Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 44 (2003), No. 1, 57-74.

Special Cohomogeneity One Isometric Actions on Irreducible Symmetric Spaces of Types I and II *

László Verhóczki

Department of Geometry, Eötvös University
Pázmány P. s. 1/c, 1117 Budapest, Hungary

Abstract. In the present paper we study isometric actions on compact symmetric spaces for which the principal orbits are tubular hypersurfaces around totally geodesic singular orbits. We show that in these cases the symmetric space can be thought of as a compact tube the radius of which is determined by the curvature tensor. Since the constant principal curvatures of the tubular orbits can explicitly be expressed, we obtain a simple method to determine volumes of symmetric spaces by using volumes of lower dimensional ones. Finally, we discuss the classical irreducible symmetric spaces of types I and II, each of which admits such special hyperpolar actions.

MSC 2000: 53C35, 53C40, 57S15

1. Introduction

Recall some basic concepts on isometric actions which will be used throughout the paper. Regarding an isometric action $\alpha : L \times N \to N$ of a compact connected Lie group L on a Riemannian manifold N, a closed (totally geodesic) submanifold C is said to be a section if C intersects orthogonally all the orbits of L, and in this case α is called polar. An isometric action is said to be hyperpolar if it admits sections which are flat totally geodesic submanifolds. In symmetric spaces the actions of isotropy subgroups present evident examples for hyperpolar ones.

*This research was supported by the Hungarian National Science and Research Foundation OTKA T032478.

Concerning the origin of the subject, first R. Bott and H. Samelson studied the so–called variationally complete actions on symmetric spaces in the paper [4].

L. Conlon has effectively discussed the variationally complete actions and the hyperpolar ones (see [8] and [9]). The connection of them is rather close, and Conlon has proved that a hyperpolar action is variationally complete. The classification of hyperpolar actions on Euclidean spaces was accomplished by J. Dadok (see [10]).

Let us take a Riemannian symmetric space $G|K$ of compact type. A connected subgroup L of the compact Lie group G is called symmetric if there exists an involutive automorphism ρ of G such that L coincides with the identity component of $G_\rho = \{g \in G \mid \rho(g) = g\}$. R. Hermann has pointed out that the action of a symmetric subgroup L on $G|K$ admits flat sections (see [15] and [16]). Later, using the construction of Hermann, J. Szenthe presented examples for hyperpolar actions on compact Lie groups (see [23]).

Concerning closed subgroups of G which are not symmetric, E. Heintze, R. S. Palais, C. L. Terng and G. Thorbergsson have given some sufficient and necessary conditions for an action to be hyperpolar in the paper [13]. Using these criteria, A. Kollross has completely classified the hyperpolar isometric actions on compact symmetric spaces (see [19]). It is important to remark that cohomogeneity one isometric actions on compact symmetric spaces are always hyperpolar (see [13]).

Let $G|K$ be a simply connected symmetric space of compact type and let σ denote the corresponding involution of G. In this paper we study the isometric action of the symmetric subgroup L on $G|K$ provided that σ and ρ commute, furthermore, the codimension of the principal orbits is equal to one and the orbit $L(o)$ of the point $o = K$ is singular. We show that in this case the orbits of L coincide with the tubular hypersurfaces around the totally geodesic orbit $L(o)$ (see Proposition 4). The whole symmetric space $G|K$ can be thought of as a compact tube the radius r of which is determined by the curvature tensor (see Theorem 1). Moreover, the other singular orbit consists of those points whose distance from $L(o)$ equals r (see Proposition 5). Since the principal curvatures of these tubular hypersurfaces can explicitly be expressed (see Proposition 6), we can compute the volumes of the principal orbits using some results of the paper [12] by A. Gray and L. Vanhecke. This yields a simple method to compute the volumes of compact symmetric spaces from the volumes of lower dimensional ones (see Section 4). Finally, we apply the idea described in Sections 3 and 4 to irreducible symmetric spaces of types I and II. Mention must be made that K. Abe and I. Yokota have already determined the volumes of all the irreducible compact symmetric spaces using a different technique (see [1]).

Throughout this paper $N = G|K$ presents a d-dimensional (simply connected) symmetric space of compact type with the relevant Riemannian metric $\langle \ , \ \rangle$ and with the Levi-Civita connection ∇. The exponential map in N defined on the tangent bundle TN will be denoted by Exp and the Riemannian curvature tensor by R. We refer to the well-known book [14] of S. Helgason for basic concepts and facts on symmetric spaces. Concerning submanifolds, the basic concepts, which are used here, can be found in the books [11] and [18]. We always take the inherited Riemannian metrics and the induced connections on the submanifolds of N. As usual, the normal vector bundle of a given submanifold M will be denoted by $\nu(M)$. Considering a smooth normal vector field ζ on M, A_ζ will denote the shape operator of M

with respect to ζ.

2. Hyperpolar actions of special symmetric subgroups

Let us take such a Riemannian symmetric pair (G,K) of compact type, where K is connected. This means that G is a connected compact semisimple Lie group and there exists an involutive automorphism $\sigma : G \to G$ such that K coincides with the identity component of the closed subgroup G_σ. This induces an involution $d\sigma$ of the semisimple Lie algebra $\mathfrak{g}$ of G. Considering the eigenspaces of $d\sigma$ with respect to the eigenvalues 1 and -1, we obtain the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}, \tag{1}$$

where $\mathfrak{k}$ coincides with the Lie algebra of the subgroup K.

Henceforth we denote the coset space $G|K$ by N and the special coset K by o, too. We can take the smooth left action $\alpha : G \times G|K \to G|K$ defined by the equality $\alpha(g, hK) = ghK$ for $g, h \in G$. It is well-known that the coset space $G|K$ can be equipped with a Riemannian metric such that the above action α turns into isometric and $G|K$ turns into a symmetric space. Therefore the elements of G can be considered as isometries of N, since to each element $g \in G$ we can assign the isometry $\alpha_g : G|K \to G|K$, where $\alpha_g(hK) = ghK$ holds for $hK \in G|K$.

As it is well-known, the subspace $\mathfrak{p}$ can be regarded as the tangent space T_oN of $N = G|K$ at $o = K$. Namely, considering the natural smooth mapping $\pi : G \to G|K$ defined by $\pi(g) = gK$, its tangent linear map $T_e\pi$ at the identity element e presents an isomorphism between $\mathfrak{p}$ and T_oN. Denoting by exp the exponential map of the Lie algebra $\mathfrak{g}$ onto G and by Exp_o the exponential map of the tangent space T_oN onto N, it is important to remark that the equality

$$exp(Y)K = Exp_o(T_e\pi(Y))$$

is valid for any $Y \in \mathfrak{p}$. In this paper $\mathfrak{p}$ and T_oN are considered to be identified by $T_e\pi$.

Then concerning the Riemannian curvature tensor at o, for any vectors v_1, v_2, v_3 in $\mathfrak{p}$ the relation

$$R(v_1, v_2)v_3 = -\big[\,[v_1, v_2], v_3\big] \tag{2}$$

is valid, where $[\ ,\ ]$ denotes the bracket operation in the Lie algebra $\mathfrak{g}$.

Hereafter we assume that G is simply connected. This implies that the closed subgroup G_σ is connected (see Chapter VII, Theorem 8.2 in [14]) and the symmetric space N is also simply connected. Let B denote the Killing form of the Lie algebra $\mathfrak{g}$. It is well-known that the quadratic form B is negative definite. In this paper we assume that for a positive number c the equality

$$\langle\, v_1, v_2 \,\rangle_o = -c \cdot B(v_1, v_2) \qquad (v_1,\ v_2 \in \mathfrak{p}) \tag{3}$$

holds, where $\langle\ ,\ \rangle_o$ denotes the inner product on the tangent space $T_oN = \mathfrak{p}$. Notice that the above relation (3) is valid if the symmetric space $G|K$ is irreducible.

