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Isothermic Surfaces and Hopf Cylinders

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Abstract. Based on the work of Pinkall, characterizations of spherical curves are given whose corresponding Hopf cylinders are isothermic surfaces in the three-dimensional sphere. Comparing these characterizations with results of Langer and Singer about elastic spherical curves we determine all isothermic Willmore Hopf tori.

Keywords: isothermic surface, Hopf cylinder, Clifford torus, Willmore surface

1. Introduction

In 1985, U. Pinkall [5] introduced Hopf cylinders in the three-dimensional unit sphere $S^3 \subset \mathbb{R}^4$ which are inverse images of spherical curves in the two-dimensional sphere $S^2 \subset \mathbb{R}^3$ by means of the Hopf map. Based on the work of J. Langer and D. A. Singer [4] about elastic curves in S^2, Pinkall further determined all Willmore Hopf tori in S^3. An example of a Willmore Hopf torus is the minimal Clifford torus in S^3. This torus is an isothermic surface: a suitable stereographic projection of it yields a torus of revolution in $\mathbb{R}^3$ with a ratio of its radii equal to $1 : \sqrt{2}$. We can now ask whether there are any more isothermic Willmore Hopf tori. Or more general:

Which Hopf cylinders are isothermic?

In the next section we introduce isothermic surfaces and in Section 3 we consider Hopf cylinders, which can be described via quaternions (see [5]).

The answer to the above question is given in Section 4. There we characterize curves in the unit sphere S^2 which belong to isothermic Hopf cylinders. Namely, the geodesic curvature of the spherical curves which correspond to isothermic Hopf cylinders is the tangent function of a linear function of their arc length. Furthermore, these curves are also characterized by a constant torsion in $\mathbb{R}^3$.

Finally in Section 5, we apply the results to Willmore Hopf tori to determine all Willmore Hopf tori which are isothermic. These are solely the minimal Clifford tori.

2. Isothermic surfaces

In the following, all considered objects are assumed to be sufficiently differentiable, e.g. in Section 4 we need differentiability class C^4 for the spherical curve.

Definition 1. *A parametrization of a local surface is called isothermic if the components of the first fundamental form have the form*

$$g_{11} = g_{22} = \lambda,\ g_{12} = 0$$

with a positive function λ. An umbilicfree local surface is called an isothermic surface if there exists a parametrization of curvature lines on the surface which is isothermic.

If the surface is second order differentiable, there always exists an isothermic parametrization (cf. [2]). Further properties and examples of isothermic surfaces can be found e.g. in the books of Blaschke ([1], p. 325 ff) and Eisenhart ([3], p. 107f, p. 226ff). A well-known lemma giving an analytical criterion for an isothermic parametrization is

Lemma 1. *On a surface there is an isothermic parametrization if and only if there are parameters (u, v) on the surface with*

$$\frac{\partial^2}{\partial u \partial v} \ln\left(\frac{g_{11}}{g_{22}}\right) = 0 \quad \textit{and} \quad g_{12} = 0. \tag{1}$$

3. Hopf cylinders

Here we cite some notations and results from [5] in short.
The unit sphere S^3 is to be considered as a set of unit quaternions $S^3 = \{q \in \mathbb{H} \,|\, \|q\| = 1\}$. S^2 can be described as the section of S^3 with a real three-dimensional linear subspace, here we take $S^2 = S^3 \cap \operatorname{lin}\{1, j, k\}$.

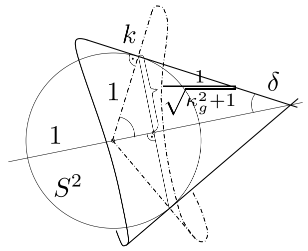
From [5] we know that the Hopf map $\pi : S^3 \to S^2$ is then given by $\pi(q) = \tilde{q}q$, with $\pi(e^{i\varphi}q) = \pi(q)$ for all $\varphi \in \mathbb{R}, q \in S^3$, where $q = q_0 + q_1 i + q_2 j + q_3 k,\ \tilde{q} = q_0 - q_1 i + q_2 j + q_3 k$, $q_m \in \mathbb{R},\ m = 0, \ldots, 3$.

