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On Mappings Preserving a Family of Star Bodies

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Abstract. The paper concerns the star mappings understood as topological embedding of $\mathbb{R}^n$ into itself preserving the class of bodies which are star shaped at point 0. The main result is full characterization of star mappings (Theorem 2.8). At the end we give a solution of some related problem.

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1. Introduction

This paper consists of two different parts, both related to [1]. Moszyńska in [1] defined a set $GS(n)$ of transformations called "generalized star mappings". They are positively homogeneous homeomorphisms of $\mathbb{R}^n$ onto itself. That class of mappings is suitable for the notion of quotient star body (comp. [1], Prop. 2.6), however (in contrary to the statement in [1], p. 47) it is not the largest possible family of maps preserving the class $\mathcal{S}^n$ of star bodies under consideration. Section 2 concerns the structure of the largest family Ω^n of maps preserving $\mathcal{S}^n$. In Section 3 we give a solution of Problem 1 in [1].

We use the following terminology and notation: By $\mathbb{R}_+$ we denote the set $\{r \in \mathbb{R};\ r \geq 0\}$, by Σ the set of topological embeddings of $\mathbb{R}_+$ into $\mathbb{R}_+$ preserving 0. For affine independent points $x_1, \ldots, x_n$ in $\mathbb{R}^n$ the simplex with vertices $x_1, \ldots, x_n$ is denoted by $\Delta(x_1, \ldots, x_n)$. As usually, $\mathbb{B}^n$ and S^{n-1} are the unit ball and the unit sphere. Let A be a nonempty compact subset of $\mathbb{R}^n$; then A is a body if and only if $A = \mathrm{cl}(\mathrm{int}(A))$; the set A is called star shaped at 0 if $\Delta(a,0) \subset A$ for every $a \in A$. The radial function $\rho_A : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}_+$ of a set A star shaped at 0 is defined by the formula

$$\rho_A(x) = \sup\{\lambda \geq 0;\ \lambda x \in A\}. \tag{1}$$

A set $A \subset \mathbb{R}^n$ will be called a star body whenever A is star shaped at 0 and its radial function restricted to S^{n-1} is continuous. The set of all star bodies in $\mathbb{R}^n$ is denoted by $\mathcal{S}^n$. The set of all halflines in $\mathbb{R}^n$ starting at 0 will be denoted by $\mathcal{L}^n$. To every $x \in \mathbb{R}^n$ such that $x \neq 0$ we assign the halfline $\operatorname{pos}(x) \in \mathcal{L}^n$ defined by the formula

$$\operatorname{pos}(x) = \left\{y \in \mathbb{R}^n;\ \exists_{\lambda\in\mathbb{R}_+}\ y = \lambda x\right\}. \tag{2}$$

2. Star mappings

In this section we shall describe the structure of the family Ω^n of *generalized star mappings* defined as follows:

Definition 2.1. *Ω^n is the family of all topological embeddings of $\mathbb{R}^n$ into itself, preserving the point* 0 *and the class $\mathcal{S}^n$.*

Lemma 2.2. *Let $\omega \in \Omega^n$. Then*

(i) *for every $B \in \mathcal{L}^n$ there exists $C \in \mathcal{L}^n$ with $\omega(B) \subset C$;*

(ii) *for every $B, B', C \in \mathcal{L}^n$ if $\omega(B) \subset C$ and $\omega(B') \subset C$, then $B = B'$.*