Let us take an involutive automorphism ρ of G such that ρ commutes σ $(\rho \neq \sigma)$ and the symmetric subgroup $L = G_\rho$. In the following we study the inherited isometric action $\alpha : L \times N \to N$. Observe that since L is a compact Lie group, all the orbits of L are submanifolds in N (see [5; pp. 301–303]). By the eigenspaces of the induced involution $d\rho$ of $\mathfrak{g}$ we get another decomposition $\mathfrak{g} = \mathfrak{l} + \mathfrak{n}$, where $\mathfrak{l}$ coincides with the Lie algebra of L. Obviously, since the involutions $d\sigma$ and $d\rho$ commute, the equalities

$$\mathfrak{l} = \mathfrak{l} \cap \mathfrak{k} + \mathfrak{l} \cap \mathfrak{p}, \qquad \mathfrak{p} = \mathfrak{p} \cap \mathfrak{l} + \mathfrak{p} \cap \mathfrak{n} \tag{4}$$

are valid, and the components of $\mathfrak{p}$ are orthogonal with respect to the inner product.

Concerning differentiable actions, basic notions and facts can be found in the book [17]. First we determine the tangent space of the orbit $L(o)$ at the point o. For this reason we introduce some further notation. By the Cartan decomposition (1) an arbitrary vector X in $\mathfrak{g}$ can uniquely be written in the form $X = X_\mathfrak{k} + X_\mathfrak{p}$, where $X_\mathfrak{k} \in \mathfrak{k}$ and $X_\mathfrak{p} \in \mathfrak{p}$.

It is obvious that the tangent space $T_oL(o)$ is spanned by the tangent vectors $\dot{\omega}_X(0)$ of the smooth curves $\omega_X : \mathbf{R} \to N$ $(X \in \mathfrak{l})$, where $\omega_X(t) = \alpha_{exp(tX)}(o) = \pi \circ exp(tX)$ is valid for $t \in \mathbf{R}$. Clearly, the tangent vector $\dot{\omega}_X(0)$ $(X \in \mathfrak{l})$ coincides with the component $X_\mathfrak{p}$. Hence, the decomposition (4) of $\mathfrak{l}$ implies the following assertion.

Proposition 1. *The tangent space $T_oL(o)$ coincides with $\mathfrak{l} \cap \mathfrak{p}$, which is a Lie triple system in $\mathfrak{p}$.*

Since $\mathfrak{l} \cap \mathfrak{p}$ is a Lie triple system in $\mathfrak{p}$, $Exp_o(\mathfrak{l} \cap \mathfrak{p})$ is a totally geodesic submanifold in N. Using the equality $Exp_o(Y) = \alpha(exp(Y), o)$ which is valid for any $Y \in \mathfrak{p}$, by Proposition 1 we obtain that $Exp_o(\mathfrak{l} \cap \mathfrak{p})$ coincides with the orbit $L(o)$. Moreover, observe that the isotropy subgroup $L_o = \{\, g \in L \mid \alpha_g(o) = o \,\}$ equals $L \cap K$.

For simplicity, hereafter the totally geodesic orbit $L(o)$ will be denoted by M, too. Therefore we get $T_oM = \mathfrak{l} \cap \mathfrak{p}$ and $\nu_oM = \mathfrak{n} \cap \mathfrak{p}$, where ν_oM denotes the normal complementary subspace of T_oM.

It is reasonable to consider the involution $\tau = \sigma \circ \rho$ of G. Hence, we can take the symmetric subgroup $H = G_\tau$ and its action on N. Denoting by $\mathfrak{h}$ the Lie algebra of H, we obtain that $\mathfrak{h} \cap \mathfrak{p} = \mathfrak{n} \cap \mathfrak{p}$ holds. As in the case of $L(o)$, it can easily be seen that the orbit $H(o)$ coincides with the totally geodesic submanifold $Exp_o(\mathfrak{h} \cap \mathfrak{p})$.

Let us take a maximal abelian subspace $\mathfrak{c}$ in $\mathfrak{n} \cap \mathfrak{p}$ and the totally geodesic submanifold $C = Exp_o(\mathfrak{c})$. This means that C is a maximal dimensional flat totally geodesic submanifold in the symmetric space $H(o)$. The following result, which is essentially due to Hermann, verifies that the isometric action $\alpha : L \times N \to N$ is hyperpolar (for proof see [16] or [13]).

Proposition 2. *The flat torus C intersects orthogonally all the orbits of L.*

Clearly, the relations $d\sigma(\mathfrak{l}) = \mathfrak{l}$, $d\sigma(\mathfrak{h}) = \mathfrak{h}$, $exp \circ d\sigma = \sigma \circ exp$ immediately imply that L and H are invariant subgroups of σ. Considering the restrictions of σ to these compact subgroups, we obtain that $L \cap K = L_{\sigma|L} = H_{\sigma|H} = H \cap K$ is valid.

As it is well-known, a connected (totally geodesic) submanifold M is called reflective if there exists an involutive isometry of N such that M is a component of its fixed point set. Lists of reflective totally geodesic submanifolds in irreducible symmetric spaces are given in the papers [20] and [7]. It is not difficult to show that $L(o)$ and $H(o)$ are reflective submanifolds in N (for details see [26]). The above statements can be summarized in the following proposition.

Proposition 3. *$(L, L\cap K)$ and $(H, H\cap K)$ present Riemannian symmetric pairs with the involution σ. The orbits $L(o) = Exp_o(\mathfrak{p}\cap\mathfrak{l})$ and $H(o) = Exp_o(\mathfrak{p}\cap\mathfrak{n})$ are reflective totally geodesic submanifolds in N, which are isometric with the symmetric spaces $L|L\cap K$ and $H|H\cap K$, respectively.*

Regarding an arbitrary element $g\in L$, it is evident that the tangent linear map $T\alpha_g$ of the isometry α_g leaves the normal vector bundle $\nu(M)$ invariant. Hence, we can take the smooth action $T\alpha : L\times\nu(M)\to\nu(M)$ of the symmetric subgroup L on $\nu(M)$ which is defined by $T\alpha(g,w) = T\alpha_g(w)$ for $g\in L$ and $w\in\nu(M)$. In what follows, Exp_M will denote the restriction of Exp to the normal bundle $\nu(M)$. Applying geodesics in N which intersect orthogonally $M = L(o)$, it can easily be seen that the relation

$$\alpha_g\circ Exp_M = Exp_M\circ T\alpha_g \tag{5}$$

is valid for each $g\in L$.

3. Cohomogeneity one isometric actions on compact symmetric spaces

Recall that the cohomogeneity of the action α is equal to the codimension of the principal orbits of L. By Proposition 2 we obtain that this number is equal to one if and only if $H(o)$ is a symmetric space of rank one.

Henceforth RP^n, CP^n and QP^n will denote the n-dimensional real, $2n$-dimensional complex and $4n$-dimensional quaternion projective spaces, respectively. Furthermore, S^n and Cay will denote the n-dimensional sphere and the Cayley projective plane. It is well-known that they present the compact symmetric spaces of rank one, furthermore, in these symmetric spaces all the geodesics are closed and have the same length.

Assume that the rank of the symmetric space $H(o) = Exp_o(\nu_o M)$ is equal to one and the dimension of $L(o)$ is less than $d-1$, where d denotes the dimension of N. Then the closed geodesics in $H(o)$ which pass through o are sections of α.

We can consider the isometric action of the isotropy subgroup $H_o = H\cap K$ on the symmetric space $H(o)$, which is a totally geodesic submanifold in N. Then the orbits of $H\cap K = L\cap K$ in $H(o)$ coincide with the geodesic spheres around o. It follows from this that the smooth action $T\alpha : L\times\nu(M)\to\nu(M)$ is transitive on the set of unit vectors in $\nu(M)$.

Let us introduce now the notation

$$\tilde{M}^t = \{\ Exp_M(w)\mid w\in\nu(M),\ \|w\| = t\ \}\qquad (t>0)$$

and call $\tilde{M}^t$ the tubular hypersurface of radius t around M. The above statement concerning $T\alpha$ and the relation (5) verify that $\tilde{M}^t$ is an orbit of L for any $t>0$. Therefore the following assertion is true.

Proposition 4. *If the cohomogeneity of the action $\alpha : L \times N \to N$ is equal to one and $L(o)$ is a singular orbit, then the other orbits of L coincide with the tubular hypersurfaces around $L(o)$.*

Hereafter we always assume that the cohomogeneity of the isometric action α is equal to one and the totally geodesic orbit $L(o)$ is singular.