Let $p : [a, b] \to S^2, t \mapsto p(t)$, be a regular spherical curve, $[a, b] \subseteq \mathbb{R}$. We choose a curve $y : [a, b] \to S^3, t \mapsto y(t)$, with $\pi \circ y = p$.

Definition 2. *The mapping $x : [a, b] \times S^1 \to S^3$ with $(t, \varphi) \mapsto e^{i\varphi}y(t)$ is called the Hopf cylinder of p in S^3. If the spherical curve p is C^2-closed, the Hopf cylinder of p is called Hopf torus of p. Especially, if p is a circle, we call the Hopf torus of p a Clifford torus.*

Later, for a geometrical interpretation we require

Lemma 2. *Let p be a spherical curve with geodesic curvature κ_g, let $k(t)$ be the circle of curvature of p at t. Then for the semi-vertex angle $\delta(t)$ of the cone touching S^2 in the circle $k(t)$ the equation*



$$\tan \delta(t) = \kappa_g(t)$$

holds.

Figures 1 and 2 show a closed spherical curve p and a stereographic projection of its Hopf torus in two different views. The parameter t can now be chosen as the arc length parameter

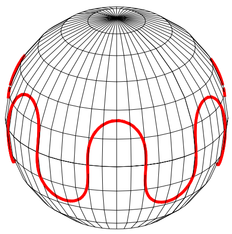


Figure 1. A closed spherical curve p with 6 periods

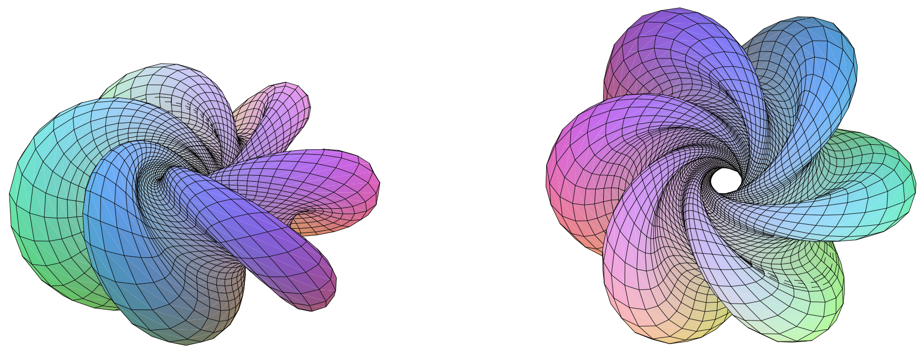


Figure 2. Stereographic image of the Hopf torus of p from Fig. 1

of y. For the arc length parameter s of $p : [0, l] \to S^2$ we get $s = 2t$.

According to [5], the metric components and the components of the second fundamental form and the Weingarten map of a Hopf cylinder are then given by

$$G = (g_{ik}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; (h_{ik}) = (h_i^k) = \begin{pmatrix} 2\kappa_g & 1 \\ 1 & 0 \end{pmatrix},$$

where κ_g is the geodesic curvature of p.

For the principal curvatures we obtain $\lambda_{1,2} = \kappa_g \pm \sqrt{\kappa_g^2 + 1} = \kappa_g \pm \kappa$, and we have $v_{1,2} = x_t + (-\kappa_g \pm \kappa)x_\varphi$ for the principal directions $v_{1,2}$ and $H = \kappa_g$ for the mean curvature H.
So a Hopf cylinder has no umbilics, and there exists a parametrization of curvature lines on it.

4. Isothermic Hopf cylinders

Now we want to apply Pinkall's calculus for Hopf cylinders to isothermic surfaces.

