F_{(\gamma,\delta)}$ be a ruled surface. For convenience, we may assume that $\|\boldsymbol{\delta}(t)\| = 1$. It is easy to show that Gaussian curvature of $F_{(\gamma,\delta)}$ is

$$K(t,u) = -\frac{(\det(\boldsymbol{\gamma}'(t), \boldsymbol{\delta}(t), \boldsymbol{\delta}'(t)))^2}{(EG - F^2)^2}$$

and mean curvature of $F_{(\gamma,\delta)}$ is

$$H(t,u) = \frac{-2(\boldsymbol{\gamma}'(t)\cdot\boldsymbol{\delta}(t))\det(\boldsymbol{\gamma}'(t), \boldsymbol{\delta}(t), \boldsymbol{\delta}'(t)) + \det(\boldsymbol{\gamma}''(t) + u\boldsymbol{\delta}''(t), \boldsymbol{\gamma}'(t) + u\boldsymbol{\delta}'(t), \boldsymbol{\delta}(t))}{2(EG - F^2)^{3/2}},$$

where

$$E = E(t,u) = \|\boldsymbol{\gamma}'(t) + u\boldsymbol{\delta}'(t)\|^2,\ F = F(t,u) = \boldsymbol{\gamma}'(t)\cdot\boldsymbol{\delta}(t),\ G = G(t,u) = 1.$$

In particular Gaussian curvature of the rectifying developable of a space curve vanishes and mean curvature of the principal normal surface of a space curve is

$$H(s,u) = \frac{u(\tau'(s) + u(\kappa'(s)\tau(s) - \tau'(s)\kappa(s)))}{(EG - F^2)^{3/2}},$$

where s is the arc-length of $\boldsymbol{\gamma}$. It follows from this fact that $H(s,u) = 0$ if and only if $u = 0$ or $\tau'(s) = u(\tau'(s)\kappa(s) - \tau(s)\kappa'(s))$. Thus, mean curvature of the principal normal surface $F_{(\gamma,n)}$ of $\boldsymbol{\gamma}$ always vanishes along $\boldsymbol{\gamma}$. If there exists a point $s_0 \in I$ such that $\tau'(s_0)\kappa(s_0) - \tau(s_0)\kappa'(s_0) = 0$, then $H(s_0,u_0) = 0$ for some $u_0 \neq 0$ if and only if $\tau'(s_0) = 0$. In this case $\kappa'(s_0) = 0$. Therefore, $H(s_0,u_0) = 0$ for some $u_0 \neq 0$ if and only if $\tau'(s_0) = \kappa'(s_0) = 0$ or

$$u_0 = \frac{\tau'(s_0)}{\tau'(s_0)\kappa(s_0) - \tau(s_0)\kappa'(s_0)}.$$

If $\tau'(s_0) \neq 0$ and $\tau'(s_0)\kappa(s_0) - \tau(s_0)\kappa'(s_0) = 0$, then $H(s_0,u) \neq 0$ for any $u \neq 0$. Moreover, under the assumption that $\tau'(s_0) = \kappa'(s_0) = 0$, $H(s_0,u) = 0$ for any u. Of course, if $\tau'(s)\kappa(s) - \tau(s)\kappa'(s) \neq 0$, mean curvature vanishes along the curve given by

$$\widetilde{\boldsymbol{\gamma}}(s) = \boldsymbol{\gamma}(s) + \frac{\tau'(s)}{\tau'(s)\kappa(s) - \tau(s)\kappa'(s)}\boldsymbol{n}(s).$$

Let $\boldsymbol{\gamma} : J \longrightarrow F_{(\gamma,\delta)}(I \times \mathbb{R}) \subset \mathbb{R}^3$ be a regular curve. We say that $\boldsymbol{\gamma}$ is the *minimal locus* of $F_{(\gamma,\delta)}$ if mean curvature H of $F_{(\gamma,\delta)}$ vanishes on $\boldsymbol{\gamma}(J)$. By the above calculation and Corollary 2.2, we have the following proposition.