Our purpose is to show that N can be regarded as a compact tube around $L(o) = M$ and to determine the radius r of this tube using the curvature tensor R. Let us take a unit vector $w \in \nu_q M$ $(q \in M)$ and the self-adjoint endomorphism $R_w : T_q N \to T_q N$ defined by $R_w(v) = R(v, w)w$ $(v \in T_q N)$. Since $\nu_q M$ is the tangent space of a totally geodesic submanifold in N, $\nu_q M$ and $T_q M$ are invariant subspaces of R_w. Furthermore, it is important to observe that in this case the eigenvalues of R_w do not depend on the choice of the unit vector w in $\nu(M)$.

Let us fix now a unit vector u in $\nu_o M$ and the geodesic $\gamma : \mathbf{R} \to N$ defined by $\gamma(\tau) = Exp_o(\tau u)$ $(\tau \in \mathbf{R})$. By Proposition 2 the closed geodesic $C = \gamma(\mathbf{R})$ is a section of the action α. Concerningthe orbits of L, Proposition 4 implies that $L(\gamma(t)) = L(\gamma(-t)) = \tilde{M}^t$ $(t > 0)$ is valid.

The eigenvalues of R_u in $T_o M$, which are non-negative numbers, will be denoted by a_i $(i = 1, \ldots, s)$ and their multiplicities will be denoted by m_i, respectively.

Let λ be the maximal sectional curvature of the symmetric space $H(o) = Exp(\nu_o M)$ with rank one. Then the eigenvalues of R_u in $\nu_o M$ are $b_1 = \lambda$, $b_2 = \frac{1}{4}\lambda$, $b_3 = 0$ with the multiplicities k_1, k_2, k_3 $(k_3 = 1)$, respectively. Obviously, $k_2 = 0$ is valid if $H(o)$ is a space of constant curvature. Moreover, $k_1 = 1$, $k_1 = 3$ and $k_1 = 7$ hold provided that $H(o) = CP^n$, $H(o) = QP^n$ and $H(o) = Cay$, respectively. (For details concerning the compact symmetric spaces of rank one see Chapter 3 of the book [2].)

Let h be the arc length of the closed geodesics in $H(o)$. Then $h = \frac{2\pi}{\sqrt{\lambda}}$ holds provided that $H(o)$ is not a real projective space. In the case of $H(o) = RP^n$ $(n \geq 2)$ we get $h = \frac{\pi}{\sqrt{\lambda}}$.

Let us consider the positive number r defined by the following relation

$$r = \text{minimum} \left(\left\{ \frac{\pi}{2\sqrt{a_i}} \mid a_i \neq 0 \ (i = 1, \ldots, s) \right\} \cup \left\{ \frac{h}{2} \right\} \right). \tag{6}$$

We can take the open tubular neighborhood

$$\nu^r(M) = \{ \ w \mid w \in \nu(M), \quad \|w\| < r \ \}$$

of radius r in the normal bundle $\nu(M)$.

In order to prove Theorem 1, which verifies that N has a tubular structure around $L(o) = M$, we need some results concerning M-Jacobi vector fields along normal geodesics (for details see [3; pp. 220–238]).

Let N be a connected compact Riemannian manifold with dimension d and let M be a connected compact submanifold of N. Take a point o of M and a unit vector u in the normal subspace $\nu_o M$. Then we can consider the geodesic $\gamma : \mathbf{R} \to N$, where $\gamma(t) = Exp_o(tu)$ holds for $t \in \mathbf{R}$. Recall that a Jacobi vector field $\xi : \mathbf{R} \to TN$ along γ is called M-Jacobi if the conditions

$$\langle\, \xi(t), \dot\gamma(t)\,\rangle = 0 \quad \text{for} \quad t\in\mathbf{R}, \qquad \xi(0) \in T_oM, \qquad \nabla_u\xi + A_u(\xi(0)) \in \nu_oM$$

are satisfied, where A_u denotes the shape operator of M with respect to u. The M-Jacobi vector fields along γ form a $(d-1)$-dimensional linear space which we denote by $\mathcal{J}(\gamma, M)$. As usual, if a vector w in $\nu(M)$ is a critical point of Exp_M, then w (respectively $Exp_M(w)$) is said to be a focal point of M in $\nu(M)$ (respectively in N). It is well-known that the vector tu $(t \neq 0)$ is a focal point of M if and only if there exists a non-trivial M-Jacobi vector field ξ along γ such that $\xi(t) = 0$ holds. If the vector εu $(\varepsilon > 0)$ is a focal point of M such that τu is not a focal one for any $\tau\in(0,\varepsilon)$, then εu is called a first focal point of M.

On the other hand, the point $\gamma(\varepsilon)$ for some $\varepsilon > 0$ is said to be the minimum point of M along γ if the following two conditions are satisfied:
Considering any value $t\in[0,\varepsilon]$, the distance between M and $\gamma(t)$ is equal to t. Furthermore, if $t > \varepsilon$ is valid, then the distance between M and $\gamma(t)$ is less than t.

Concerning minimum points of submanifolds, we can state the assertion below the proof of which is analogous to the proof of the theorem characterizing the cut points of a given point (see [3; pp. 237–238]). To give a complete proof we have to use the fact that M is a compact submanifold of N.

Lemma 1. *If $\gamma(\varepsilon)$ $(\varepsilon > 0)$ yields the minimum point of M along γ, then at least one of the following statements is true.*
(1) *The vector εu is a first focal point of M.*
(2) *There is a unit vector $w\in\nu(M)$ different from u such that $Exp_M(\varepsilon w) = \gamma(\varepsilon)$ holds.*

Let us return to the discussion of the cohomogeneity one isometric action $\alpha : L \times N \to N$. The following theorem verifies that the simply connected symmetric space N is a compact tube of radius r around $L(o) = M$.

Theorem 1. *The restriction of Exp_M to $\nu^r(M)$ is a diffeomorphism, and the relation $Exp_M(\nu^r(M)) \cup L(\gamma(r)) = N$ is valid.*

Proof. First we show that the smooth map $Exp_M : \nu^r(M) \to N$ is regular by using the method of M-Jacobi vector fields. Since the submanifold $L(o) = M$ is totally geodesic, the shape operator A_u vanishes. Let us take a non-zero vector v_i $(i = 1, \dots, s)$ in T_oM such that $R_u(v_i) = a_i\, v_i$ holds and the parallel vector field η_i along γ, where $\eta_i(0) = v_i$. Then the vector field $\xi_i : \mathbf{R} \to TN$ defined by $\xi_i(t) = \cos(\sqrt{a_i}\, t)\, \eta_i(t)$ is M-Jacobi. Moreover, let $\hat v_j$ $(j = 1, 2)$ be a non-zero vector in ν_oM such that $R_u(\hat v_j) = b_j\, \hat v_j$, and consider the parallel vector field $\hat\eta_j$ along γ, where $\hat\eta_j(0) = \hat v_j$. Obviously, the vector field $\hat\xi_j$ defined by $\hat\xi_j(t) = \sin(\sqrt{b_j}\, t)\ \hat\eta_j(t)$ is also M-Jacobi. These vector fields generate the linear space $\mathcal{J}(\gamma, M)$. Regarding the formula (6) which presents the radius r, we can see that for any non-trivial M-Jacobi vector field ξ along γ the relation $\xi(t) \neq 0$ is valid provided that $t \in (0, r)$. Since L acts transitively on the set of unit vectors in $\nu(M)$ by the tangent linear maps, we obtain that the restriction of the smooth map Exp_M to $\nu^r(M)$ is regular. Moreover, Proposition 4 implies that the orbits $L(\gamma(t)) = \tilde M^t$ $(0 < t < r)$ are $(d-1)$-dimensional.

After this we prove that the map $Exp_M : \nu^r(M) \to N$ is injective which follows from the statement below.

Considering a point $p = \gamma(t) \quad (0 < t < r)$, the distance between M and p is equal to t and $\gamma([0,t])$ presents the unique minimizing geodesic segment which joins M and p.