Let the Hopf cylinder be parametrized by means of curvature line parameters (u, v), i.e. we have a curvature line parametrization $\tilde{x}(u, v) = x(t(u, v), \varphi(u, v))$ with partial derivatives proportional to the principal curvature directions v_1, v_2

$$\begin{aligned} \tilde{x}_u &= \alpha(u,v)v_1 = \frac{\partial t}{\partial u}x_t + \frac{\partial \varphi}{\partial u}x_\varphi \\ \tilde{x}_v &= \beta(u,v)v_2 = \frac{\partial t}{\partial v}x_t + \frac{\partial \varphi}{\partial v}x_\varphi \ , \end{aligned}$$

where the proportionality factors α, β do not vanish. Here x_t, x_φ denote the derivative of x with respect to t, φ. After comparison of coefficients we obtain

$$\frac{\partial t}{\partial u} = \alpha \ , \quad \frac{\partial t}{\partial v} = \beta \tag{2}$$

and

$$\frac{\partial \varphi}{\partial u} = \alpha(-\kappa_g + \kappa) \ , \quad \frac{\partial \varphi}{\partial v} = -\beta(\kappa_g + \kappa) \ . \tag{3}$$

As we know that there locally exists a curvature line parametrization, the following integrability conditions must hold

$$\frac{\partial}{\partial v}\left(\frac{\partial t}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial t}{\partial v}\right) \quad \text{and} \quad \frac{\partial}{\partial u}\left(\frac{\partial \varphi}{\partial v}\right) = \frac{\partial}{\partial v}\left(\frac{\partial \varphi}{\partial u}\right) .$$

Inserting (2) and (3), the conditions are equivalent to $(\kappa_g' := \frac{d\kappa_g}{dt}\ ,\ \kappa' := \frac{d\kappa}{dt})$ $\alpha_v = \beta_u$ and

$$\beta_u(\kappa_g + \kappa) + (\kappa_g' + \kappa')\alpha\beta + \alpha_v(-\kappa_g + \kappa) + (-\kappa_g' + \kappa')\alpha\beta = 0 \ .$$

The second equation can be transformed by the first one and we get

$$\alpha_v = \beta_u \quad \text{and} \quad \alpha_v\kappa + \kappa'\alpha\beta = 0 \ . \tag{4}$$

The matrix of metric components, now in curvature line parametrization, has the form

$$\tilde{G} = (\tilde{g}_{ik}) = 2\kappa \begin{pmatrix} \alpha^2(\kappa - \kappa_g) & 0 \\ 0 & \beta^2(\kappa + \kappa_g) \end{pmatrix} \tag{5}$$

where $\kappa_g = \kappa_g(t),\ \kappa = \kappa(t)$ and $t = t(u, v)$.

If we differentiate the second equation of (4) partially with respect to v, we obtain

$$\alpha_{vv}\kappa + 2\alpha_v\beta\kappa' + \alpha\beta_v\kappa' + \alpha\beta^2\kappa'' = 0 \ . \tag{6}$$

Now we can show

Proposition 1. *Exactly those Hopf cylinders are isothermic whose image under the Hopf map π is a spherical curve $p:[0,l]\to S^2$ with geodesic curvature*

$$\kappa_g(s)=\tan\left(\frac{c}{2}s+d\right)$$

where $c,d\in\mathbb{R}$ are suitable constants and s is the arc length parameter of p. Especially, the Clifford tori ($c=0$) are the only isothermic Hopf tori.

Proof. If κ' vanishes everywhere, then $\kappa=$const and hence we have $|\kappa_g|=\sqrt{\kappa^2-1}=$ const. The curve p is then a part of a circle and the assertion holds with $c=0$.