Proposition 3.1. *Let $\boldsymbol{\gamma}$ be a Bertrand curve and $\bar{\boldsymbol{\gamma}}$ be the Bertrand mate of $\boldsymbol{\gamma}$. Then $\bar{\boldsymbol{\gamma}}$ is the minimal locus of the principal normal surface of $\boldsymbol{\gamma}$.*

Proof. By Corollary 2.2, if $\bar{\boldsymbol{\gamma}}$ is the Bertrand mate of $\boldsymbol{\gamma}$, then there exists a real number A such that $A(\tau'(s)\kappa(s) - \tau(s)\kappa'(s)) - \tau'(s) = 0$. and $\bar{\boldsymbol{\gamma}}(s) = \boldsymbol{\gamma}(s) + A\boldsymbol{n}(s)$. This means that $H(\bar{\boldsymbol{\gamma}}(s)) = H(s, A) = 0$. This completes the proof. $\square$

By definition, $\boldsymbol{\gamma}$ is a geodesic of the rectifying developable and a asymptotic curve of the principal normal surface of $\boldsymbol{\gamma}$ itself. The following proposition has been known as the Bonnet's theorem for non-cylindrical ruled surfaces. The assertion, however, holds even for general ruled surfaces.

Proposition 3.2. *Let $F_{(\gamma,\delta)}(s,u) = \boldsymbol{\gamma}(s) + u\boldsymbol{\delta}(s)$ be a ruled surface with $\|\boldsymbol{\delta}(s)\| = 1$. Let $\boldsymbol{\sigma}(s) = \boldsymbol{\gamma}(s) + u(s)\boldsymbol{\delta}(s)$ be a curve on $F_{(\gamma,\delta)}$, where s is the arc-length of $\boldsymbol{\sigma}(s)$. Consider the following three conditions on $\boldsymbol{\sigma}$:*

(1) *$\boldsymbol{\sigma}(s)$ is a line of striction of $F_{(\gamma,\delta)}$.*

(2) *$\boldsymbol{\sigma}(s)$ is a geodesic of $F_{(\gamma,\delta)}$.*

(3) *The angles between $\boldsymbol{\sigma}'(s)$ and $\boldsymbol{\delta}(s)$ are constant.*

If we assume that any two of the above three conditions hold, then the other condition holds.

We remark that the above conditions are respectively equivalent to the following conditions:

$(1)'$ $\boldsymbol{\sigma}'(s) \cdot \boldsymbol{\delta}'(s) = 0$.

$(2)'$ $\boldsymbol{\sigma}''(s) \cdot \boldsymbol{\delta}(s) = 0$.

$(3)'$ $\boldsymbol{\sigma}'(s) \cdot \boldsymbol{\delta}(s) = \text{constant}$.

The assertion follows from the fact that $(\boldsymbol{\sigma}'(s) \cdot \boldsymbol{\delta}(s))' = \boldsymbol{\sigma}''(s) \cdot \boldsymbol{\delta}(s) + \boldsymbol{\sigma}'(s) \cdot \boldsymbol{\delta}'(s)$.

We have the following corollary.

Corollary 3.3. *Suppose that there exist two disjoint geodesics $\boldsymbol{\sigma}_i(s)$ $(i = 1, 2)$ on a ruled surface $F_{(\gamma,\delta)}(s,u) = \boldsymbol{\gamma}(s) + u\boldsymbol{\delta}(s)$ such that the angles between $\boldsymbol{\sigma}_i'(s)$ and $\boldsymbol{\delta}(s)$ are constant. Then the ruled surface $F_{(\gamma,\delta)}(s,u)$ is a cylindrical surface and both of $\boldsymbol{\sigma}_i(s)$ are cylindrical helices. Moreover, the direction of $\boldsymbol{\delta}(s)$ is equal to the direction of the Darboux vector of $\boldsymbol{\sigma}_i(s)$.*

Proof. By the proposition, $\boldsymbol{\sigma}_i(s)$ are lines of striction of $F_{(\gamma,\delta)}$. If the point $F_{(\gamma,\delta)}(s)$ is a non-cylindrical, then $\boldsymbol{\sigma}_1(s) = \boldsymbol{\sigma}_2(s)$ by the uniqueness of the line of striction, so that the ruled surface is a cylindrical surface. Since $\boldsymbol{\sigma}_i(s)$ are geodesics of $F_{(\gamma,\delta)}$, these are cylindrical helices and the rectifying plane of $\boldsymbol{\sigma}_i(s)$ is the tangent plane of $F_{(\gamma,\delta)}$. This means that $F_{(\gamma,\delta)}$ is the rectifying developable of $\boldsymbol{\sigma}_i(s)$. $\square$

Corollary 3.3 gives a characterization of cylindrical surfaces by the existence of geodesics with special properties. Especially, a cylindrical surface is the rectifying developable of a cylindrical helix which is a geodesic of the original surface. We now consider the question when a ruled surface is the rectifying developable of a curve.