Indirectly, suppose that the above assertion is not true for some $t \in (0,r)$. Since τu is not a focal point of M for any $\tau \in (0,t)$, by Lemma 1 this implies that we can find a number $\varepsilon \quad (\varepsilon \leq t)$ and a unit vector $w \in \nu_q M \quad (q \in M,\ q \neq o)$ such that $Exp_M(\varepsilon u) = Exp_M(\varepsilon w)$ holds. Assume that ε is the least positive number having this property. Then the orbit $\tilde{M}^\tau = L(\gamma(\tau)) \quad (0 < \tau < \varepsilon)$ is principal and the isotropy subgroup $L_{\gamma(\tau)}$ does not depend on the choice of $\tau \in (0,\varepsilon)$. Since $L(o) = L(\gamma(0))$ is a singular orbit, the isotropy subgroup $L_{\gamma(\tau)}$ is included in $L_o = L \cap K$. Let us take now an element $g \in L$ such that $T\alpha_g(w) = u$ holds. Since $\alpha_g(q) = o$ and $\alpha_g(\gamma(\varepsilon)) = \gamma(\varepsilon)$ are true, we obtain that the isotropy subgroup $L_{\gamma(\varepsilon)}$ of L at the point $\gamma(\varepsilon)$ is larger than $L_{\gamma(\tau)}$ because g is not contained by L_o. Therefore we have got the $(d-1)$-dimensional orbit $L(\gamma(\varepsilon))$ which is not principal. However, since the symmetric space N is simply connected, by one of the results of Conlon all the maximal dimensional orbits of L are principal (see Proposition 2.2 in [8]). This contradiction verifies that γ presents the unique minimizing geodesic segment which joins M and $\gamma(t)$. Since the action $T\alpha : L \times \nu(M) \to \nu(M)$ is transitive on the set of unit vectors of $\nu(M)$, the relation (5) implies that the mapping Exp_M is injective on $\nu^r(M)$.

It remained only to prove the second assertion of the theorem. We can easily show that the distance of each point of N from the submanifold M is not greater than r. Considering a point p of N, let $\chi : [0,\delta] \to N$ be a minimizing geodesic segment which joins M and p, where $\dot\chi(0) = w$ is a unit vector in $\nu(M)$ and $\chi(\delta) = p$. Then the relations $\delta \leq \frac{\pi}{2\sqrt{a_i}}$ $(i = 1, \ldots, s)$ hold since the vector τw $(0 < \tau < \delta)$ is not a focal point of M, furthermore, $\delta \leq \frac{h}{2}$ is also valid. It follows from this that the inequality $\delta \leq r$ is true. Therefore the set $Exp_M(\nu^r(M)) \cup \tilde{M}^r$ coincides with the symmetric space N.

Finally, observe that in consequence of the above facts the elements of $\tilde{M}^r$ are minimum points of the submanifold $L(o) = M$. $\square$

The following proposition shows that the symmetric subgroup L has two singular orbits.

Proposition 5. *$L(\gamma(r)) = \tilde{M}^r$ is another singular orbit of L.*

Proof. Considering an element $X \in \mathfrak{l}$ and a point $p \in N$, let us take the smooth curve $\omega_{X,p} : \mathbf{R} \to N$ defined by $\omega_{X,p}(\tau) = \alpha_{exp(\tau X)}(p)$ for $\tau \in \mathbf{R}$. Denoting by $\dot\omega_{X,p}(0)$ the tangent vector of this curve at 0, we get

$$T_pL(p) = \{\ \dot\omega_{X,p}(0) \mid X \in \mathfrak{l}\ \}.$$

Regarding an element $X \in \mathfrak{l}$, we can take the vector field $\xi_X : \mathbf{R} \to TN$ along the fixed geodesic γ, where $\xi_X(t) = \dot\omega_{X,\gamma(t)}(0)$ is valid. It is well-known that the transversal vector fields of a geodesic variation are Jacobi vector fields along the geodesics. Consider the geodesic variation $\Gamma_X : \mathbf{R} \times (-\epsilon,\epsilon) \to N$ of γ defined by $\Gamma_X(t,\tau) = \alpha_{exp(\tau X)}(\gamma(t))$, where ϵ is a positive number and $t \in \mathbf{R},\ \tau \in (-\epsilon,\epsilon)$. Then ξ_X coincides with the transversal vector field of this geodesic variation, and it can be seen that ξ_X is an M-Jacobi vector field along γ. Therefore we obtain the equality

$$T_{\gamma(t)}\tilde{M}^t = \{ \xi(t) \mid \xi \in \mathcal{J}(\gamma, M) \} \qquad (t > 0). \tag{7}$$

The relations (6) and (7) imply that if either $H(o)$ is not a real projective space or $2r < h$ holds, then $\gamma(r)$ is a focal point of $L(o) = M$. Hence, the codimension of the submanifold $L(\gamma(r))$ is greater than one.

Finally, assume that $H(o) = RP^n$ holds and $\gamma(r)$ is an antipodal point of $o = \gamma(0)$ on the closed geodesic C. Regarding the isotropy subgroup of L at $\gamma(r)$, it can easily be seen that the orbit $L(\gamma(r))$ is not principal. As we mentioned it in the proof of Theorem 1, since N is simply connected, all the maximal dimensional orbits of L are principal. This implies that the orbit $L(\gamma(r))$ is singular, and $L(\gamma(r))$ consists of the focal points of M. $\square$

Remark 1. The even integer $\frac{h}{r}$ presents the number of points of the intersection of the circle $C = \gamma(\mathbf{R})$ and a principal orbit $L(\gamma(t)) = \tilde{M}^t$ $(0 < t < r)$.

Let ζ denote the smooth unit normal vector field on the hypersurface $\tilde{M}^t$ $(0 < t < r)$ defined by the condition $\zeta(\gamma(t)) = \dot{\gamma}(t)$. Clearly, the shape operator $\tilde{A}_\zeta$ of $\tilde{M}^t$ has constant eigenvalues, which are called principal curvatures of $\tilde{M}^t$. Using the endomorphism R_u, we can explicitly express these eigenvalues by the following statement which is a special case of Theorem 1 given in the paper [25]. Notice that by our notational convention $b_1 = \lambda$ and $b_2 = \frac{1}{4}\lambda$ hold in Proposition 6 described below.

Proposition 6. *The constant principal curvatures of the hypersurface* $\tilde{M}^t = L(\gamma(t))$ $(0 < t < r)$ *are* $\mu_i(t) = \sqrt{a_i}\,\tan(\sqrt{a_i}\,t)$ $(i = 1, \dots, s)$ *with multiplicities* m_i *and* $\hat{\mu}_j(t) = -\sqrt{b_j}\,\cot(\sqrt{b_j}\,t)$ $(j = 1, 2)$ *with multiplicities* k_j*, respectively.*

4. Volumes of tubular hypersurfaces around $L(o)$

In this section first we review some basic formulae concerning volumes of tubes around a compact submanifold. For details and proof see the paper [12] and the book [11].

Let N be an orientable complete Riemannian manifold with dimension d and let M be an m-dimensional connected orientable submanifold with compact closure $(1 \leq m \leq d-2)$. Denote by k the codimension of M in N $(k = d - m)$. Assume that the restriction of Exp_M to $\nu^r(M)$ is a diffeomorphism for a suitable number r $(r > 0)$. Let us take a unit vector $u \in \nu_q M$ and the normal geodesic γ defined by $\gamma(t) = Exp_M(tu)$ for $t \in \mathbf{R}$. Denote by ω_N the volume form of N and by ω_ν the canonical volume form of the normal bundle $\nu(M)$. Then for a suitable number $\vartheta_u(t)$ the equality

$$(Exp_M)^*\omega_N\,(tu) = \vartheta_u(t) \cdot \omega_\nu(tu)$$

holds, where $(Exp_M)^*\omega_N$ denotes the transform of ω_N by Exp_M. The mapping $\vartheta_u : \mathbf{R} \to \mathbf{R}$ is called the infinitesimal change of volume function corresponding to u. Obviously, $\vartheta_u(0) = 1$ is valid. Denote by $Tr\,\tilde{A}_{\dot{\gamma}(t)}$ the trace of the shape operator of the tubular hypersurface $\tilde{M}^t$

with respect to $\dot\gamma(t)$. Then the restriction of the function ϑ_u to $(0,r)$ satisfies the differential equation

$$\frac{\vartheta_u'(t)}{\vartheta_u(t)} = -\frac{k-1}{t} - Tr\,\tilde{A}_{\dot\gamma(t)} \qquad (0<t<r). \tag{8}$$

The volume of the hypersurface $\tilde{M}^t$ $(0<t<r)$ is given by the formula

$$vol(\tilde{M}^t) = t^{k-1}\cdot \int_M \Big(\int_{S^{k-1}[1]} \vartheta_u(t)\,du\Big)\,dm, \tag{9}$$

where $S^{k-1}[1]$ denotes the unit spheres in the normal subspaces $\nu_q M$ $(q\in M)$ and du denotes the volume forms on them, furthermore, dm denotes the volume form of M.