So we can assume that we can find an interval with non-vanishing κ'. We insert $\tilde G$ in condition (1) for an isothermic surface and we get

$$\begin{aligned}
0 &= \frac{\partial^2}{\partial u\partial v}\ln\left(\frac{\tilde g_{11}}{\tilde g_{22}}\right)\\
&= \frac{\partial^2}{\partial u\partial v}\left(2\ln\alpha-2\ln\beta+\ln(\kappa-\kappa_g)-\ln(\kappa+\kappa_g)\right) \qquad \text{(with (5))}\\
&= \frac{\partial}{\partial u}\left(2\frac{\alpha_v}{\alpha}-2\frac{\beta_v}{\beta}+\frac{(\kappa-\kappa_g)'\beta}{\kappa-\kappa_g}-\frac{(\kappa+\kappa_g)'\beta}{\kappa+\kappa_g}\right)\\
&= \frac{\partial}{\partial u}\left(\beta(-2\frac{\kappa'}{\kappa}+2(\kappa_g\kappa'-\kappa_g'\kappa))-2\frac{\beta_v}{\beta}\right)
\end{aligned}$$

using $\kappa^2-\kappa_g^2=1$ and (4). Furthermore,

$$\begin{aligned}
0 &= 2\left(\beta_u(-\frac{\kappa'}{\kappa}+\kappa_g\kappa'-\kappa_g'\kappa)+\beta\alpha(-\frac{\kappa'}{\kappa}+\kappa_g\kappa'-\kappa_g'\kappa)'-\frac{\beta_{uv}\beta-\beta_u\beta_v}{\beta^2}\right)\\
&= 2\left(\alpha_v(-\frac{\kappa'}{\kappa}+\kappa_g\kappa'-\kappa_g'\kappa)-\alpha_v\frac{\kappa}{\kappa'}(-\frac{\kappa'}{\kappa}+\kappa_g\kappa'-\kappa_g'\kappa)'-\frac{\alpha_{vv}\beta-\alpha_v\beta_v}{\beta^2}\right)\\
&\qquad \text{(with (4))}\\
&= \frac{2}{\beta^2\kappa}\left(\alpha_v\beta^2\kappa(-\frac{\kappa'}{\kappa}+\kappa_g\kappa'-\kappa_g'\kappa)-\alpha_v\beta^2\frac{\kappa^2}{\kappa'}(-\frac{\kappa'}{\kappa}+\kappa_g\kappa'-\kappa_g'\kappa)'+\right.\\
&\qquad \left.+\beta(2\alpha_v\beta\kappa'+\alpha\beta_v\kappa'+\alpha\beta^2\kappa'')+\alpha_v\beta_v\kappa\right) \qquad \text{(with (4) and (6))}\\
&= \frac{2}{\beta^2\kappa}\left(\alpha_v\beta^2\kappa(\frac{\kappa'}{\kappa}+\kappa_g\kappa'-\kappa_g'\kappa)-\alpha_v\beta^2\frac{\kappa^2}{\kappa'}(-\frac{\kappa'}{\kappa}+\kappa_g\kappa'-\kappa_g'\kappa)'+\right.\\
&\qquad \left.+(-\alpha_v\beta^2\frac{\kappa}{\kappa'})\kappa''\right) \qquad \text{(with (4))}\\
&= \frac{2\alpha_v}{\kappa\kappa'}\left(\kappa\kappa'(\frac{\kappa'}{\kappa}+\kappa_g\kappa'-\kappa_g'\kappa)-\kappa^2(-\frac{\kappa'}{\kappa}+\kappa_g\kappa'-\kappa_g'\kappa)'-\kappa\kappa''\right). \qquad (7)
\end{aligned}$$

From the equation $\kappa'=\dfrac{\kappa_g\kappa_g'}{\kappa}$ (Derivation of $\kappa=\sqrt{\kappa_g^2+1}$) as well as

$$\kappa''=\frac{\kappa_g\kappa_g''\kappa^2+(\kappa_g')^2}{\kappa^3}$$

it follows

$$(\kappa')^2\kappa = \frac{\kappa_g^2(\kappa_g')^2}{\kappa} \quad \text{and} \quad \kappa'\kappa^2 = \kappa_g\kappa_g'\kappa \tag{8}$$

and after differentiation we get

$$(-\frac{\kappa'}{\kappa} + \kappa_g\kappa' - \kappa_g'\kappa)' = -\frac{\kappa''}{\kappa} + \frac{(\kappa')^2}{\kappa^2} + \kappa_g\kappa'' - \kappa_g''\kappa \, . \tag{9}$$