Proof. First we prove that the image of every closed segment starting at 0 is again a closed segment starting at 0. Let $g \in \mathbb{R}^n$, $g \neq 0$ and $h = \omega(g)$. Let $G = \Delta(0,g)$ and $H = \Delta(0,h)$. Since, evidently, for every $\varepsilon > 0$ the set G_ε belongs to $\mathcal{S}^n$, it follows that $\omega(G_\varepsilon) \in \mathcal{S}^n$. Further, since $h = \omega(g) \in \omega(G) \subset \omega(G_\varepsilon)$, we get:

$$\forall_{\varepsilon>0}\ H \subset \omega(G_\varepsilon). \tag{3}$$

Let $\{\varepsilon_k\}_{k=0}^\infty$ be a sequence convergent to 0 such that $\varepsilon_k > 0$ for every k. The set G is compact; thus $G = \bigcap_{k=0}^\infty G_{\varepsilon_k}$. The mapping ω is a topological embedding; thus

$$\omega(G) = \bigcap_{k=0}^{\infty} \omega(G_{\varepsilon_k}). \tag{4}$$

From (3) and (4) we obtain:

$$H \subset \omega(G). \tag{5}$$

Since the arc $\omega(G)$ and the segment H have common ends, if follows that

$$H = \omega(G). \tag{6}$$

If the second part of the lemma is false, then there exist points $g, h \neq 0$ such that $\omega(g)$, $\omega(h)$ belong to one halfline from $\mathcal{L}^n$ thought g, h do not belong to such a halfline. We may assume that $\|\omega(g)\| \geq \|\omega(h)\|$. Using the first part of lemma, we get $\omega(h) \in \omega(\Delta(0,g))$. So there exists $c \in \Delta(0, g)$ such that $\omega(h) = \omega(c)$. Since ω is a topological embedding, it follows that $c = h$. Hence g, h belong to one halfline. □

Corollary 2.3. *Let $\omega \in \Omega^n$ then for all $x, y \in \mathbb{R}^n$*

$$\frac{x}{\|x\|} = \frac{y}{\|y\|} \Leftrightarrow \frac{\omega(x)}{\|\omega(x)\|} = \frac{\omega(y)}{\|\omega(y)\|}.$$

Proof. ($\Rightarrow$) Let us assume that $\frac{x}{\|x\|} = \frac{y}{\|y\|}$. That means that x and y belong to one element of $\mathcal{L}^n$. Using Lemma 2.2(i) we infer that also $\omega(x)$ and $\omega(y)$ belong to one element of $\mathcal{L}^n$. This is equivalent to the condition $\frac{\omega(x)}{\|\omega(x)\|} = \frac{\omega(y)}{\|\omega(y)\|}$.

($\Leftarrow$) If we assume now that $\frac{x}{\|x\|} \neq \frac{y}{\|y\|}$, then in similar way, using Lemma 2.2(ii), we get $\frac{\omega(x)}{\|\omega(x)\|} \neq \frac{\omega(y)}{\|\omega(y)\|}$. □

It is now clear that we can look at $\mathbb{R}^n$ as the union of halflines starting at 0, which will be called "hairs". What $\omega \in \Omega^n$ can do with a hair? First, it can move any point different from 0 along the hair. It can even map a hair on a subset of some hair of a finite length. The way the points are moved along the hair will be described in terms of mappings from the family Φ^n defined as follows:

Definition 2.4. $\Phi^n = \left\{\phi : S^{n-1} \to \Sigma;\ \forall_{r \in \mathbb{R}_+} \forall_{u_k, u \in S^{n-1}}\, u_k \to u \Rightarrow (\phi(u_k))(r) \to (\phi(u))(r)\right\}$.

The arguments of Φ are points in S^{n-1}, which determine the hair. Each value is a topological embedding of $\mathbb{R}^n$ into itself; it gives us full information about $\|x\|$ and $\|\omega(x)\|$ for any $x \in \mathbb{R}^n$. A hair can also change its direction. To describe it we shall use mappings from the class Ψ^n defined as follows:

Definition 2.5. *Let Ψ^n be the class of homeomorphisms of S^{n-1} onto itself.*

The information stored in such mappings is direction of a hair and its image under ω. To every mapping from Φ^n and Ψ^n we shall assign mappings of $\mathbb{R}^n$ into itself.