We can apply the above formulae to cohomogeneity one hyperpolar actions discussed in the preceding section. In this case the tubular hypersurfaces around $M = L(o)$ are isoparametric and the function ϑ_u does not depend on the choice of the unit vector u.

Using Proposition 6 and the equation (8), we can easily verify the equality

$$\vartheta(t) = 2^{k_2}\,\lambda^{\frac{1-k}{2}}\,t^{1-k}\cdot\sin^{k_1}(\sqrt{\lambda}\,t)\cdot\sin^{k_2}(\frac{1}{2}\sqrt{\lambda}\,t)\cdot\prod_{i=1}^{s}\cos^{m_i}(\sqrt{a_i}\,t), \tag{10}$$

where $k_1+k_2 = k-1$. Notice that the above expression (10) can be regarded as a special case of Proposition 3.1 of the paper [21]. Concerning the volumes of the principal orbits $L(\gamma(t)) = \tilde{M}^t$ $\;(0<t<r)$, the relation (9) implies

$$vol(\tilde{M}^t) = vol(M)\cdot vol(S^{k-1}[1])\cdot t^{k-1}\vartheta(t). \tag{11}$$

Hence, by Theorem 1 we obtain the equality

$$vol(N) = \int_0^r vol(\tilde{M}^t)\,dt = vol(M)\cdot vol(S^{k-1}[1])\cdot \int_0^r t^{k-1}\vartheta(t)\,dt. \tag{12}$$

Since $M = L(o)$ is a lower dimensional symmetric space, the above formula presents a simple method for computation of volumes of several symmetric spaces.

Remark 2. K. Abe and I. Yokota have computed the volumes of all the compact irreducible symmetric spaces in a different way (see [1]). Their method is based on the results of S. A. Broughton (see [6]).

Remark 3. Assume that $N = G|K$ is an irreducible compact symmetric space with dimension d. Then the Riemannian metric of N is given by the equality (3). Let κ be the maximal sectional curvature in N. It is clear that in this case the products $\kappa^{\frac{d}{2}}\cdot vol(N)$ and $\kappa^{\frac{m}{2}}\cdot vol(M)$ do not depend on the choice of the positive factor c.

5. Examples for special cohomogeneity one isometric actions on irreducible symmetric spaces of type I

In this section we apply the earlier results of the paper to classical irreducible symmetric spaces of type I. Some concrete hyperpolar actions on the classical structures will be discussed in detail.

We always take an action $\alpha : L \times N \to N$ such that $M = L(o)$ is a totally geodesic singular orbit of the symmetric subgroup L and $H(o)$ is a symmetric space of rank one. Hence, the closed geodesics in $H(o)$ which pass through o present sections of α, furthermore, by Theorem 1 N is a compact tube around $L(o)$. Among others, we compute the radii of the tubes and the functions $\vartheta : (0, r) \to \mathbf{R}$ which determine the volumes of the principal orbits by the relation (11).

As earlier, κ and λ will denote the maximal sectional curvatures of $N = G|K$ and $H(o)$, respectively. On several occasions the maximal curvature of a given symmetric space will be indicated as a subscript (for instance $N_\kappa,\ H(o)_\lambda$).

Using the isotropy subgroups, we get evident examples for isometric actions on the symmetric spaces of rank one, where the principal orbits coincide with the geodesic spheres. Therefore we consider only those Grassmannian manifolds the ranks of which are not less than 2.

Concerning matrix Lie groups, we use the notation of the book [14] (in particular, see Chapter X). Regarding an element X of the complex matrix group $Gl(n, \mathbf{C})$, $\bar{X}$ and X^T will denote its conjugate and its transpose, respectively. Furthermore, E_s will denote that matrix in $Gl(n, \mathbf{C})$, where the entry in s-th row and s-th column is equal to 1 and all the other entries vanish. The identity element of $Gl(n, \mathbf{C})$ will be denoted by I_n. Moreover, we shall use the notation

$I_{p,q} = (E_1 + \cdots + E_p) - (E_{p+1} + \cdots + E_n)$ with $p + q = n$,

$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ and $F_{p,q} = \begin{pmatrix} I_{p,q} & 0 \\ 0 & I_{p,q} \end{pmatrix}$, where $F_{p,q},\ J_n \in Gl(2n, \mathbf{C})$. The isomorphism of two Lie groups will be denoted by the sign $\approx$.

5.1. Compact symmetric spaces $SU(n)|SO(n)$ $(n \geq 3)$ of type AI

In this case we have the equalities $G = SU(n),\ \sigma(X) = \bar{X}$ for $X \in G$ and $K = G_\sigma = SO(n)$. For simplicity, the symmetric space $SU(n)|SO(n)$ will be denoted by $\mathcal{A}I(n)$. Let us take the involutive automorphism $\rho : G \to G$ defined by $\rho(X) = I_{n-1,1} X I_{n-1,1}$, which commutes σ. Then we obtain the symmetric subgroups $L = G_\rho = S(U_{n-1} \times U_1)$, $H = G_\tau \approx SO(n)$ and $L \cap K = S(O_{n-1} \times O_1)$.

In order to characterize the orbit $L(o)$ we need the vector $Z = i(E_1 + \cdots + E_{n-1} - (n-1)E_n)$ of $\mathfrak{l} \cap \mathfrak{p}$ and the subspace $\mathfrak{a} = \{\ X \in \mathfrak{l} \cap \mathfrak{p} \mid B(X, Z) = 0\ \}$ which is a Lie triple system in $\mathfrak{p}$. Then $Exp_o(\mathfrak{a})$ is a totally geodesic submanifold in N which is isometric with $\mathcal{A}I(n-1)_\kappa$. The closed geodesic $S^1 = Exp_o(\mathbf{R}Z)$ has the arc length $l\,(S^1) = \sqrt{2n(n-1)}\,\pi\kappa^{-\frac{1}{2}}$. It can be shown that $L(o) = M$ is covered by the product of $Exp_o(\mathfrak{a})$ and $Exp_o(\mathbf{R}Z)$, more precisely, M is isometric with $(\mathcal{A}I(n-1)_\kappa \times S^1)|\mathbf{Z}_{n-1}$, where $\mathbf{Z}_{n-1}$ denotes the cyclic group of order

$n-1$. Therefore we get

$$vol(M) = vol(\mathcal{A}I(n-1)_\kappa) \cdot \frac{\sqrt{2n}}{\sqrt{n-1}}\,\pi\,\kappa^{-\frac{1}{2}}.$$

It is easy to verify that $H(o)$ coincides with the real projective space RP_λ^{n-1}.

Using the equality (2), we can determine the eigenvalues of the restricted endomorphism $R_u|T_oM$ (and their multiplicities), which are $a_1 = \kappa$ $(m_1 = 1)$, $a_2 = \frac{1}{4}\kappa$ $(m_2 = n-2)$ and $a_3 = 0$. Furthermore, by virtue of (6) and (10) it can be shown that the equalities

$$\kappa = \frac{1}{c \cdot n}, \quad \lambda = \frac{1}{4}\kappa, \quad r = \frac{\pi}{2\sqrt{\kappa}}, \quad \vartheta(t) = \frac{1}{(\sqrt{\kappa}\,t)^{n-2}} \sin^{n-2}(\sqrt{\kappa}\,t)\,\cos(\sqrt{\kappa}\,t)$$

hold. Finally, by the relation (12) we obtain the recursive formula

$$vol(\mathcal{A}I(n)_\kappa) = vol(\mathcal{A}I(n-1)_\kappa) \cdot vol(S^{n-2}[1]) \cdot \frac{\sqrt{2n}}{(n-1)^{\frac{3}{2}}}\,\pi\,\kappa^{-\frac{n}{2}} \qquad (n \geq 3),$$

where $vol(\mathcal{A}I(2)_\kappa) = 4\pi\kappa^{-1}$ is true because of $\mathcal{A}I(2) = S^2$.