Inserting (9) in (7) and using (8) we obtain

$$\begin{aligned}
0 &= \alpha_v \left((\kappa')^2\kappa\kappa_g - \kappa_g'\kappa^2\kappa' - \kappa_g\kappa''\kappa^2 + \kappa_g''\kappa^3\right) \\
&= \alpha_v \left(\frac{\kappa_g^3(\kappa_g')^2}{\kappa} - \kappa_g(\kappa_g')^2\kappa - \frac{\kappa_g}{\kappa}(\kappa_g\kappa_g''\kappa^2 + (\kappa_g')^2) + \kappa_g''\kappa^3\right) \\
&= \frac{\alpha_v}{\kappa}\left(\kappa_g^3(\kappa_g')^2 - \kappa_g(\kappa_g')^2\kappa^2 - \kappa_g^2\kappa_g''\kappa^2 - \kappa_g(\kappa_g')^2 + \kappa_g''\kappa^4\right) \\
&= \frac{\alpha_v}{\kappa}\left(-2\kappa_g(\kappa_g')^2 + \kappa_g''(1+\kappa_g^2)\right) \\
&\qquad \text{(with several applications of } \kappa^2 = 1+\kappa_g^2\text{)} \\
&= \frac{\alpha_v(1+\kappa_g^2)^2}{\kappa}\left(\frac{\kappa_g'}{1+\kappa_g^2}\right)' .
\end{aligned}$$

This equation is fulfilled iff $\alpha = \alpha(u)$ is only dependent on u or $\frac{\kappa_g'}{1+\kappa_g^2} = c$ holds with $c =$ const.$\in \mathbb{R}$. In the first case, we have $\kappa_g =$const. because of (4), and p is again a part of a circle. In the second case, we integrate once more and get

$$\kappa_g(t) = \tan(ct+d) \, ,$$

where $c, d \in \mathbb{R}$ are suitable integration constants with $|ct+d| < \frac{\pi}{2}$ and $s = 2t$. □

4.1. Characterizations of p corresponding to an isothermic Hopf cylinder

Proposition 2. *A spherical curve p corresponding to an isothermic Hopf cylinder is characterized by a constant torsion τ as a space curve in $\mathbb{R}^3$. For its curvature κ we get*

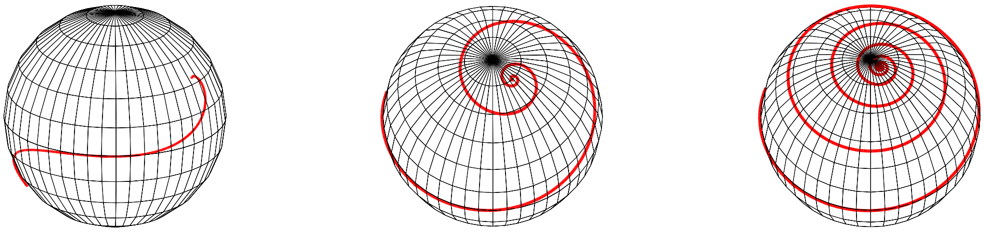
$$\kappa(s) = \frac{1}{\cos(\tau s + d)} \, .$$

Proof. The assertion immediately follows with $2\tau = 2\frac{\frac{d\kappa_g}{ds}}{\kappa_g^2+1} = \frac{\kappa_g'}{\kappa_g^2+1} = c$ and $\kappa = \sqrt{\kappa_g^2+1}$ after inserting the equations of Proposition 1 with $|\tau s + d| < \frac{\pi}{2}$. □

