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On the Dimension of Finite Permutation Group Actions

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Abstract. The dimension (or "minimal base size") of a finite permutation group action is defined to be the smallest power of the action that contains a regular orbit. Although the concept has appeared before in various contexts, the intention of the current paper is to survey it from a slightly different viewpoint, with particular emphasis on its behaviour with respect to G-set constructions. Elementary inequalities relate the dimension to the degree and closure properties of the action. The dimension is also expressed exactly in terms of the Möbius function of the subgroup lattice of the permutation group. For geometric permutation actions, the dimension is related to the geometric dimension of the space being acted on. The behaviour of the dimension is studied with respect to disjoint unions, Cartesian products, and wreath products of actions. Use of the wreath product construction exhibits permutation group actions with arbitrary positive integral dimension and degree-to-dimension ratio.

1. Introduction

Let G be a finite group. A *G-set* (X, G) or *permutation representation* of the group G consists of a set X, together with a (right) action of G on X via a homomorphism

$$G \to X!; g \mapsto (x \mapsto xg) \tag{1.1}$$

from G into the group $X!$ of all permutations of the set X. The cardinality of the set X is described as the *degree* $\deg(X, G)$ or $\deg X$. A G-set (X, G) may be construed as an algebra of unary operations on the set X. It is said to be *transitive* if it is non-empty, but has no proper non-empty subalgebras. Transitive subalgebras of a G-set (X, G) are described as its *orbits*. The set of orbits of (X, G) is denoted by X/G. For each positive integer r, the *direct power* $(X, G)^r$ of a G-set (X, G) is the direct power X^r with the *diagonal action*

$$g : (x_1, \dots, x_r) \mapsto (x_1 g, \dots, x_r g) \tag{1.2}$$

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of the elements g of G. The G-set (X, G) is said to be *faithful* if (1.1) injects. In this case, one often identifies G with its image under (1.1) and refers to (X, G) as a *permutation action*, describing G as a *permutation group* on the set X.

For a subgroup H of a group G, let $H\backslash G = \{Hx \mid x \in G\}$ denote the set of right cosets of H in G. The transitive G-set $(H\backslash G, G)$ is then defined by $g : H\backslash G \to H\backslash G; Hx \mapsto Hxg$ for x, g in G. Each transitive G-set (X, G) is isomorphic to the G-set $(\mathrm{Stab}_G(x) \setminus G, G)$ for the *stabilizer* $\mathrm{Stab}_G(x) = \{g \in G \mid xg = x\}$ of an element x of X. A transitive G-set (X, G) is said to be *regular* if it is isomorphic to the G-set $(\{1\} \setminus G, G)$.

The concepts of faithfulness and regularity are related: a transitive G-set (X, G) is faithful if and only if some power of (X, G), strictly less than the degree $\deg(X, G)$, contains a regular orbit (Proposition 2.2). This observation raises the following:

Problem 1.1. [8, Problem 5.3]. Determine the smallest power of a permutation action that contains a regular orbit.

For transitive actions, Problem 1.1 may be answered in principle (Theorem 3.2) in terms of the action of the permutation group by conjugation on its lattice of subgroups. Unless the point stabilizer is relatively small, however (cf. Theorem 4.2), this method is not very practicable. A more elementary and conceptual approach seems desirable. The current paper takes such an approach, working with the *dimension* of a permutation action as its smallest power containing a regular orbit (Definition 2.3). Section 3 presents some fundamental inequalities relating the dimension to the degree, permutation group order, and to closure properties of the permutation group. (Compare also [1, Proposition], [2], [6].) Section 4 examines the dimensions of cyclic, symmetric, alternating, and dihedral actions. Particular attention focuses on the relative merits of the inequalities of Section 3 when applied to these cases: The results are summarized in Table 4.1.

Section 5 computes the dimensions of the actions of automorphism groups of finite linear, affine, and projective geometries, showing how the permutation group dimension concept tracks the usual geometric dimensions. The final two sections investigate the behaviour of the permutation group dimension with respect to three permutation group constructions: disjoint unions, Cartesian products, and wreath products. As an application of the wreath product construction, Corollary 7.4 shows how to design a permutation action with an arbitrary dimension and an arbitrary (integral) ratio of degree to dimension. In particular, correcting a misleading impression that one might gain from Table 4.1, it yields a series of permutation actions for which the inequality (3.2) is increasingly ineffective.