Definition 2.6. *To every $\phi \in \Phi^n$ we assign the mapping $\widehat{\phi} : \mathbb{R}^n \to \mathbb{R}^n$ satisfying the condition*

$$\forall_{u \in S^{n-1}} \forall_{r \in \mathbb{R}_+} \widehat{\phi}(ru) = (\phi(u))(r)\, u. \tag{7}$$

Similarly, for every $\psi \in \Psi^n$ the mapping $\widetilde{\psi} : \mathbb{R}^n \to \mathbb{R}^n$ is defined by the condition

$$\forall_{u \in S^{n-1}} \forall_{r \in \mathbb{R}_+} \widetilde{\psi}(ru) = r\psi(u). \tag{8}$$

It seems to be clear that

$$\forall_{\phi \in \Psi^n} \forall_{\psi \in \Psi^n} \forall_{u \in S^{n-1}} (\phi(u))(0)\, u = 0 = 0\psi(u); \tag{9}$$

so we do not have to worry about the choice of u when $x = 0$ in Definition 2.6. These mappings have a property which is very useful for us:

Lemma 2.7. *For every $\phi \in \Phi^n$ and $\psi \in \Psi^n$ the mappings $\widehat{\phi}, \widetilde{\psi}$ defined in* 2.6 *are in the class* Ω^n.

Proof. First we prove that $\widehat{\phi}, \widetilde{\psi}$ are topological embeddings. We begin with $\widehat{\phi}$ proving that

$$\forall_{x_k, x \in \mathbb{R}^n}\, x_k \to x \Leftrightarrow \widehat{\phi}\,(x_k) \to \widehat{\phi}\,(x)\,. \tag{10}$$

Case 1. Let $x = 0$ then $\widehat{\phi}(x) = 0$.
($\Rightarrow$)For every $\varepsilon > 0$ let us consider the function $h^\varepsilon : S^{n-1} \to \mathbb{R}_+$ defined as follows:

$$h^\varepsilon\,(v) = \phi\,(v)\,(\varepsilon)\,.$$

Evidently, $h^\varepsilon\,(v) = \left\|\widehat{\phi}\,(\varepsilon v)\right\|$. By Definition 2.4 the function h^ε is positive and continuous. Moreover

$$\forall_{v \in S^{n-1}} \forall_{\alpha,\beta>0}\, \alpha \geq \beta \Leftrightarrow h^\alpha\,(v) \geq h^\beta\,(v)\,, \tag{11}$$

$$\forall_{v \in S^{n-1}} \lim_{\alpha \searrow 0} h^\alpha\,(v) = 0. \tag{12}$$

So $\lim_{\varepsilon \searrow 0} h^\varepsilon = 0$ pointwise; by (11), since S^{n-1} is compact, the convergence is uniform. We get:

$$\forall_{\delta>0} \exists_{\mu>0} \forall_{0 \leq \varepsilon < \mu} \forall_{v \in S^{n-1}} h^\varepsilon\,(v) < \delta. \tag{13}$$

Since $h^\varepsilon\,(v) = \left\|\widehat{\phi}\,(\varepsilon v)\right\|$, the previous condition can be reformulated as follows:

$$\forall_{\delta>0} \exists_{\mu>0} \forall_{z \in \mathbb{R}^n}\, \|z\| < \mu \Rightarrow \left\|\widehat{\phi}\,(z)\right\| < \delta. \tag{14}$$

($\Leftarrow$)Let $\delta > 0$ and $\mu = \min_{v \in S^{n-1}} h^\delta\,(v) = \min_{v \in S^{n-1}} \left\|\widehat{\phi}\,(\delta v)\right\|$. By Definition 2.4 mapping $v \mapsto \left\|\widehat{\phi}\,(\delta v)\right\| = \phi\,(v)\,(\delta)$ is continuous, moreover the set S^{n-1} is compact so $\mu > 0$. Let $z \in \mathbb{R}^n$, $u \in S^{n-1}$, $r \in \mathbb{R}_+$ be such that $z = ru$ and $\left\|\widehat{\phi}\,(z)\right\| < \mu$. Since $\left\|\widehat{\phi}\,(\delta u)\right\| > \mu$, it follows that $\|\widehat{\varphi}\,(\delta u)\| > \|\widehat{\varphi}\,(z)\|$. Using (11) we obtain $\|z\| < \|\delta u\| = \delta$. Thus