Proposition 3. *A spherical curve p corresponding to an isothermic Hopf cylinder is further characterized by the fact that the semi-vertex angles of the cones touching S^2 in curvature circles of p are linear functions of its arc length.*

Proof. A comparison of Proposition 1 with Lemma 2 yields $\delta(s) = \frac{c}{2}s + d = \tau s + d$ with $c, d \in \mathbb{R}$ suitably chosen. □

Figure 3 shows the shape of p in S^2 for some values of $c \neq 0$ and $d = 0$ which is similar to a clothoid. Figure 4 presents a stereographic projection of the Hopf cylinder of p, on the left hand side the whole surface, on the right hand side a part of it.



(a) $c = 2$ (b) $c = \frac{1}{2}$ (c) $c = \frac{1}{4}$

Figure 3. The spherical curve p from Proposition 1 with initial conditions: p touches the equator for $s = 0$ and $\kappa_g(0) = 0$

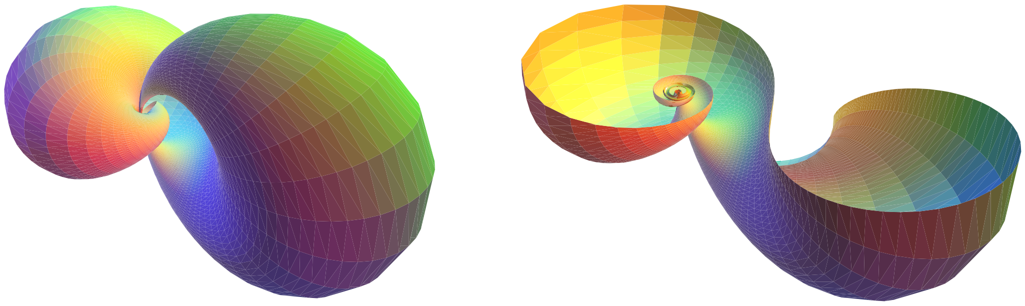


Figure 4. Stereographic projection of the Hopf cylinder of the curve p from Fig. 3, (b)

5. Willmore Hopf tori

Let M be a closed orientiable two-dimensional manifold. Hence for simply C^2-closed spherical curves p we know from [5]:

$f(M)$ is a Willmore Hopf torus iff p is a critical point of $\oint \kappa^2(s)ds$.

In 1987, such curves p called *elastic curves* were found by Langer and Singer [4]. They got the following result:

Proposition 4. *There are infinitely many simply C^∞-closed elastic curves p in S^2 with geodesic curvature*

$$\kappa_g(s) = k_0 \,\mathrm{cn}\!\left(\frac{k_0 s}{2\omega}, \omega\right) ,$$

where the maximal geodesic curvature k_0 is given by $k_0 = \frac{\sqrt{2}\omega}{\sqrt{1-2\omega^2}}$ with a certain ω where $\omega^2 \in [0, \frac{1}{2})$ and cn *denotes the amplitude cosine (Jacobi's elliptic cosine).*

Figures 1 and 2 show a closed elastic spherical curve for $\omega \approx 0.6894$ ($k_0 \approx 4.3838$, 6 periods) and a stereographic image of its Willmore Hopf torus.

A comparison of Propositions 1 and 4 yields for a C^∞-closed p

Corollary 1. *In the class of Willmore Hopf tori the minimal Clifford tori are the only isothermic surfaces.*

Proof. Both formulas for κ_g in Proposition 1 and Proposition 4 must coincide, this is only possible for $\kappa_g = 0$, i.e. for p is a great circle. The torus is therefore a minimal Clifford torus. $\square$

As the properties of being an isothermic surface and of being a Willmore surface are Möbius invariants, we get

Corollary 2. *Let $f : S^3 \to S^3$ be a conformal transformation, M a Hopf torus and $f(M)$ an isothermic (Willmore) surface. Then M is a (minimal) Clifford torus.*

Figures 1 to 4 were built with the computer algebra system Maple.

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