2. The dimension

In order to facilitate consideration of the dimension concept, it is convenient to list some conditions equivalent to regularity and faithfulness of G-sets for a finite group G.

Proposition 2.1. *Let (X, G) be a transitive G-set. Then the following conditions are equivalent:*

(a) *(X, G) is regular;*

(b) *$\forall x \in X,\ \forall g \in G,\ xg = x \Rightarrow g = 1$;*

(c) $\exists x \in X.\ \forall 1 \neq g \in G,\ xg \neq x$;
(d) $\deg X = |G|$. □

It is interesting to contrast (c) of Proposition 2.1 with (b) of Proposition 2.2 below. The distinction between regularity and faithfulness reduces to a question of the order of quantifiers, analogous to the distinction between continuity and uniform continuity.

Proposition 2.2. *Let (X, G) be a non-empty G-set. Then the following conditions are equivalent:*

(a) *(X, G) is faithful;*
(b) *$\forall 1 \neq g \in G,\ \exists x \in X.\ xg \neq x$;*
(c) *$\exists r < \deg X.\ (X, G)^r$ contains a regular orbit.*

Proof. The equivalence of (a) and (b) is immediate. Now if (b) does not hold, there is a non-identity element g of G such that $xg = x$ for all x in X. But then no element $(x_1, \ldots, x_r)$ of any power $(X, G)^r$ could lie in a regular orbit, since the equation $(x_1, \ldots, x_r)g = (x_1, \ldots, x_r)$ would contradict (b) of Proposition 2.1. Conversely, suppose that (X, G) is faithful. Consider $r = \deg X - 1$ and $(x_1, \ldots, x_r) \in (X, G)^r$ with $|\{x_1, \ldots, x_r\}| = r$. Suppose $(x_1, \ldots, x_r)g = (x_1, \ldots, x_r)$ for g in G. Then $x_1 g = x_1, \ldots, x_r g = x_r$ by (1.2). Moreover, for the unique element x of the complement of $\{x_1, \ldots, x_r\}$ in X, one has $xg \notin \{x_1, \ldots, x_r\}$, whence $xg = x$. Thus $g = 1$. Proposition 2.1 (c) then shows that $(x_1, \ldots, x_r)$ lies in a regular orbit of $(X, G)^r$. □

Proposition 2.2 shows that some power of a permutation action contains a regular orbit. The dimension of the action is defined to be the least such power.

Definition 2.3. *Let (X, G) be a (finite) permutation action, i.e. a finite, faithful G-set X. Then the dimension $\dim X$ or $\dim(X, G)$ is defined to be the least power r for which $(X, G)^r$ contains a regular orbit. An element $(x_1, \ldots, x_s)$ of a regular orbit, and the set $\{x_1, \ldots, x_s\}$ of components of such an element, are said to span the action (X, G).*

Note that the concept of spanning is well-defined: If $(x_1, \ldots, x_s)$ lies in a regular orbit, and $\{x_1, \ldots, x_s\} = \{y_1, \ldots, y_t\}$, then $(y_1, \ldots, y_t)$ also lies in a regular orbit. Following Sims [7, §2] (cf. eg., [6, §3], [2, §1]), the term "base" has been used for "spanning set", and the term "(minimal) base size" for "dimension". The usage proposed in Definition 2.3 is closer to that customary in linear algebra (cf. Theorem 5.1 below).

3. Equalities and inequalities

This section records various equalities and inequalities to which the dimension $\dim X$ of a finite, faithful action (X, G) is subject. Firstly, note that Proposition 2.2(c) yields the *Diversity Inequality*

$$\dim X < \deg X. \tag{3.1}$$

On the other hand, Proposition 2.1(d) yields the *Cardinality Inequality*

$$\dim X \geq \log_{\deg X} |G|, \tag{3.2}$$

since $X^{\dim X}$ contains the $|G|$ elements of a regular orbit.

Recall that the *k-closure* of a permutation group G on a set X is the automorphism group $\mathrm{Aut}(X, X^k/G)$ of the relational structure on X given by the orbits of $(X, G)^k$ [11, Def. 5.3]. The dimension of the permutation action (X, G) is then related to the closure properties of G as follows.