Theorem 3.4. *Let $F_{(\gamma,\delta)}(s,u) = \boldsymbol{\gamma}(s) + u\boldsymbol{\delta}(s)$ be a non-singular ruled surface with $\|\boldsymbol{\delta}(s)\| = 1$. Let $\boldsymbol{\sigma}(s) = \boldsymbol{\gamma}(s) + u(s)\boldsymbol{\delta}(s)$ be a curve on $F_{(\gamma,\delta)}$ with $\kappa(s) \neq 0$. Then the following conditions are equivalent:*

(1) $F_{(\gamma,\delta)}$ *is the rectifying developable of* $\boldsymbol{\sigma}(s)$.
(2) $\boldsymbol{\sigma}(s)$ *is a geodesic of* $F_{(\gamma,\delta)}$ *which is transversal to rulings and* $F_{(\gamma,\delta)}$ *is a developable surface.*
(3) $\boldsymbol{\sigma}(s)$ *is a geodesic of* $F_{(\gamma,\delta)}$ *which is transversal to rulings and Gaussian curvature of* $F_{(\gamma,\delta)}$ *vanishes along* $\boldsymbol{\sigma}(s)$.

Proof. Since the Darboux vector field always transverse to rulings, the condition (2) holds under the assumption of the condition (1). It is trivial that the condition (3) follows from the condition (2). We assume that the condition (3) holds. Since $\boldsymbol{\sigma}(s)$ is transverse to rulings, we may assume that $\boldsymbol{\sigma}(s) = \boldsymbol{\gamma}(s)$. Gaussian curvature of $F_{(\gamma,\delta)}$ is given by

$$K(s,u) = -\frac{\det(\boldsymbol{\gamma}'(s), \boldsymbol{\delta}(s), \boldsymbol{\delta}'(s))^2}{(EG - F^2)^2},$$

then it vanishes along $\boldsymbol{\gamma}(s)$ if and only if $\det(\boldsymbol{\gamma}'(s), \boldsymbol{\delta}(s), \boldsymbol{\delta}'(s)) = 0$. Since $\boldsymbol{\gamma}(s)$ is a geodesic of $F_{(\gamma,\delta)}$, $\boldsymbol{\delta}(s)$ is contained in the rectifying plane of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(s)$. There exists $\lambda(s), \mu(s)$ such that $\boldsymbol{\delta}(s) = \lambda(s)\boldsymbol{t}(s) + \mu(s)\boldsymbol{b}(s)$, where $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ and $\boldsymbol{b}(s)$ is the binormal vector of $\boldsymbol{\gamma}$. By the Frenet-Serret formulae, we have

$$\boldsymbol{\delta}'(s) = \lambda'(s)\boldsymbol{t}(s) + \mu'(s)\boldsymbol{b}(s) + (\lambda(s)\kappa(s) - \mu(s)\tau(s))\boldsymbol{n}(s).$$

It follows from this formula that $\det(\boldsymbol{\gamma}'(s), \boldsymbol{\delta}(s), \boldsymbol{\delta}'(s)) = (\mu(s)\tau(s) - \lambda(s)\kappa(s))\mu(s)$. If there exists a point s_0 such that $\mu(s_0) = 0$, then $\boldsymbol{\delta}(s_0) = \lambda(s_0)\boldsymbol{t}(s_0)$. This contradicts to the assumption that $\boldsymbol{\gamma}$ is transversal to rulings.

Hence, we have $\mu(s)\tau(s) - \lambda(s)\kappa(s) = 0$, so that

$$\tau(s)\boldsymbol{\delta}(s) = \tau(s)\lambda(s)\boldsymbol{t}(s) + \kappa(s)\lambda(s)\boldsymbol{b}(s) = \lambda(s)D(s),$$

where $D(s)$ is the Darboux vector field along $\boldsymbol{\gamma}$. □

Since the rectifying developable of a cylindrical helix is a cylindrical surface, we have the following other characterization of cylindrical surfaces as a simple corollary of Theorem 3.4.