$$\forall_{z \in \mathbb{R}^n} \left\|\widehat{\phi}\,(z)\right\| < \mu \Rightarrow \|z\| < \delta, \tag{15}$$

which completes the proof of (10) for $x = 0$.
Case 2. Let $x \neq 0$; then $\widehat{\phi}(x) \neq 0$.
($\Leftarrow$) We may assume that $x_k \neq 0$ which implies $\widehat{\phi}\,(x_k) \neq 0$. By Definition 2.5

$$\forall_{y \in \mathbb{R}^n \setminus \{0\}} \frac{y}{\|y\|} = \frac{\widehat{\phi}\,(y)}{\left\|\widehat{\phi}\,(y)\right\|}. \tag{16}$$

Thus

$$\frac{x_k}{\|x_k\|} \to \frac{x}{\|x\|} \Leftrightarrow \frac{\widehat{\phi}(x_k)}{\left\|\widehat{\phi}\,(x_k)\right\|} \to \frac{\widehat{\phi}\,(x)}{\left\|\widehat{\phi}\,(x)\right\|}. \tag{17}$$

It remains to prove that

$$\|x_k\| \to \|x\| \Leftrightarrow \left\|\widehat{\phi}\left(x_k\right)\right\| \to \left\|\widehat{\phi}\left(x\right)\right\|. \tag{18}$$

Let $r = \|x\|$ and $u \in S^{n-1}$ be such that $x = ru$. For every $\varepsilon \in (0; r)$ we consider the functions $h_1^\varepsilon, h_2^\varepsilon : S^{n-1} \to \mathbb{R}_+$ defined as follows:

$$h_1^\varepsilon\left(v\right) = \phi\left(v\right)\left(r+\varepsilon\right) - \phi\left(v\right)\left(r\right), \tag{19}$$

$$h_1^\varepsilon\left(v\right) = \phi\left(v\right)\left(r\right) - \phi\left(v\right)\left(r-\varepsilon\right). \tag{20}$$

Like the mapping h^ε, both $h_1^\varepsilon, h_2^\varepsilon$ are positive, continuous and uniformly convergent to 0 when $\varepsilon \searrow 0$. For every $\delta > 0$ there exists $\mu > 0$ such that $\varepsilon < \mu \Rightarrow h_i^\varepsilon < \frac{\delta}{2}$ for $i \in \{1, 2\}$. This means that

$$\forall_{y\in\mathbb{R}^n}\ 0 \le \|y\| - r < \mu \Rightarrow \left|\left\|\widehat{\phi}\left(y\right)\right\| - \left\|\widehat{\phi}\left(r\frac{y}{\|y\|}\right)\right\|\right| = h_1^{\|y\|-r}\left(\frac{y}{\|y\|}\right) < \frac{\delta}{2}, \tag{21}$$

$$\forall_{y\in\mathbb{R}^n}\ 0 \le r - \|y\| < \mu \Rightarrow \left|\left\|\widehat{\phi}\left(y\right)\right\| - \left\|\widehat{\phi}\left(r\frac{y}{\|y\|}\right)\right\|\right| = h_2^{r-\|y\|}\left(\frac{y}{\|y\|}\right) < \frac{\delta}{2}. \tag{22}$$

So

$$\forall_{y\in\mathbb{R}^n}\ \left|\|y\| - r\right| < \mu \Rightarrow \left|\left\|\widehat{\phi}\left(y\right)\right\| - \left\|\widehat{\phi}\left(r\frac{y}{\|y\|}\right)\right\|\right| < \frac{\delta}{2}, \tag{23}$$

First we prove $(\Rightarrow)$ in (18). Since the mapping $\phi\left(\bullet\right)\left(r\right)$ is continuous, it follows that there exists $U \subset S^{n-1}$, an open neighborhood of u, satisfying

$$\forall_{v\in U}\ \left|\phi\left(u\right)\left(r\right) - \phi\left(v\right)\left(r\right)\right| < \frac{\delta}{2}. \tag{24}$$