Theorem 3.1. *For a finite, faithful G-set (X, G),*

$$1 + \dim(X, G) \geq \min\{k \mid G = \mathrm{Aut}(X, X^k/G)\}. \tag{3.3}$$

Proof. Let $r = \dim X$, and let $(x_1, \ldots, x_r)$ lie in a regular orbit of $(X, G)^r$. Then $\mathrm{Stab}_G\ ((x_1, \ldots, x_r)) = \{1\}$. By [11, Th. 5.12], it follows that $G = \mathrm{Aut}\ (X, X^{r+1}/G)$. □

The inequality (3.3) is known as the *Closure Inequality.*

Now let (X, G) be a faithful, finite transitive action. Let H be the stabilizer of an element of X. Under the uniform probability distribution on G, let

$$\mathrm{P}(K^g \leq H) \tag{3.4}$$

denote the probability that a random conjugate K^g of a subgroup K of G lies in the point stabilizer H. The dimension of (X, G) is then specified exactly in terms of the Möbius function

$$\mu(K, L) = \sum_{k=0}^{\infty} (-1)^k |\{(K_1, \ldots, K_k) \mid K = K_0 < K_1 < \ldots < K_k = L\}| \tag{3.5}$$

of the set of subgroups of G ordered by containment.

Theorem 3.2. *For a finite, transitive, faithful G-set (X, G) with point stabilizer H,*

$$\dim\ (X, G) = \min\{r \mid 0 < \sum_{K \leq H} \mu(1, K) \mathrm{P}(K^g \leq H)^{r-1}\}. \tag{3.6}$$

Proof. Consider the uniform probability distribution on X^r. By [8, (5.2)], the probability that a random element of X^r lies in a regular orbit is given as

$$\sum_{K \leq H} \mu(1, K) \mathrm{P}(K^g \leq H)^{r-1}. \tag{3.7}$$

□

In the notation of [2, §3], (3.7) specifies $B(G, r)$.

4. Elementary examples

This section computes the dimensions of cyclic actions C_n, symmetric actions S_n, alternating actions A_n, and dihedral actions D_n of degree n. These elementary examples give some idea of the relative strengths and weaknesses of the inequalities listed in the previous section. Firstly, note that

$$\dim C_n = 1, \tag{4.1}$$

since any transitive abelian permutation group is regular. In this case, equality holds in (3.2) and (3.3) [5, 8.1.24], [10], while (3.1) is as bad as possible.

Theorem 4.1. (a) $\dim S_n = n - 1$ *for* $n > 1$. (b) $\dim A_n = n - 2$ *for* $n > 2$.

Proof. (a) No set of $n - 2$ elements can span S_n, since such a set is fixed pointwise by the transposition interchanging the remaining two elements. Thus $\dim S_n > n - 2$. The only option left by (3.1) is $\dim S_n = n - 1$.

(b) No set of $n - 3$ elements can span A_n, since such a set is fixed pointwise by a 3-cycle permuting the remaining three elements. On the other hand, each set of $n - 2$ elements spans, since the only non-identity permutation pointwise fixing such a set is odd. □

As noted in the proof of Theorem 4.1(a), (3.1) is sharp for S_n. On the other hand, S_n is its own 1-closure, so that (3.3) is maximally bad for S_n. For A_n, the inequality (3.1) is almost sharp, while equality holds in (3.3) [4, 1.5.21]. In the cases of both S_n and A_n, the inequality (3.2) is asymptotically good for large n, by Stirling's Formula.

Theorem 4.2. $\dim D_n = 2$ *for* $n > 2$.

Proof. Let H be a point stabilizer in the action D_n. If n is even, D_n has n subgroups of order 2. One half of these, being point stabilizers, are conjugates of H. Thus
$0 = 1 - 1 = \sum_{K \leq H} \mu(1, K)\mathrm{P}(K^g \leq H)^{1-1}$, while $\sum_{K \leq H} \mu(1, K)\mathrm{P}(K^g \leq H)^{2-1} = 1 - \frac{2}{n} > 0$,
whence $\dim D_n = 2$ by Theorem 3.2. If n is odd, the dihedral group D_n again has n subgroups of order 2. This time, however, all are point stabilizers, and thus conjugate to H. Then
$\sum_{K \leq H} \mu(1, K)\mathrm{P}(K^g \leq H)^{1-1} = 1 - 1 = 0$, while $\sum_{K \leq H} \mu(1, K)\mathrm{P}(K^g \leq H)^{2-1} = 1 - \frac{1}{n} > 0$.
Theorem 3.2 again shows that $\dim D_n = 2$ in this case. □

For D_n, the inequality (3.2) takes the form

$$\dim D_n \geq 1 + \log_n 2. \tag{4.2}$$

Subject to the integrality constraint for $\dim D_n$, the inequality becomes sharp. Now D_n, as the automorphism group of the undirected graph on $\mathbb{Z}_n$ with edge set $\{\{x, x+1\} | x \in \mathbb{Z}_n\}$, is its own 2-closure. Thus (3.3) is almost sharp for D_n. On the other hand, (3.1) for D_n deteriorates steadily as n increases. Table 4.1 summarizes the answers to the question "Is the inequality to the left sharp for the action shown at the top?"