Corollary 3.5. *Suppose that* $F_{(\gamma,\delta)}$ *is a non-singular developable surface. If there exists a cylindrical helix with non-zero curvature on* $F_{(\gamma,\delta)}$ *which is a geodesic of* $F_{(\gamma,\delta)}$*, then* $F_{(\gamma,\delta)}$ *is a cylindrical surface.*

Moreover, we also have another characterization of cylindrical surfaces.

Corollary 3.6. *Let* $F_{(\gamma,\delta)}(s,u) = \boldsymbol{\gamma}(s) + u\boldsymbol{\delta}(s)$ *be a non-singular ruled surface. If there exists a planar geodesic of* $F_{(\gamma,\delta)}$ *with non-zero curvature which is perpendicular to rulings at any point, then* $F_{(\gamma,\delta)}$ *is a cylindrical surface.*

Proof. By the Frenet-Serret formulae, a planar geodesic is a line of curvature. Since the tangent vector of such a geodesic is perpendicular to the ruling, the direction of the ruling is also the principal direction. This means that $F_{(\gamma,\delta)}$ is a developable surface. Since any plane curve is a helix, the assertion follows from Corollary 3.5. □

On the other hand, we now consider asymptotic curves on ruled surfaces. We prepare the following simple lemma on Euclidean plane.

Lemma 3.7. *Let $\boldsymbol{e}_1, \boldsymbol{e}_2$ be the canonical basis of Euclidean plane $\mathbb{R}^2$. Let $\boldsymbol{v}_1, \boldsymbol{v}_2$ be unit vectors in $\mathbb{R}^2$. We assume that $\lambda > 0$ and α are chosen such that $\boldsymbol{v}_1 = \lambda(\boldsymbol{e}_1 + \alpha\boldsymbol{e}_2)$. Then $\boldsymbol{v}_2 = \lambda(\boldsymbol{e}_1 - \alpha\boldsymbol{e}_2)$ if and only if*

$$\boldsymbol{v}_2 \cdot \boldsymbol{e}_1 = \boldsymbol{v}_1 \cdot \boldsymbol{e}_1 \text{ and } \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = \frac{1-\alpha^2}{1+\alpha^2}.$$

Let $F_{(\gamma,\delta)}(s,u) = \boldsymbol{\gamma}(s) + u\boldsymbol{\delta}(s)$ be a ruled surface which is non-singular on $\boldsymbol{\gamma}(s)$. In this case, $\boldsymbol{\gamma}(s)$ is transversal to rulings. If Gaussian curvature is negative along $\boldsymbol{\gamma}(s)$, then there exist two different principal directions $\boldsymbol{e}_1(s), \boldsymbol{e}_2(s)$ along $\boldsymbol{\gamma}(s)$ with principal curvatures $\kappa_1(s), \kappa_2(s)$ respectively. We may assume that $\|\boldsymbol{e}_i(s)\| = 1$. We have the following proposition.

Proposition 3.8. *Under the same situation as the above, $\boldsymbol{\gamma}(s)$ is an asymptotic curve if and only if*

$$\boldsymbol{\gamma}'(s) \cdot \boldsymbol{e}_1(s) = \boldsymbol{\delta}(s) \cdot \boldsymbol{e}_1(s) \text{ and } \boldsymbol{\gamma}(s) \cdot \boldsymbol{\delta}(s) = \frac{\kappa_1(s)+\kappa_2(s)}{\kappa_1(s)-\kappa_2(s)}.$$

Proof. We now consider two tangent vectors at $\boldsymbol{\gamma}(s)$ which are given by

$$\boldsymbol{v}_1 = \boldsymbol{e}_1(s) + \sqrt{-\frac{\kappa_1(s)}{\kappa_2(s)}}\boldsymbol{e}_2(s),\ \boldsymbol{v}_2 = \boldsymbol{e}_1(s) - \sqrt{-\frac{\kappa_1(s)}{\kappa_2(s)}}\boldsymbol{e}_2(s).$$