5.2. Compact symmetric spaces $SU(2n)|Sp(n)$ $(n \geq 3)$ of type AII

In this case the equalities $G = SU(2n)$, $\sigma(X) = J_n \bar{X} J_n^T$ for $X \in G$ and $K = Sp(n) = SU(2n) \cap Sp(n, \mathbf{C})$ are valid. For brevity, $\mathcal{A}II(n)$ will denote the symmetric space $SU(2n)|Sp(n)$. Consider the involution ρ defined by $\rho(X) = F_{n-1,1} X F_{n-1,1}$ for $X \in G$, which satisfies the condition $\sigma \circ \rho = \rho \circ \sigma$. Therefore we get the symmetric subgroups $L \approx S(U_{2n-2} \times U_2)$, $H \approx Sp(n)$ and $L \cap K \approx Sp(n-1) \times Sp(1)$.

For describing the orbit $L(o)$ we need the vector

$$Z = i(E_1 + \cdots + E_{n-1} - (n-1)E_n) + i(E_{n+1} + \cdots + E_{2n-1} - (n-1)E_{2n})$$

and the subspace $\mathfrak{a} = \{\, X \in \mathfrak{l} \cap \mathfrak{p} \mid B(X, Z) = 0 \,\}$ which is a Lie triple system in $\mathfrak{p}$. Then the totally geodesic submanifold $Exp_o(\mathfrak{a})$ is isometric with $\mathcal{A}II(n-1)_\kappa$.
As in the previous case, we obtain that $L(o) = M$ is isometric with $(\mathcal{A}II(n-1)_\kappa \times S^1)|\mathbf{Z}_{n-1}$, where $S^1 = Exp_o(\mathbf{R}Z)$. The other totally geodesic orbit $H(o)$ coincides with the $4(n-1)$-dimensional quaternion projective space QP_λ^{n-1}.

The eigenvalues of the self-adjoint operator R_u in T_oM are $a_1 = \kappa$ $(m_1 = 1)$, $a_2 = \frac{1}{4}\kappa$ $(m_2 = 4n-8)$ and $a_3 = 0$. Considering the relations (6) and (10), by straightforward calculation we get

$$\kappa = \frac{1}{c \cdot 4n}, \quad \lambda = \kappa, \quad r = \frac{\pi}{2\sqrt{\kappa}}, \quad \vartheta(t) = \frac{1}{(\sqrt{\kappa}\,t)^{4n-5}} \sin^{4n-5}(\sqrt{\kappa}\,t)\,\cos(\sqrt{\kappa}\,t).$$

Hence, (12) and the equality $vol(S^{4n-5}[1]) = \frac{2\,\pi^{2n-2}}{(2n-3)!}$ imply the formula

$$vol(\mathcal{A}II(n)_\kappa) = vol(\mathcal{A}II(n-1)_\kappa) \cdot \frac{\sqrt{2n}}{\sqrt{n-1}}\,\frac{\pi^{2n-1}}{(2n-2)!}\,\kappa^{\frac{3}{2}-2n} \qquad (n \geq 3).$$

Remark that $vol(\mathcal{A}II(2)_\kappa) = \pi^3\kappa^{-\frac{5}{2}}$ holds because $\mathcal{A}II(2) = S^5$ is true.

5.3. Compact symmetric spaces $SU(p+q)|S(U_p\times U_q)$ $(p\geq q\geq 2)$ of type $AIII$

We need the Lie group $G=SU(n)$ with $n=p+q$, the involution defined by $\sigma(X)=I_{p,q}XI_{p,q}$ for $X\in G$ and the symmetric subgroup $K=S(U_p\times U_q)$. The complex Grassmannian manifold $SU(p+q)|S(U_p\times U_q)$ will be denoted by $\mathcal{G}^C(p,q)$. Let us consider the involution $\rho: G\to G$, where $\rho(X)=I_{n-1,1}XI_{n-1,1}$. Hence, we get the symmetric subgroups $L=S(U_{n-1}\times U_1)$ and $H\approx S(U_{p+1}\times U_{q-1})$. It can be seen that $L(o)=M$ is isometric with $\mathcal{G}^C(p,q-1)_\kappa$, and $H(o)$ presents the $2p$-dimensional complex projective space CP^p_λ.

The eigenvalues of the restricted endomorphism $R_u|T_oM$ are $a_1=\frac{1}{4}\kappa$ $(m_1=2q-2)$ and $a_2=0$. Moreover, we can verify the equalities

$$\kappa=\frac{1}{c\,(p+q)},\quad \lambda=\kappa,\quad r=\frac{\pi}{\sqrt{\kappa}},\quad \vartheta(t)=\frac{2^{2p-1}}{(\sqrt{\kappa}\,t)^{2p-1}}\sin^{2p-1}(\frac{1}{2}\sqrt{\kappa}\,t)\cos^{2q-1}(\frac{1}{2}\sqrt{\kappa}\,t).$$

5.4. Compact symmetric spaces $SO(p+q)|SO(p)\times SO(q)$ $(p\geq q\geq 2)$ of type BDI

Let us consider the Lie group $G=SO(n)$ with $n=p+q$, the involution defined by $\sigma(X)=I_{p,q}XI_{p,q}$ for $X\in SO(n)$ and the identity component $K=SO(p)\times SO(q)$ of the subgroup G_σ. The oriented real Grassmannian manifold $SO(p+q)|SO(p)\times SO(q)$ will be denoted by $\mathcal{G}^R(p,q)$. Take the involution $\rho: G\to G$, where $\rho(X)=I_{n-1,1}XI_{n-1,1}$ is valid. Then the identity components of the symmetric subgroups G_ρ and G_τ are $L=SO(n-1)$ and $H\approx SO(p+1)\times SO(q-1)$, respectively. It can be seen that $L(o)=M$ is isometric with $\mathcal{G}^R(p,q-1)_\kappa$ provided that $q\geq 3$ holds, and $L(o)=M$ coincides with the sphere S^p of constant curvature $\frac{\kappa}{2}$ if $q=2$ is valid. $H(o)$ always gives the p-dimensional sphere S^p_λ.

The eigenvalues of the restricted endomorphism $R_u|T_oM$ are $a_1=\frac{1}{2}\kappa$ $(m_1=q-1)$ and $a_2=0$. By means of the relations (6) and (10) it can be shown that the equalities

$$\kappa=\frac{1}{c\,(p+q-2)},\quad \lambda=\frac{1}{2}\kappa,\quad r=\frac{\pi}{2\sqrt{\lambda}},\quad \vartheta(t)=\frac{1}{(\sqrt{\lambda}\,t)^{p-1}}\sin^{p-1}(\sqrt{\lambda}\,t)\cos^{q-1}(\sqrt{\lambda}\,t)$$

are valid.

5.5. Compact symmetric spaces $SO(2n)|U(n)$ $(n\geq 3)$ of type $DIII$

In this case we have $G=SO(2n)$, $\sigma(X)=J_nXJ_n^T$ for $X\in G$ and $K=G_\sigma\approx U(n)$. For simplicity, the symmetric space $G|K$ will be denoted by $\mathcal{D}III(n)$. Let us take the involution $\rho: G\to G$ defined by $\rho(X)=F_{n-1,1}XF_{n-1,1}$, which commutes σ. Then we obtain the symmetric subgroups $G_\rho\approx S(O_{2n-2}\times O_2)$ and $H=G_\tau\approx U(n)$. Consider the identity component $L\approx SO(2n-2)\times SO(2)$ of G_ρ and its isometric action on $\mathcal{D}III(n)$. It can be seen that $L(o)=M$ is isometric with $\mathcal{D}III(n-1)_\kappa$. Moreover, $H(o)$ coincides with the complex projective space CP^{n-1}_λ.

The eigenvalues of the self-adjoint operator R_u in T_oM are $a_1=\frac{1}{4}\kappa$ $(m_1=2n-4)$ and $a_2=0$. Furthermore, we can verify the relations

$$\kappa=\frac{1}{c\,(2n-2)},\quad \lambda=\kappa,\quad r=\frac{\pi}{\sqrt{\kappa}},\quad \vartheta(t)=\frac{1}{(\sqrt{\kappa}\,t)^{2n-3}}\sin^{2n-3}(\sqrt{\kappa}\,t).$$

If we calculate the volume of $\mathcal{D}III(n)$ $(n \geq 3)$ by using the equality (12), then $\mathcal{D}III(2) = S^2$ and $vol(\mathcal{D}III(2)_\kappa) = 4\pi\kappa^{-1}$ are needed.