Since $x = \lim x_k$, there exists l such that

$$\forall_{k\ge l}\ \frac{x_k}{\|x_k\|} \in U \ \&\ \left|\|x_k\| - r\right| < \mu \tag{25}$$

and for any $k \ge l$

$$\left|\left\|\widehat{\phi}(x_k)\right\| - \left\|\widehat{\phi}\left(x\right)\right\|\right| \le \left|\left\|\widehat{\phi}\left(x_k\right)\right\| - \left\|\widehat{\phi}\left(r\frac{x_k}{\|x_k\|}\right)\right\|\right| + \left|\left\|\widehat{\phi}\left(r\frac{x_k}{\|x_k\|}\right)\right\| - \left\|\widehat{\phi}\left(x\right)\right\|\right| \le$$

$$\le \frac{\delta}{2} + \left|\phi\left(\frac{x_k}{\|x_k\|}\right)\left(r\right) - \phi\left(u\right)\left(r\right)\right| \le \delta. \tag{26}$$

This proves $(\Rightarrow)$ in (18).

Let now $\delta \in (0; r)$. Let $\mu = \frac{1}{2}\min\left\{\min_{v\in S^{n-1}} h_1^\delta\left(v\right), \min_{v\in S^{n-1}} h_2^\delta\left(v\right)\right\}$ and $U \subset S^{n-1}$ be an open neighborhood of u such that

$$\forall_{v\in U}\ \left|\phi\left(u\right)\left(r\right) - \phi\left(v\right)\left(r\right)\right| < \mu. \tag{27}$$

We get:

$$\forall_{v\in U}\forall_{t\geq r+\delta}\ \left|\left\|\widehat{\phi}\left(tv\right)\right\|-\left\|\widehat{\phi}\left(x\right)\right\|\right|=\left|\phi\left(v\right)\left(t\right)-\phi\left(u\right)\left(r\right)\right|\geq$$

$$\geq\left|\phi\left(v\right)\left(t\right)-\phi\left(v\right)\left(r\right)\right|-\left|\phi\left(v\right)\left(r\right)-\phi\left(u\right)\left(r\right)\right|\geq\left|\phi\left(v\right)\left(r+\delta\right)-\phi\left(u\right)\left(r\right)\right|-\mu\geq\left(2\mu\right)-\mu=\mu \tag{28}$$

and

$$\forall_{v\in U}\forall_{t\leq r-\delta}\ \left|\left\|\widehat{\phi}\left(tv\right)\right\|-\left\|\widehat{\phi}\left(x\right)\right\|\right|=\left|\phi\left(v\right)\left(t\right)-\phi\left(u\right)\left(r\right)\right|\geq$$

$$\geq\left|\phi\left(v\right)\left(t\right)-\phi\left(v\right)\left(r\right)\right|-\left|\phi\left(v\right)\left(r\right)-\phi\left(u\right)\left(r\right)\right|\geq\left|\phi(v)\left(r-\delta\right)-\phi\left(u\right)\left(r\right)\right|-\mu\geq\left(2\mu\right)-\mu=\mu \tag{29}$$

By (28) and (29) we obtain:

$$\forall_{v\in U}\forall_{t>0}\ \left|t-r\right|\geq\delta\Rightarrow\left|\left\|\widehat{\phi}\left(tv\right)\right\|-\left\|\widehat{\phi}\left(x\right)\right\|\right|\geq\mu. \tag{30}$$

In other words

$$\frac{\widehat{\phi}\left(x_k\right)}{\left\|\widehat{\phi}\left(x_k\right)\right\|}\in U\ \&\ \left|\left\|\widehat{\phi}\left(x_k\right)\right\|-\left\|\widehat{\phi}\left(x\right)\right\|\right|<\mu\Rightarrow\left|\left\|x_k\right\|-\left\|x\right\|\right|<\delta. \tag{31}$$