	C_n	S_n	A_n	D_n
Diversity	no	yes	almost	no
Cardinality	yes	asymptotically	asymptotically	yes
Closure	yes	no	yes	almost

Table 4.1.

5. Geometric actions

The finite permutation actions (X, G) studied in this section are all *geometric*, in that X is a *space* (linear, affine, or projective), and that G consists of automorphisms of the space X. The goal is to relate the geometric dimension of the space X to the dimension of the permutation action in the sense of Definition 2.3. Notation is as follows. Let F be a field of prime power order. The *linear space* of geometric dimension n is the vector space F^n. Its group of automorphisms is the *general linear group* $\mathrm{GL}_n(F)$. The *general affine group* $\mathrm{GA}_n(F)$ is the split extension $F^n]\ \mathrm{GL}_n(F)$ of F^n by $\mathrm{GL}_n(F)$, and the *affine space* $\mathrm{A}_n(F)$ of dimension n is the geometry of cosets of subspaces in the vector space F^n. Note that $\mathrm{GA}_n(F)$ acts faithfully on $\mathrm{A}_n(F)$ [3, Theorem III.1]. The *projective linear group* $\mathrm{PGL}_n(F)$ is the quotient $\mathrm{GL}_n(F)/F^*$ of $\mathrm{GL}_n(F)$ by its centre, the group F^* of homotheties. The *projective space* $\mathrm{PG}_{n-1}(F)$ of projective dimension $n-1$ is the set $(F^n - \{0\})/F^*$ of orbits of non-zero points of F^n, with subspaces corresponding to positive-dimensional subspaces of F^n. The action $\mathrm{GL}_n(F) \to \{F^n - \{0\}\}!$ of the linear group factors to an action $\mathrm{PGL}_n(F) \to \mathrm{PG}_{n-1}(F)!$ of the projective linear group. This action is also faithful [3, Theorem III.2].

Theorem 5.1. $\dim(F^n, \mathrm{GL}_n(F)) = n$.

Proof. Consider $1 \leq r < n$, and an element $(x_1, \dots, x_r)$ of $(F^n,\ \mathrm{GL}_n(F))^r$. The set $\{x_1, \dots, x_r\}$ spans a proper subspace U of F^n. Choose a basis $\{e_1, \dots, e_s\}$ for U, and extend to a basis $\{e_1, \dots, e_s, e_{s+1}, \dots, e_n\}$ for F^n. Define a non-identity element g of $\mathrm{Gl}_n(F)$ by

$$e_i g = \textbf{ if } i = n \textbf{ then } e_1 + e_n \textbf{ else } e_i. \tag{5.1}$$

Then $(x_1, \dots, x_r)g = (x_1, \dots, x_r)$, so there is no regular orbit in $(F^n,\ \mathrm{GL}_n(F))^r$. On the other hand, for a basis $\{f_1, f_2, \dots, f_n\}$ of F^n, the stabilizer of $(f_1, f_2, \dots, f_n)$ in $\mathrm{GL}_n(F)$ is trivial. □

Theorem 5.2. $\dim(\mathrm{A}_n(F), \mathrm{GA}_n(F)) = n + 1$.

Proof. Consider $0 \leq r < n$, and an element $(y_0, y_1, \dots, y_r)$ of $(\mathrm{A}_n(F),\ \mathrm{GA}_n(F))^{r+1}$. Consider the translation $t : F^n \to F^n;\ y \mapsto y - y_0$, an element of $\mathrm{GA}_n(F)$. For $1 \leq i \leq r$, define $x_i = y_i t$. Define the non-identity element g of $\mathrm{GL}_n(F) \leq \mathrm{GA}_n(F)$ by (5.1).