5.6. Compact symmetric spaces $Sp(n)|U(n)$ $(n \geq 2)$ of type CI

In this case the equalities $G = Sp(n)$, $\sigma(X) = \bar{X}$ for $X \in G$, $K = G_\sigma \approx U(n)$ are valid. Henceforth, the symmetric space $G|K$ will be denoted by $\mathcal{C}I(n)$. Consider the involutive automorphism $\rho : G \to G$ defined by $\rho(X) = F_{n-1,1} X F_{n-1,1}$. Then we obtain the symmetric subgroups $L = G_\rho = Sp(n-1) \times Sp(1)$, $H = G_\tau \approx U(n)$ and $L \cap K \approx U(n-1) \times U(1)$. The totally geodesic orbit $M = L(o)$ is isometric with the product $\mathcal{C}I(n-1)_\kappa \times S^2_\kappa$, and $H(o)$ presents the complex projective space CP^{n-1}_λ.

Using the equality (2), we can show that the eigenvalues of the self-adjoint operator $R_u|T_oM$ are $a_1 = \frac{1}{2}\kappa$ $(m_1 = 2)$, $a_2 = \frac{1}{8}\kappa$ $(m_2 = 2n-4)$ and $a_3 = 0$. By virtue of the relations (6) and (10) we obtain

$$\kappa = \frac{1}{c\,(n+1)}, \quad \lambda = \frac{1}{2}\kappa, \quad r = \frac{\pi}{2\sqrt{\lambda}}, \quad \vartheta(t) = \frac{1}{(\sqrt{\lambda}\,t)^{2n-3}} \sin^{2n-3}(\sqrt{\lambda}\,t)\cos^2(\sqrt{\lambda}\,t).$$

Concerning the formula (12) on the volume of $\mathcal{C}I(n)_\kappa$ $(n \geq 2)$, observe that $\mathcal{C}I(1) = S^2$ is valid.

5.7. Compact symmetric spaces $Sp(p+q)|Sp(p) \times Sp(q)$ $(p \geq q \geq 2)$ of type CII

Let us consider the Lie group $G = Sp(n)$ with $n = p+q$, the involution $\sigma(X) = F_{p,q} X F_{p,q}$ for $X \in Sp(n)$ and the symmetric subgroup $K = S(p) \times Sp(q)$. The quaternion Grassmannian manifold $Sp(p+q)|Sp(p) \times Sp(q)$ will be denoted by $\mathcal{G}^Q(p,q)$. It is reasonable to take the involution $\rho : G \to G$, where $\rho(X) = F_{n-1,1} X F_{n-1,1}$ is valid. Hence, we get the symmetric subgroups $L = Sp(n-1) \times Sp(1)$ and $H \approx Sp(p+1) \times Sp(q-1)$. It can be seen that $L(o) = M$ is isometric with $\mathcal{G}^Q(p,q-1)_\kappa$ provided that $q \geq 3$ holds, and $L(o) = M$ coincides with QP^p having the maximal curvature $\frac{\kappa}{2}$ if $q = 2$ is valid. Moreover, $H(o)$ gives the quaternion projective space QP^p_λ.

The eigenvalues of the restricted endomorphism $R_u|T_oM$ are $a_1 = \frac{1}{8}\kappa$ $(m_1 = 4q-4)$ and $a_2 = 0$. Furthermore, we can verify the equalities below

$$\kappa = \frac{1}{c\,(n+1)}, \quad \lambda = \frac{1}{2}\kappa, \quad r = \frac{\pi}{\sqrt{\lambda}}, \quad \vartheta(t) = \frac{2^{4p-1}}{(\sqrt{\lambda}\,t)^{4p-1}} \sin^{4p-1}(\frac{1}{2}\sqrt{\lambda}\,t)\cos^{4q-1}(\frac{1}{2}\sqrt{\lambda}\,t).$$

As in the other cases, by means of the relation (12) the volume of $\mathcal{G}^Q(p,q)_\kappa$ can be expressed from the value $vol(\mathcal{G}^Q(p,q-1)_\kappa)$.

Finally, some results of Section 5 are summarized in the following table.

N	L	$M = L(o)$	$H(o)$	$\frac{\kappa}{\lambda}$	r
$\mathcal{A}I(n)$	$S(U_{n-1} \times U_1)$	$(\mathcal{A}I(n-1) \times S^1) \vert \mathbf{Z}_{n-1}$	RP^{n-1}	4	$\frac{\pi}{4\sqrt{\lambda}}$
$\mathcal{A}II(n)$	$S(U_{2n-2} \times U_2)$	$(\mathcal{A}II(n-1) \times S^1) \vert \mathbf{Z}_{n-1}$	QP^{n-1}	1	$\frac{\pi}{2\sqrt{\lambda}}$
$\mathcal{G}^C(p,q)$	$S(U_{n-1} \times U_1)$	$\mathcal{G}^C(p,q-1)$	CP^p	1	$\frac{\pi}{\sqrt{\lambda}}$
$\mathcal{G}^R(p,q)$	$SO(n-1)$	$\mathcal{G}^R(p,q-1)$	S^p	2	$\frac{\pi}{2\sqrt{\lambda}}$
$\mathcal{D}III(n)$	$SO(2n-2) \times SO(2)$	$\mathcal{D}III(n-1)$	CP^{n-1}	1	$\frac{\pi}{\sqrt{\lambda}}$
$\mathcal{C}I(n)$	$Sp(n-1) \times Sp(1)$	$\mathcal{C}I(n-1) \times S^2$	CP^{n-1}	2	$\frac{\pi}{2\sqrt{\lambda}}$
$\mathcal{G}^Q(p,q)$	$Sp(n-1) \times Sp(1)$	$\mathcal{G}^Q(p,q-1)$	QP^p	2	$\frac{\pi}{\sqrt{\lambda}}$

Table 1.

6. Examples for special cohomogeneity one isometric actions on irreducible symmetric spaces of type II

Let us consider a connected compact Lie group G with semisimple Lie algebra $\mathfrak{g}$. Regarding the product group $\hat{G} = G \times G$, we can take the smooth action $\hat{\alpha} : \hat{G} \times G \to G$ which is defined by $\hat{\alpha}((g_1, g_2), h) = g_1 h (g_2)^{-1}$ for $g_1,\ g_2,\ h \in G$. Endow G with a biinvariant Riemannian metric $\langle\ ,\ \rangle$ which is derived from the inner product

$$\langle\ ,\ \rangle_e = -c \cdot B \qquad (c \in \mathbf{R},\ c > 0)$$

on the tangent space $T_e G = \mathfrak{g}$ at the identity element e. Then G turns into a symmetric space of compact type, and the mappings $Exp_e,\ exp$ defined on $\mathfrak{g}$ coincide.

On the other hand, we can take the canonical involution $\hat{\sigma} : \hat{G} \to \hat{G}$ defined by $\hat{\sigma}(g_1, g_2) = (g_2, g_1)$ for $g_1,\ g_2 \in G$. Then the subgroup of the fixed elements coincides with $\triangle G = \{\ (g,g) \mid g \in G\ \}$, and $(\hat{G}, \triangle G)$ presents a special Riemannian symmetric pair. As it is well-known, the coset space $\hat{G} | \triangle G$ can naturally be identified with G.

For simplicity, assume that G is simply connected. Let us consider an involutive automorphism ρ of G and the connected compact subgroup $L = G_\rho$. Then we can take the symmetric subgroup $\hat{L} = L \times L$ of $\hat{G}$ and the inherited isometric action $\hat{\alpha} : \hat{L} \times G \to G$. Using the formalism of Section 2, in this case we have $N = G, \quad o = e, \quad M = \hat{L}(e) = L$ and $\nu_e M = \mathfrak{n}$. It can be seen that the totally geodesic submanifold $exp(\mathfrak{n})$ coincides with the orbit $\hat{H}(e)$ of the other symmetric subgroup $\hat{H} = \{\ (g, \rho(g)) \mid g \in G\ \}$. Obviously, the maximal dimensional flat totally geodesic submanifolds of $\hat{H}(e)$ which pass through e are sections of $\hat{\alpha}$.