This proves (18); thus the proof of (10) is complete. Now we look at the mapping $\widetilde{\psi}$. We are going to show that:

$$\forall_{x_k,x\in\mathbb{R}^n}\ x_k\to x\Leftrightarrow\widetilde{\psi}\left(x_k\right)\to\widetilde{\psi}\left(x\right). \tag{32}$$

From (8) we get

$$\forall_{y\in\mathbb{R}^n}\ \left\|\widetilde{\psi}\left(y\right)\right\|=\left\|y\right\|, \tag{33}$$

which implies (32) for $x=0$ (i.e. for $\widetilde{\psi}\left(x\right)=0$). Let $x\neq 0$. From (8) we get

$$\forall_{y\in\mathbb{R}^n\setminus\{0\}}\ \frac{\widetilde{\psi}\left(y\right)}{\left\|\widetilde{\psi}\left(y\right)\right\|}=\psi\left(\frac{y}{\left\|y\right\|}\right). \tag{34}$$

By 2.5, (8), and (34) we obtain

$$\frac{x_k}{\left\|x_k\right\|}\to\frac{x}{\left\|x\right\|}\Leftrightarrow\frac{\widetilde{\psi}\left(x_k\right)}{\left\|\widetilde{\psi}\left(x_k\right)\right\|}\to\frac{\widetilde{\psi}\left(x\right)}{\left\|\widetilde{\psi}\left(x\right)\right\|}. \tag{35}$$

Combining (33) and (35), we get (32) for $x\neq 0$.

We proved that both $\widehat{\phi}$ and $\widetilde{\psi}$ are topological embeddings. So if $A=\mathrm{cl}\left(\mathrm{int}\left(A\right)\right)$, then $\widehat{\phi}\left(A\right)=\mathrm{cl}\left(\mathrm{int}\left(\widehat{\phi}\left(A\right)\right)\right)$ and $\widetilde{\psi}\left(A\right)=\mathrm{cl}\left(\mathrm{int}\left(\widetilde{\psi}\left(A\right)\right)\right)$. Moreover, it is easy to show that both $\widehat{\phi}$ and $\widetilde{\psi}$ take every segment starting at 0 to segment starting at 0 too. Hence if A is star shaped at 0 then $\widehat{\phi}\left(A\right)$ and $\widetilde{\psi}\left(A\right)$ are star shaped at 0 too. Finally, it can be proved that for every $v\in S^{n-1}$:

$$\rho_{\widehat{\phi}(A)}\left(v\right)=\phi\left(v\right)\left(\rho_A\left(v\right)\right) \tag{36}$$

and

$$\rho_{\widetilde{\psi}(A)}(v) = \rho_A\left(\psi^{-1}(v)\right). \tag{37}$$

So if A is a star body and $\rho_A|_{S^{n-1}}$ is continuous, then $\rho_{\widehat{\phi}(A)}|_{S^{n-1}}$ and $\rho_{\widetilde{\psi}(A)}|_{S^{n-1}}$ are continuous. □

Now we are ready to prove the main result:

Theorem 2.8. *The mapping* $(\phi, \psi) \mapsto \left(\widetilde{\psi} \circ \widehat{\phi}\right)$ *is a biunique correspondence between the sets* $(\Phi^n \times \Psi^n)$ *and* Ω^n.

Proof. Let $\omega \in \Omega^n$. The mappings ϕ and ψ will be defined as follows:

$$\phi(v)(r) = \|\omega(rv)\|, \tag{38}$$

$$\psi(v) = \frac{\omega(v)}{\|\omega(v)\|} \tag{39}$$

for any $v \in S^{n-1}$ and $r \in \mathbb{R}_+$. At the beginning we shall prove that $\phi \in \Phi^n$. Let $u \in S^{n-1}$; let $B = \operatorname{pos}(u)$ and $C = \operatorname{pos}(\omega(u))$. The mapping $\phi(u)(\bullet)$ can be expressed as the composition of three mappings. The first of them goes from $\mathbb{R}_+$ to B. It is given by the formula $t \mapsto tu$. The second is the restriction of ω to B. The third one $x \mapsto \|x\|$ goes from C to $\mathbb{R}_+$. The first and the last one are homeomorphisms. The second is a topological embedding. That means that $\phi(u)(\bullet) \in \Sigma$. Let $u_k \in S^{n-1}$; $r > 0$. We know that