Then $(y_0, y_1, \ldots, y_r)tgt^{-1} = (y_0, y_1, \ldots, y_r)$ with $1 \neq tgt^{-1} \in \mathrm{GA}_n(F)$, so that $(\mathrm{A}_n(F), \mathrm{GA}_n(F))^{r+1}$ contains no regular orbit. On the other hand, for a basis $\{f_1, \ldots, f_n\}$ of F^n, the stabilizer of $(0, f_1, \ldots, f_n)$ in $\mathrm{GA}_n(F)$ is trivial [3, Theorem III.1]. □

Theorem 5.3. $\dim(\mathrm{PG}_{n-1}(F), \mathrm{PGL}_n(F)) = n + 1$ for $|F| > 2$.

Proof. Consider $r \leq n$, and an element $(x_1F^*, \ldots, x_rF^*)$ of $(\mathrm{PG}_{n-1}(F), \mathrm{PGL}_n(F))^r$. If $\{x_1, \ldots, x_r\}$ spans a proper subspace U of F^n, define the non-identity element g of $\mathrm{GL}_n(F)$ by (5.1). Then $(x_1F^*, \ldots, x_rF^*)gF^* = (x_1F^*, \ldots, x_rF^*)$. If $r = n$ and $\{x_1, \ldots, x_n\}$ is a basis for F^n, choose $1 \neq k \in F^*$. Then define an element h of $\mathrm{GL}_n(F)$ by

$$x_ih = \textbf{ if } i = 1 \textbf{ then } x_i \textbf{ else } kx_i. \tag{5.2}$$

Note that hF^* is a non-identity element of $\mathrm{PGL}_n(F)$. However, $(x_1F^*, \ldots, x_nF^*)hF^* = (x_1F^*, \ldots, x_nF^*)$. Thus $(\mathrm{PG}_{n-1}(F), \mathrm{PGL}_n(F))^r$ contains no regular orbit. On the other hand, for a basis $\{f_1, \ldots, f_n\}$ of F^n, the stabilizer of $(f_1F^*, \ldots, f_nF^*, (f_1 + \ldots + f_n)F^*)$ in $\mathrm{PGL}_n(F)$ is trivial [3, Theorem II.2]. □

For $|F| = 2$, note $(\mathrm{PG}_{n-1}(F), \mathrm{PGL}_n(F)) = (F^n - \{0\}, \mathrm{GL}_n(F))$, so that

$$|F| = 2 \ \Rightarrow\ \dim(\mathrm{PG}_{n-1}(F), \mathrm{PGL}_n(F)) = n \tag{5.3}$$

by Theorem 5.1.

6. Disjoint unions and Cartesian products

Throughout this section, let (X, G) be a finite permutation action of dimension d spanned by a set $\{a_1, \ldots, a_d\}$, and let (Y, H) be a finite permutation action of dimension e spanned by a set $\{b_1, \ldots, b_e\}$. The aim is to show that the disjoint union and Cartesian product of (X, G) and (Y, H) are faithful permutation actions, having dimensions expressible in terms of $\dim X$ and $\dim Y$.

The *disjoint union* $(X, G) + (Y, H) = (X + Y, G \times H)$ of (X, G) and (Y, H) is the $(G \times H)$-set on the disjoint union $X + Y$ of X with Y given by the action

$$z(g, h) = \textbf{ if } z \in X \textbf{ then } zg \textbf{ else } zh \tag{6.1}$$

of elements (g, h) of $G \times H$ [4, 1.7.8], [9, p.37].

Lemma 6.1. *For $r < \dim X + \dim Y$, the power $(X + Y)^r$ contains no regular orbit of $G \times H$.*

Proof. Consider $(z_1, \ldots, z_r)$ in $(X + Y)^r$. Now $|X \cap \{z_1, \ldots, z_r\}| < d$ or $|Y \cap \{z_1, \ldots, z_r\}| < e$. In the first case, Proposition 2.1(c) gives a non-identity element g of G fixing $X \cap \{z_1, \ldots, z_r\}$ pointwise. Then $(z_1, \ldots, z_r)(g, 1) = (z_1, \ldots, z_r)$. In the second case, there is a non-identity element h of H with $(z_1, \ldots, z_r)(1, h) = (z_1, \ldots, z_r)$. Proposition 2.1(c) then shows that $(X + Y)^r$ contains no regular orbit. □

Lemma 6.2. *The disjoint union* $(X+Y, G\times H)$ *is faithful, being spanned by* $(a_1,\dots,a_d,b_1,\dots,b_e)$.

