Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 44 (2003), No. 1, 189-201.

Decomposing Four-Manifolds up to Homotopy Type

Alberto Cavicchioli **Beatrice Ruini** **Fulvia Spaggiari** *

Dipartimento di Matematica, Università di Modena e Reggio Emilia
Via Campi 213/B, 41100 Modena, Italy
e-mail: cavicchioli.alberto@unimo.it

Abstract. Let M be a closed connected oriented topological 4-manifold with fundamental group π_1. Let Λ be the integral group ring of π_1. Suppose that $f : M \to P$ is a degree one map inducing an isomorphism on π_1. We give a homological condition on the intersection forms $\lambda_M^{\mathbb{Z}}$ and λ_M^{Λ} under which M is homotopy equivalent to a connected sum $P\#M'$ for some simply-connected closed (non-trivial) topological 4-manifold M'. This gives a partial solution to a conjecture of Hillman [16] on the classification of closed 4-manifolds with vanishing second homotopy group. Then some splitting results for closed 4-manifolds with special homotopy complete the paper.

MSC 2000: 57N65, 57R67, 57Q10
Keywords: four-manifolds, homotopy type, connected sum, obstruction theory, intersection forms, homology with local coefficients, degree one map

1. Introduction

Let M^4 be a closed connected orientable topological 4-manifold with fundamental group $\pi_1 = \pi_1(M)$. We assume that all manifolds have a CW-structure with one 4-cell and one 0-cell, hence the 3-skeleton $M^{(3)}$ of M is obtained from M by removing an open 4-cell.

*Work performed under the auspices of the G.N.S.A.G.A. of the C.N.R. (National Research Council) of Italy, partially supported by the Ministero per la Ricerca Scientifica e Tecnologica of Italy within the project "Proprietà Geometriche delle Varietà Reali e Complesse", and by a research grant of the University of Modena and Reggio Emilia.

Let $\Lambda = \mathbb{Z}[\pi]$ be the integral group ring of π. If $[M] \in H_4(M;\mathbb{Z}) \cong \mathbb{Z}$ is the fundamental class of the orientation, then there are Poincaré duality isomorphisms

$$\cap[M] : H^q(M;\Lambda) \to H_{4-q}(M;\Lambda)$$

where $H_*(M;\Lambda) \cong H_*(C_*(\widetilde{M}) \otimes_\Lambda \Lambda) \cong H_*(\widetilde{M})$ is the integral homology of the universal covering $\widetilde{M}$ of M.

The first goal of the paper is to study when M is homotopy equivalent to a connected sum $P\#M'$, where M' is a simply-connected non-trivial closed 4-manifold (by *non-trivial* we mean that M' is not homeomorphic to the standard 4-sphere). It would be interesting to do this when the manifold P is minimal in some sense. Section 4 will be devoted to state precisely the concept of minimality on P, and to find such minimal models for many classes of closed topological 4-manifolds with special homotopy. A result along this line is contained implicitly in Theorem 10.3 of [10] (corrected version as given in [23] and [24]), and a sketch of the relevant consequence of that theorem will be presented in Section 3. However, we shall prove our result in an independent way by using only standard techniques from obstruction theory (with local coefficients).

Suppose there exists a degree one map

$$f : M \to P$$

between closed topological 4-manifolds which induces an isomorphism on π, i.e. $f_* : \pi_1(M) \cong \pi_1(P)$.

Then we have split exact sequences

$$0 \longrightarrow K_q(f,\Lambda) \longrightarrow H_q(M;\Lambda) \xrightarrow{f_*^\Lambda} H_q(P;\Lambda) \longrightarrow 0$$

where $K_q(f,\Lambda)$ denotes the kernel of f_*^Λ. The same holds for other (local) coefficients; in particular, we have split exact sequences

$$0 \longrightarrow K_q(f,\mathbb{Z}) \longrightarrow H_q(M;\mathbb{Z}) \xrightarrow{f_*^\mathbb{Z}} H_q(P;\mathbb{Z}) \longrightarrow 0.$$

Moreover, for $q = 2$, these sequences give splittings of the intersection forms (over Λ or $\mathbb{Z}$, respectively), and $K_2(f,\Lambda)$ is stably Λ-free (and s-based), i.e. $K_2(f,\Lambda) \oplus \Lambda^r \cong \Lambda^s$ for some non-negative integers r and s (see [26]). Moreover, we have the isomorphism of Λ-modules

$$K_2(f,\Lambda) \cong K_2(f,\mathbb{Z}) \otimes_\mathbb{Z} \Lambda.$$

By the celebrated work of Freedman [8] (see also [9] and [10]) there is a simply-connected closed topological 4-manifold M' realizing the restriction of the integral intersection form $\lambda_M^\mathbb{Z}$ on $K_2(f,\mathbb{Z})$, i.e.

$$\lambda_M^\mathbb{Z}|_{K_2(f,\mathbb{Z})} \cong \lambda_{M'}^\mathbb{Z}.$$

Up to homeomorphism, there are at most two such M'; exactly two in the case of an odd intersection form, and they are M' and the unique non-smoothable 4-manifold homotopy equivalent to M'.