Let $\boldsymbol{N}$ be the unit normal of $F_{(\gamma,\delta)}$ at $\boldsymbol{\gamma}(s)$. Since $(-d\boldsymbol{N})\boldsymbol{e}_i(s) = \kappa_i(s)\boldsymbol{e}_i$ $(i = 1,2)$, we have

$$(-d\boldsymbol{N})\boldsymbol{v}_i \cdot \boldsymbol{v}_i = \kappa_1(s) - \frac{\kappa_1(s)}{\kappa_2(s)}\kappa_2(s) = 0.$$

This means that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ give asymptotic directions at $\boldsymbol{\gamma}(s)$. Since Gaussian curvature is negative at $\boldsymbol{\gamma}(s)$ and $\boldsymbol{\delta}(s)$ gives an asymptotic direction, we may assume that

$$\boldsymbol{\delta}(s) = \lambda(s)\left(\boldsymbol{e}_1(s) + \sqrt{-\frac{\kappa_1(s)}{\kappa_2(s)}}\boldsymbol{e}_2(s)\right),$$

where $\lambda(s) = 1/\sqrt{1-\kappa_1(s)/\kappa_2(s)}$. If $\alpha = \sqrt{-\kappa_1(s)/\kappa_2(s)}$, then

$$\frac{1-\alpha^2}{1+\alpha^2} = \frac{\kappa_2(s)+\kappa_1(s)}{\kappa_2(s)-\kappa_1(s)}.$$

The assertion follows directly from the above lemma. □

We have the following corollary which is analogous result to Bonnet's theorem on geodesics of ruled surfaces.

Corollary 3.9. *Let $F_{(\gamma,\delta)}(s,u) = \boldsymbol{\gamma}(s) + u\boldsymbol{\delta}(s)$ be a ruled surface which is nonsingular on $\boldsymbol{\gamma}(s)$. We assume that $\boldsymbol{\gamma}(s)$ is an asymptotic curve of $F_{(\gamma,\delta)}$ and we denote that $\kappa_i(s)$ $(i = 1,2)$ are two different principal curvatures at $\boldsymbol{\gamma}(s)$. Then the following conditions are equivalent:*
(1) The angle between $\boldsymbol{\gamma}'(s)$ and $\boldsymbol{\delta}(s)$ is constant.
(2) $(\kappa_1/\kappa_2)(s)$ is constant.

In the first paragraph of this section we have shown that the mean curvature of the principal normal surface $F_{(\gamma,n)}$ vanishes along $\boldsymbol{\gamma}$ which is an asymptotic curve of $F_{(\gamma,n)}$. We can show the converse assertion is also true as a corollary of Proposition 3.8. We say that a curve on a surface is a *minimal asymptotic curve* if it is an asymptotic curve and the mean curvature vanishes along the curve.

Theorem 3.10. *Let $F_{(\gamma,\delta)}(s,u) = \boldsymbol{\gamma}(s) + u\boldsymbol{\delta}(s)$ be a ruled surface and $\boldsymbol{\sigma}(s)$ be a curve on $F_{(\gamma,\delta)}$. Then the following conditions are equivalent:*
(1) $F_{(\gamma,\delta)}$ is the principal normal surface of $\boldsymbol{\sigma}(s)$.
(2) The curve $\boldsymbol{\sigma}(s)$ is a minimal asymptotic curve of $F_{(\gamma,\delta)}$ which is transversal to rulings.

Proof. Suppose that condition (2) holds, then $\kappa_2(s) + \kappa_1(s) = 0$. By Proposition 3.8, $\boldsymbol{\sigma}'(s)$ is perpendicular to $\boldsymbol{\delta}(s)$. Since $\boldsymbol{\sigma}(s)$ is an asymptotic curve, this means that $\boldsymbol{\delta}(s)$ is parallel to the principal normal direction of $\boldsymbol{\sigma}(s)$. The converse assertion has already been proved. □

On the other hand, we have another proof of Theorem 3.10 as follows: Let $\boldsymbol{\sigma}(s) = \boldsymbol{\gamma}(s) + u(s)\boldsymbol{\delta}(s)$ be a curve on $F_{(\gamma,\delta)}$. Suppose that $F_{(\gamma,\delta)}$ is non-singular on $\boldsymbol{\sigma}(s)$. This means that $\boldsymbol{\sigma}(s)$ is transversal to rulings. Since

$$\boldsymbol{\sigma}'(s) = \frac{\partial F_{(\gamma,\delta)}}{\partial s}(s,u(s)) + u'(s)\frac{\partial F_{(\gamma,\delta)}}{\partial u}(s,u(s)),$$