Proof. Suppose $(a_1,\dots,a_d,b_1,\dots,b_e)(g,h) = (a_1,\dots,a_d,b_1,\dots,b_e)$. Then $(a_1,\dots,a_d)g = (a_2,\dots,a_d)$ by (6.1), whence $g=1$ by Proposition 2.1(b). Similarly, $(b_1,\dots,b_e)h = (b_1,\dots,b_e)$ and $h=1$. Thus $(a_1,\dots,a_d,b_1,\dots,b_e)$ spans $X+Y$ by Proposition 2.1(c). In particular, $X+Y$ is faithful by Proposition 2.2(c). □

Theorem 6.3. *The disjoint union* $(X,G)+(Y,H)$ *of finite permutation actions is again a permutation action, with*

$$\dim(X+Y) = \dim X + \dim Y. \tag{6.2}$$

Proof. Combine Lemmas 6.1 and 6.2. □

The *Cartesian product* $(X,G)\times(Y,H) = (X\times Y, G\times H)$ of (X,G) and (Y,H) is the $(G\times H)$-set on the Cartesian product $X\times Y$ given by the action

$$(x,y)(g,h) = (xg,yh) \tag{6.3}$$

of elements (g,h) of $G\times H$ [4, 1.7.9], [9, p. 37].

Lemma 6.4. *For* $r<\max\{\dim X,\dim Y\}$, *the power* $(X\times Y)^r$ *contains no regular orbit of* $G\times H$.

Proof. Without loss of generality, suppose $r<\dim X$. Consider an element $((x_1,y_1),\dots,(x_r,y_r))$ of $(X\times Y)^r$. By Proposition 2.1(c), there is a non-identity element g of G with $(x_1,\dots,x_r)g = (x_1,\dots,x_r)$. Then by (6.3), one has $((x_1,y_1),\dots,(x_r,y_r))(g,1) = ((x_1,y_1),\dots,(x_r,y_r))$ with $(g,1)\neq(1,1)$. Proposition 2.1(c) then shows that $(X\times Y)^r$ contains no regular orbit. □

Lemma 6.5. *The Cartesian product* $(X\times Y, G\times H)$ *is faithful, being spanned by a* $\max\{d,e\}$*-subset of* $X\times Y$.

Proof. Without loss of generality, suppose $\dim X\geq \dim Y$. If $d>e$, pick $b_{e+1},\dots,b_d$ arbitrarily from Y. Suppose $((a_1,b_1),\dots,(a_d,b_d))(g,h) = ((a_1,b_1),\dots,(a_d,b_d))$. Then $(a_1,\dots,a_d)g = (a_1,\dots,a_d)$ implies $g=1$ by Proposition 2.1(b), and similarly $(b_1,\dots,b_d)h = (b_1,\dots,b_d) \Rightarrow (b_1,\dots b_e)h = (b_1,\dots,b_e) \Rightarrow h=1$. Thus $(g,h)=(1,1)$, and Proposition 2.1(c) shows that $((a_1,b_1),\dots,(a_d,b_d))$ spans $X\times Y$. In particular, $X\times Y$ is faithful by Proposition 2.2(c). □

Theorem 6.6. *The Cartesian product* $(X,G)\times(Y,H)$ *of finite permutation actions is again a permutation action, with*

$$\dim(X\times Y) = \max\{\dim X,\dim Y\}. \tag{6.4}$$

Proof. Combine Lemmas 6.4 and 6.5. □

7. Wreath products

Continue the notation of the previous section. The *wreath product group* $G \wr (Y, H)$ has $G^Y \times H$ as its underlying set. Elements of G^Y are written as functions $s : Y \to G$. The group multiplication in $G \wr (Y, H)$ is defined by

$$(s, h)(t, h') = (y \mapsto ys \cdot yht, hh'). \tag{7.1}$$

The *wreath product action* $(X, G) \wr (Y, H) = (X \times Y, G \wr (Y, H))$ is the $(G \wr (Y, H))$-set on the Cartesian product $X \times Y$ given by the action

$$(x, y)(s, h) = (x \cdot y^s, yh) \tag{7.2}$$

of elements (s, h) of $G \wr (Y, H)$ [4, 1.7C], [9, p.40]. The wreath product action $(X, G) \wr (Y, H)$ will be shown to be faithful, and its dimension will be determined.