Our first result is the following

Theorem 1. *With the above notation, let $f\colon M \to P$ be a degree one map which induces an isomorphism on the fundamental group $\pi = \pi_1$. Then M is homotopy equivalent to a connected sum $P \# M'$ if and only if*

$$\lambda_M^{\Lambda}|_{K_2(f,\Lambda)} \cong \lambda_M^{\mathbb{Z}}|_{K_2(f,\mathbb{Z})} \otimes_{\mathbb{Z}} \Lambda.$$

The following conjecture was stated by Hillman in [16].

Conjecture. Suppose that $\pi = \pi_1(M)$ is torsion free and infinite, and $\pi_2(M) \cong 0$. Then M is topologically homeomorphic to a connected sum of aspherical closed 4-manifolds and factors $\mathbb{S}^1 \times \mathbb{S}^3$.

Using Theorem 1 we give a partial solution of the conjecture, up to homotopy type.

Theorem 2. *Let M^4 be a closed connected orientable topological 4-manifold such that π_1 is torsion free and infinite, and $\pi_2(M) \cong 0$. Then M is homotopy equivalent to a connected sum of aspherical closed 4-manifolds with factors $\mathbb{S}^1 \times \mathbb{S}^3$.*

As a consequence of Theorem 1, we also obtain simple alternative proofs of well-known algebraic characterizations of $\mathbb{S}^1 \times \mathbb{S}^3$, $\mathbb{R}^4/\mathbb{Z}^4 \cong \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, and an $\mathbb{S}^2$-bundle over the torus among closed orientable 4-manifolds.

Theorem 3. *A closed connected orientable 4-manifold M is topologically homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^3$ (resp. $\mathbb{R}^4/\mathbb{Z}^4$, and an $\mathbb{S}^2$-bundle over the torus) if and only if the Euler characteristic of M vanishes and $\pi_1(M) \cong \mathbb{Z}$ (resp. $\mathbb{Z}^4$, and $\mathbb{Z}^2$).*

For other results on connected sum decomposition of 4-manifolds we refer to [11], [16], [17], [20], and [23]–[25]. A splitting theorem for homotopy equivalent smooth 4-manifolds can be found in [6].

2. Proof of Theorem 1

Let $\pi = \pi_1(M)$ be the fundamental group of M. Suppose there exists a degree one map $f : M \to P$ inducing an isomorphism on π. We are going to construct a map

$$\alpha : P^{(3)} = P \backslash \overset{\circ}{D}{}^4 \to M$$

such that the composition

$$P^{(3)} \xrightarrow{\alpha} M \xrightarrow{f} P$$

is homotopic to the inclusion $P^{(3)} \subset P$. Since $\pi \cong \pi_1(P)$, there is a map from the wedge $\vee \mathbb{S}^1 \simeq P^{(1)}$ to M such that the composition with $f : M \to P$ is the canonical inclusion. There is an obstruction map

$$H_2(\widetilde{P}^{(2)}, \widetilde{P}^{(1)}) \to \pi = \pi_1(M)$$

for extending over the 2-skeleton of P. Composing this map with

$$f_* : \pi_1(M) \underset{\cong}{\to} \pi_1(P)$$

yields that we can extend it over the 2-skeleton of P, i.e. there is a map $P^{(2)} \to M$ inducing an isomorphism on π. We extend it to $P^{(3)}$ by using obstruction theory with local coefficients. The obstruction is an element $\theta \in H^3(P; \pi_2(M))$.

Since there are isomorphisms

$$\pi_2(M) \cong \pi_2(\widetilde{M}) \cong H_2(\widetilde{M}) \cong H_2(M;\Lambda) \cong H_2(P;\Lambda) \oplus K_2(f,\Lambda)$$

and $K_2(f,\Lambda)$ is stably Λ-free, we have

$$\begin{aligned} H^3(P;\pi_2(M)) &\cong H^3(P; H_2(M;\Lambda)) \cong H_1(P; H_2(M;\Lambda)) \\ &\cong \mathrm{Tor}_1^{\Lambda}(\mathbb{Z}, H_2(M;\Lambda)) \cong \mathrm{Tor}_1^{\Lambda}(\mathbb{Z}, H_2(P;\Lambda) \oplus K_2(f,\Lambda)) \\ &\cong \mathrm{Tor}_1^{\Lambda}(\mathbb{Z}, H_2(P;\Lambda)) \cong H_1(P; H_2(P;\Lambda)) \cong H^3(P;\pi_2(P)). \end{aligned}$$

So the obstruction $\theta \in H^3(P;\pi_2(M))$ corresponds bijectively to the obstruction

$$\overline{\theta} \in H^3(P;\pi_2(P))$$

for extending the composite map

$$P^{(2)} \xrightarrow{\alpha} M \xrightarrow{f} P$$

to a map $P^{(3)} \to P$. But $f \circ \alpha$ is homotopic to the inclusion $P^{(2)} \subset P$ which extends to $P^{(3)}$. Then we have $\theta = 0$, and there is a map

$$\alpha : P^{(3)} \to M$$

such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & P \\ {\scriptstyle\alpha}\uparrow & & \| \\ P^{(3)} & \xrightarrow[i]{} & P \end{array}$$

commutes, up to homotopy.