$\boldsymbol{\sigma}(s)$ is an asymptotic curve if and only if

$$\det(\boldsymbol{\gamma}''(s) + u(s)\boldsymbol{\delta}''(s), \boldsymbol{\gamma}'(s) + u(s)\boldsymbol{\delta}'(s), \boldsymbol{\delta}(s)) + 2\det(\boldsymbol{\delta}'(s), \boldsymbol{\gamma}'(s), \boldsymbol{\delta}(s))u'(s) = 0.$$

Under the assumption that $K(s,u(s)) < 0$, $\boldsymbol{\sigma}(s)$ is a minimal asymptotic curve if and only if $u'(s) = -\boldsymbol{\delta}(s)\cdot\boldsymbol{\gamma}'(s)$. We remark that $\boldsymbol{\sigma}'(s)\cdot\boldsymbol{\delta}(s) = 0$ if and only if $u'(s) = -\boldsymbol{\delta}(s)\cdot\boldsymbol{\gamma}'(s)$. This completes the alternate proof of Theorem 3.10.

By using this method, we have the following characterization of helicoids.

Proposition 3.11. *Let $F_{(\gamma,\delta)}(s,u) = \boldsymbol{\gamma}(s) + u\boldsymbol{\delta}(s)$ be a non-singular ruled surface. If there exist three disjoint minimal asymptotic curves on $F_{(\gamma,\delta)}$ which are transversal to rulings, then $F_{(\gamma,\delta)}$ is a helicoid. In this case minimal asymptotic curves which are transversal to rulings are circular helices.*

Proof. By the previous calculation, we remark that mean curvature of $F_{(\gamma,\delta)}$ is a quadratic function of the u variable. If there exist three disjoint minimal asymptotic curves on $F_{(\gamma,\delta)}$ which are transversal to rulings, then mean curvature always vanishes. This means that the surface $F_{(\gamma,\delta)}$ is a minimal surface. It has been classically known that a minimal ruled surface is a helicoid or a plane. In this case each minimal asymptotic curve which is transversal to rulings is a circular helix. □

Finally we give a characterization of Bertrand curves as curves on ruled surfaces.

Proposition 3.12. *Let $F_{(\gamma,\delta)}(s,u) = \boldsymbol{\gamma}(s) + u\boldsymbol{\delta}(s)$ be a non-singular ruled surface. If there exist two disjoint minimal asymptotic curves on $F_{(\gamma,\delta)}$ which are transversal to rulings, then these curves are a Bertrand curve and the Bertrand mate of each other.*

Proof. Let $\boldsymbol{\sigma}_i(s) = \boldsymbol{\gamma}(s) + u_i(s)\boldsymbol{\delta}(s)$ $(i = 1, 2)$ be minimal asymptotic curves which are transversal to rulings. By Theorem 4.10, $F_{(\gamma,\delta)}$ is the principal normal surface of $\boldsymbol{\sigma}_i(s)$. By the previous argument, we have $u_i'(s) = -\boldsymbol{\delta}(s) \cdot \boldsymbol{\gamma}'(s)$, so that $(u_1 - u_2)'(s) = 0$. Thus there exists a constant A such that $u_1(s) = u_2(s) + A$. It follows from this fact that $\boldsymbol{\sigma}_1(s) = \boldsymbol{\sigma}_2(s) + A\boldsymbol{\delta}(s)$. We may choose s as the arc-length parameter of $\sigma_2(s)$. In this case $\boldsymbol{\delta}(s)$ can be considered as the unite principal normal of $\boldsymbol{\sigma}_2(s)$. By the calculation of mean curvature of the principal normal surface of $\boldsymbol{\sigma}_2(s)$ and Corollary 2.2, $\boldsymbol{\sigma}_1(s)$ is a Bertrand curve and $\boldsymbol{\sigma}_2(s)$ is the Bertrand mate of $\boldsymbol{\sigma}_1(s)$. $\square$

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Received April 12, 2001; revised version July 17, 2001