$$u_k \to u \Leftrightarrow ru_k \to ru \Leftrightarrow \omega(ru_k) \to \omega(ru) \Rightarrow \|\omega(ru_k)\| \to \|\omega(ru)\| \Leftrightarrow \phi(r)(u_k) \to \phi(r)(u).$$

So $\phi \in \Phi^n$.

Now it is time to prove that $\psi \in \Psi^n$. We know that $0 = \omega(0) \in \omega(\operatorname{int}(\mathbb{B}^n))$ and $\omega(\mathbb{B}^n)$ is bounded. So for every halfline C starting at 0 the set $C \cap \operatorname{bd}(\omega(\mathbb{B}^n))$ is not empty. Since $\operatorname{bd}(\omega(\mathbb{B}^n)) = \omega(S^{n-1})$ then ψ is surjective. By Corollary 2.3, ψ is injective. By (39), ψ is continuous. Let $\lambda > 0$ be such that $2\lambda\mathbb{B}^n \subset \omega(\mathbb{R}^n)$ and mapping $h : S^{n-1} \to S^{n-1}$ satisfies the condition

$$h(v) = \frac{\omega^{-1}(\lambda v)}{\|\omega^{-1}(\lambda v)\|}. \tag{40}$$

The mapping h is continuous. Moreover:

$$\psi h(v) = \frac{\omega\left(\frac{\omega^{-1}(\lambda v)}{\|\omega^{-1}(\lambda v)\|}\right)}{\left\|\omega\left(\frac{\omega^{-1}(\lambda v)}{\|\omega^{-1}(\lambda v)\|}\right)\right\|}. \tag{41}$$

By Corollary 2.3

$$\frac{\omega\left(\frac{1}{\|\omega^{-1}(\lambda v)\|}\omega^{-1}(\lambda v)\right)}{\left\|\omega\left(\frac{1}{\|\omega^{-1}(\lambda v)\|}\omega^{-1}(\lambda v)\right)\right\|} = \frac{\omega(\omega^{-1}(\lambda v))}{\|\omega(\omega^{-1}(\lambda v))\|} = \frac{\lambda v}{\|\lambda v\|} = v; \tag{42}$$

so ψ^{-1} is continuous. That means that $\psi \in \Psi^n$. Let us notice that

$$\widetilde{\psi} \circ \widehat{\varphi}(ru) = \widetilde{\psi}(\widehat{\varphi}(ru)) = \widetilde{\psi}(\varphi(u)(r)u) = \varphi(u)(r)\psi(u) =$$

$$= \|\omega(ru)\| \frac{\omega(u)}{\|\omega(u)\|} = \|\omega(ru)\| \frac{\omega(ru)}{\|\omega(ru)\|} = \omega(ru). \tag{43}$$

We proved that the mapping $(\varphi, \psi) \mapsto \widetilde{\psi} \circ \widehat{\phi}$ is surjective. It remains to show that it is injective. Let $\phi_1, \phi_2 \in \Phi^n$, $\psi_1, \psi_2 \in \Psi^n$, and

$$\widetilde{\psi_1} \circ \widehat{\phi_1} = \widetilde{\psi_2} \circ \widehat{\phi_2}. \tag{44}$$

Then

$$\widetilde{\psi_2}^{-1} \circ \widetilde{\psi_1} \circ \widehat{\phi_1} = \widehat{\phi_2}. \tag{45}$$