Lemma 7.1. For $r < \dim X \cdot \deg Y$ and $|G| \neq 1$, the power $(X \times Y)^r$ contains no regular orbit of $G \wr (Y, H)$.

Proof. Consider an element $((x_1, y_1), \ldots, (x_r, y_r))$ of $(X \times Y)^r$. If $|\{y_1, \ldots, y_r\}| < \deg Y$, define $s : Y \to G$, using a non-identity element of g of G, by

$$ys = \textbf{ if } y \in \{y_1, \ldots, y_r\} \textbf{ then } 1 \textbf{ else } g. \tag{7.3}$$

Then $(s, 1)$ is a non-identity element of $G \wr (Y, H)$, but $((x_1, y_1), \ldots, (x_r, y_r))(s, 1) = ((x_1, y_1), \ldots, (x_r, y_r))$ by (7.2). Now if $\{y_1, \ldots, y_r\} = Y$, there is an element y of Y such that $|\{i \mid y_i = y\}| < \dim(X, G)$. Let $\{i_j \mid 1 \leq j \leq t\} = \{i \mid y_i = y\}$. Then by Proposition 2.1(c), there is a non-identity element g of G with $(x_{i_1}, \ldots, x_{i_t})g = (x_{i_1}, \ldots, x_{i_t})$. Define $s : Y \to G$ by

$$y_i s = \textbf{ if } y_j = y \textbf{ then } g \textbf{ else } 1. \tag{7.4}$$

Certainly $(s, 1)$ is a non-identity element of $G \wr (Y, H)$, but $((x_1, y_1), \ldots, (x_r, y_r))(s, 1) = ((x_1, y_1), \ldots, (x_r, y_r))$ by (7.2). Proposition 2.1(c) then shows that $(X \times Y)^r$ contains no regular orbit. □

Lemma 7.2. *The wreath product $(X \times Y, G \wr (Y, H))$ is faithful, being spanned by the direct product of the spanning set $\{a_1, \ldots, a_d\}$ for X with Y.*

Proof. Suppose that the element (s, h) of $G \wr (Y, H)$ fixes $\{a_1, \ldots, a_d\} \times Y$ pointwise, so that

$$\forall 1 \leq i \leq d,\ \forall y \in Y,\ (a_i \cdot y^s, yh) = (a_i, y) \tag{7.5}$$

by (7.2). Certainly $h = 1$, by the second component of the equality in (7.5). Now for each element y of Y, (7.5) yields $(a_1, \ldots, a_d)y^s = (a_1, \ldots, a_d)$. Since $(a_1, \ldots, a_d)$ spans (X, G), it follows that $y^s = 1$. Thus (s, h) is the identity element of $G \wr (Y, H)$. □

Theorem 7.3. *The wreath product $(X, G) \wr (Y, H)$ of finite permutation actions is again a permutation action. Moreover,*

$$|G| = 1 \ \Rightarrow\ \dim(X, G) \wr (Y, H) = \dim(Y, H), \tag{7.6}$$

while

$$|G| > 1 \ \Rightarrow\ \dim(X, G) \wr (Y, H) = \dim(X, G) \cdot \deg(Y, H). \tag{7.7}$$

Proof. The faithfulness of the wreath product follows by Lemma 7.2. If G is trivial, then $G \wr (Y, H)$ is isomorphic to H and $(X, G) \wr (Y, H)$ is similar to (Y, H), so that (7.6) holds. On the other hand, (7.7) follows by Lemmas 7.1 and 7.2. □

Corollary 7.4. *For $m > 0$ and $n > 1$, one has*

$$\dim C_n \wr C_m = m \tag{7.8}$$

and

$$\frac{\deg C_n \wr C_m}{\dim C_n \wr C_m} = n. \tag{7.9}$$

Proof. Use (4.1) and (7.7). □

Studying Table 4.1, one might be led to believe that the Cardinality Inequality (3.2) is always effective for large permutation groups. However, for the permutation actions $X = C_2 \wr C_m$, Corollary 7.4 shows that

$$(\dim X)^{-1} \log_{\deg X} |G| = m^{-1} \log_{2m} 2^m m \to 0 \tag{7.10}$$

as $m \to \infty$. Thus (3.2) weakens steadily for these actions: If m is large, there are many powers of $C_2 \wr C_m$ that do not contain any regular orbit, even though their cardinality would be large enough to contain one.

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