Recall that the irreducible symmetric spaces of type II are the compact Lie groups with simple Lie algebras. In this paper we consider only the classical matrix groups $SU(n)$ $(n \geq 3)$, $Sp(n)$ $(n \geq 2)$ and $SO(n)$ $(n \geq 5)$. Although the Lie group $SO(n)$ is not simply connected, we can apply the method described in Section 3 to $SO(n)$, too. Using the relevant cohomogeneity one actions, it can be seen that the symmetric spaces $SU(n)$, $Sp(n)$ and $SO(n)$ are compact tubes around the "totally geodesic orbits" $S(U_{n-1} \times U_1)$, $Sp(n-1) \times Sp(1)$ and $SO(n-1)$, respectively.

Some results relating to these special symmetric spaces are summarized in the following table, where r denotes the radius of the tube, furthermore, κ and λ denote the maximal sectional curvatures of G and $\hat{H}(e)$, respectively.

G	$\frac{1}{c\cdot\kappa}$	$L=\hat{L}(e)$	$\hat{H}(e)$	$\frac{\kappa}{\lambda}$	r
$SU(n)$	$4n$	$S(U_{n-1}\times U_1)$	CP^{n-1}	1	$\frac{\pi}{2\sqrt{\lambda}}$
$Sp(n)$	$4(n+1)$	$Sp(n-1)\times Sp(1)$	QP^{n-1}	2	$\frac{\pi}{2\sqrt{\lambda}}$
$SO(n)$	$4(n-2)$	$SO(n-1)$	RP^{n-1}	2	$\frac{\pi}{2\sqrt{\lambda}}$

Table 2.

Concerning the functions $\vartheta:(0,r)\to\mathbf{R}$ which present the volumes of the principal orbits by the equality (11), we obtain

$$\vartheta(t)=\frac{1}{(\sqrt{\lambda}\,t)^{2n-3}}\sin^{2n-3}(\sqrt{\lambda}\,t)\,\cos(\sqrt{\lambda}\,t)\qquad\text{if}\quad G=SU(n),$$
$$\vartheta(t)=\frac{1}{(\sqrt{\lambda}\,t)^{4n-5}}\sin^{4n-5}(\sqrt{\lambda}\,t)\cos^{3}(\sqrt{\lambda}\,t)\qquad\text{if}\quad G=Sp(n),$$
$$\vartheta(t)=\frac{1}{(2\sqrt{\lambda}\,t)^{n-2}}\sin^{n-2}(2\sqrt{\lambda}\,t)\qquad\text{if}\quad G=SO(n).$$

Hence, among others we can verify the following recursive formulae

$$vol(SU(n)_\kappa)=vol(SU(n-1)_\kappa)\cdot\frac{\sqrt{2n}}{(n-1)^{\frac{3}{2}}\,(n-2)!}\,\pi^n\,\kappa^{\frac{1}{2}-n}\qquad(n\geq 3),$$
$$vol(Sp(n)_\kappa)=vol(Sp(n-1)_\kappa)\cdot\frac{2^{2n-1}}{(2n-1)!}\,\pi^{2n}\,\kappa^{\frac{1}{2}-2n}\qquad(n\geq 2).$$

Regarding the above relations, observe that $SU(2)=Sp(1)=S^3$ and $vol(S^3_\kappa)=2\pi^2\kappa^{-\frac{3}{2}}$ are valid.

References

[1] Abe, K.; Yokota, I.: *Volumes of compact symmetric spaces.* Tokyo J. Math. **20** (1997), 87–105. Zbl 0884.53040

[2] Besse, A. L.: *Manifolds all of whose geodesics are closed.* Springer-Verlag, Berlin 1978. Zbl 0387.53010

[3] Bishop, R. L.; Crittenden, R. J.: *Geometry of manifolds.* Academic Press, New York 1964. Zbl 0132.16003

[4] Bott, R.; Samelson, H.: *Applications of the theory of Morse to symmetric spaces.* American J. Math. **80** (1958), 964–1029. Zbl 0101.39702

[5] Bredon, G. E.: *Introduction to compact transformation groups.* Academic Press, New York 1972. Zbl 0246.57017

[6] Broughton, S. A.: *Volumes of subgroups of compact Lie groups.* Algebras Groups Geom. **4** (1987), 325–364. Zbl 0641.22001

[7] Chen, B. Y.; Nagano, T.: *Totally geodesic submanifolds of symmetric spaces II.* Duke Math. J. **45** (1978), 405–425. Zbl 0384.53024

[8] Conlon, L.: *Variational completeness and K–transversal domains.* J. Diff. Geom. **5** (1971), 135–147. Zbl 0213.48602

[9] Conlon, L.: *A class of variationally complete representations.* J. Diff. Geom. **7** (1972), 149–160. Zbl 0276.53038

[10] Dadok, J.: *Polar coordinates induced by actions of compact Lie groups.* Trans. Amer. Math. Soc. **288** (1985), 125–137. Zbl 0565.22010

[11] Gray, A.: *Tubes.* Addison-Wesley, Redwood City 1990. Zbl 0692.53001

[12] Gray, A.; Vanhecke, L.: *The volumes of tubes in a Riemannian manifold.* Rend. Sem. Mat. Univ. Politec. Torino **39** (1981), 1–50. Zbl 0511.53059

[13] Heintze, E.; Palais R. S.; Terng, C. L.; Thorbergsson, G.: *Hyperpolar actions on symmetric spaces.* Geometry, topology and physics, Conf. Proc. Lecture Notes Geom. Topology VI, edited by S. T. Yau, Internat. Press, Cambridge 1995, 214–245. Zbl 0871.57035

[14] Helgason, S.: *Differential geometry, Lie groups and symmetric spaces.* Academic Press, New York 1978. Zbl 0451.53038

[15] Hermann, R.: *Variational completeness for compact symmetric spaces.* Proc. Amer. Math. Soc. **11** (1960), 544–546. Zbl 0098.36603

[16] Hermann, R.: *Totally geodesic orbits of groups of isometries.* Nederl. Akad. Wetensch. **65** (1962), 291–298. Zbl 0124.14502

[17] Jänich, K.: *Differenzierbare G-Mannigfaltigkeiten.* Springer-Verlag, Berlin 1968. Zbl 0159.53701

[18] Kobayashi, S.; Nomizu, K.: *Foundations of differential geometry II.* Interscience Publishers, New York 1969. Zbl 0175.48504

[19] Kollross, A.: *A classification of hyperpolar and cohomogeneity one actions.* Ph.D. thesis, Augsburg 1998.

[20] Leung, D. S. P.: *On the classification of reflective submanifolds of Riemannian symmetric spaces.* Indiana Univ. Math. J. **24** (1974), 327–339. Zbl 0296.53039

[21] Naveira, A. M.; Gual, X.: *The volume of geodesic balls and tubes about totally geodesic submanifolds in compact symmetric spaces.* Diff. Geom. Appl. **7** (1997), 101–113. Zbl 0876.53033

[22] Szenthe, J.: *Orthogonally transversal submanifolds and the generalizations of the Weyl group.* Period. Math. Hung. **15** (1984), 281–299. Zbl 0583.53035

[23] Szenthe, J.: *Some isometric actions with orthogonally transversal submanifolds on Riemannian symmetric spaces.* Studia Scient. Math. Hung. **21** (1986), 175–179. Zbl 0621.53033

[24] Vanhecke, L.: *Geometry in normal and tubular neighborhoods.* Rend. Sem. Fac. Sci. Univ. Cagliari, Suppl. al vol. **58** (1988), 73–176.

[25] Verhóczki, L.: *Principal curvatures of special hypersurfaces in symmetric spaces.* Acta Scient. Math. (Szeged) **58** (1993), 349–361. Zbl 0795.53052

[26] Verhóczki, L.: *On orbits of symmetric subgroups in Riemannian symmetric spaces.* New Developments in Differential Geometry, Proc. Conf. Budapest 1996. Kluwer Academic Publishers, Dordrecht 1999, 485–501. Zbl 0940.53029

Received March 13, 2001