Since $\widehat{\phi_2}$ preserves directions, it follows that so does $\widetilde{\psi_2}^{-1} \circ \widetilde{\psi_1} \circ \widehat{\phi_1}$. So $\widetilde{\psi_2}^{-1} \circ \widetilde{\psi_1} = \mathrm{id}$; hence $\widehat{\phi_1} = \widehat{\phi_2}$. □

It may seem that the mappings from the classes Φ^n and Ψ^n are very simple. So we may think that so are the mappings from Ω^n. To help the reader to realize how complicated they are we shall give two examples:

Example 2.9. Let $\phi(u)(r) = 1 - \exp(-r)$ and $\omega = \widehat{\phi}$. It is easy to see that

$$\forall_{u \in S^{n-1}} \forall_{r \in \mathbb{R}_+} \; \|\omega(ru)\| = \phi(u)(r) < 1. \tag{46}$$

So we cannot expect that $\omega(\mathbb{R}^n) = \mathbb{R}^n$. I think that it is not very surprising because after applying ω image of every hair has finite lengths. It seems to be more interesting that there exists a generalized star mapping ω for which images of some hairs have finite lengths and images of another hairs have infinite lengths.

Example 2.10. Let $n = 2$ and $e_1 = (1,0)$. Let the $\mu(u) = \angle(u, e_1)$. Let mapping $\phi(u)(r) = \mu(u)r + (1 - \exp(-r))$ and $\omega = \widehat{\phi}$. It is easy to see that if $u \neq e_1$, then $\phi(u)(r)$ can be arbitrarily large while $\phi(e_1)(r) < 1$ for every r.

3. A solution of Problem 1 in [1]

To every $A \in \mathcal{S}^n$ we assign the subset S_A of the unit sphere:

$$S_A = \left\{u \in S^{n-1};\ \rho_A(u) > 0\right\}. \tag{47}$$

M. Moszyńska proved that if $A, B \in \mathcal{S}^n$ and there exists $\omega \in GS(n)$ such that $\omega(A) = B$, then S_A is homeomorphic to S_B. She asked if the existence of a homeomorphism between S_A and S_B suffices for A, B to be star equivalent in sense of [1].

Proposition 3.1. *If $A, B \in \mathcal{S}^n$ and there exists $\omega \in \Omega^n$ such that $\omega(A) = B$, then $S^{n-1} \setminus S_A$ is homeomorphic to $S^{n-1} \setminus S_B$.*

Proof. By Theorem 1.8 there exist $\phi \in \Phi^n$ and $\psi \in \Psi^n$ such that $\widetilde{\psi} \circ \widehat{\phi} = \omega$. It can be easily proved that if $\omega(A) = B$ then $\psi(S^{n-1} \setminus S_A) = S^{n-1} \setminus S_B$ (and $\psi(S_A) = S_B$). The mapping ψ is a homeomorphism; thus the restriction of ψ to $S^{n-1} \setminus S_A$ is a homeomorphism as well. This completes the proof. □

The following example shows that the answer to the above question is negative even for the larger family Ω^n.

Example 3.2. Let $n = 2$ and let $A, B \in \mathcal{S}^n$ be defined by the values of their radial functions restricted to S^{n-1}:

$$\rho_A(u) = |\langle u; e_1\rangle|, \tag{48}$$

$$\rho_B(u) = \max\left\{|\langle u; e_1\rangle| - \frac{1}{2},\, 0\right\}. \tag{49}$$

The set $S^{n-1} \setminus S_A$ consists of two points, while the set $S^{n-1} \setminus S_B$ consists of two closed arcs. In this case there is no star mapping $\omega \in \Omega^n$ such that $\omega(A) = B$. On the other hand, each of S_A and S_B consists of two open arcs. So they are homeomorphic.

This example can be generalized to any $n \geq 2$ showing that the answer is no even if we look in family Ω^n. Moreover it can be proved that even if S_A is homeomorphic to S_B and $S^{n-1} \setminus S_A$ is homeomorphic to $S^{n-1} \setminus S_B$ we cannot expect that A, B are star equivalent.

References

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