In fact, the obstructions for homotopy are in

$$H^2(P;\pi_2(M)) \cong H_0(P;\pi_2(M)) \cong 0$$

and

$$H^3(P;\pi_3(M)) \cong H_1(P;\pi_3(M)).$$

Looking at the diagram

$$\begin{array}{ccc} \pi_3(M) & \xrightarrow{f_*} & \pi_3(P) \\ {\scriptstyle\text{epi}}\downarrow & & \downarrow{\scriptstyle\text{epi}} \\ H_3(M;\Lambda) & \xrightarrow[\text{epi}]{f_*^{\Lambda}} & H_3(P;\Lambda) \end{array}$$

it follows that f_* is onto. So it is possible to construct an extension

$$\alpha : P^{(3)} \to M$$

such that $f \circ \alpha$ is homotopic to the inclusion $P^{(3)} \subset P$.

Now we construct a map

$$\beta : M' \backslash \overset{\circ}{D}{}^4 = (M')^{(3)} \to M.$$

Recall the split exact sequence

$$0 \longrightarrow K_2(f, \mathbb{Z}) \longrightarrow H_2(M; \mathbb{Z}) \xrightarrow{f_*^{\mathbb{Z}}} H_2(P; \mathbb{Z}) \longrightarrow 0.$$

We have a splitting of the integral intersection forms

$$\lambda_M^{\mathbb{Z}} \cong \lambda_P^{\mathbb{Z}} \oplus \lambda_M^{\mathbb{Z}}|_{K_2(f,\mathbb{Z})}.$$

By Freedman's theorem, there is a simply-connected closed topological 4-manifold M' such that

$$\lambda_{M'}^{\mathbb{Z}} \cong \lambda_M^{\mathbb{Z}}|_{K_2(f,\mathbb{Z})}.$$

Of course, $M' \backslash \overset{\circ}{D}{}^4 = (M')^{(3)}$ is homotopy equivalent to a wedge $\vee_r \mathbb{S}^2$, and

$$H_2(M'; \mathbb{Z}) \cong K_2(f, \mathbb{Z}) \cong \oplus_r \mathbb{Z}$$

is $\mathbb{Z}$-free. Since

$$H_2(M'; \mathbb{Z}) \cong \oplus_r \mathbb{Z} \subset H_2(M; \mathbb{Z}) \cong H_2(\widetilde{M}) \otimes_\Lambda \mathbb{Z} \cong \pi_2(M) \otimes_\Lambda \mathbb{Z},$$

we can represent a set of generators of $H_2(M'; \mathbb{Z})$ by maps of 2-spheres into M (compare also with [7]). Then there exists a map

$$\beta \colon M' \backslash \overset{\circ}{D}{}^4 = (M')^{(3)} \simeq \vee_r \mathbb{S}^2 \to M.$$

Obviously, the induced homomorphism

$$\beta_* : H_2((M')^{(3)}) \to H_2(M)$$

is injective. Thus we have constructed a map

$$\varphi = \alpha \vee \beta : P^{(3)} \vee (M')^{(3)} \to M.$$

Since the wedge $P^{(3)} \vee (M')^{(3)}$ is homotopy equivalent to

$$(P \# M')^{(3)} = (P \# M') \backslash \overset{\circ}{D}{}^4_1,$$

we have a map (also denoted by φ)

$$\varphi : (P \# M')^{(3)} \to M.$$

In general, the map φ can not be extended over $P\#M'$. The obstruction for extending φ to a map from $P\#M'$ to M is a cohomology class

$$\begin{aligned}\xi \in & H^4(P\#M'; \pi_3(M)) \cong H_0(P\#M'; \pi_3(M)) \\ & \cong \mathrm{Tor}_0^{\Lambda}(\mathbb{Z}; \pi_3(M)) \cong \pi_3(M) \otimes_\Lambda \mathbb{Z} \\ & \cong \Gamma(\pi_2(M)) \otimes_\Lambda \mathbb{Z}\end{aligned}$$

where $\Gamma(\pi_2)$ is the quadratic Γ-functor applied to the abelian group $\pi_2(M) \cong H_2(M; \Lambda)$ (see [26] and [27]). Moreover, $\Gamma(\pi_2)$ is a Λ-submodule of $\pi_2 \otimes_\Lambda \pi_2$ (the module of symmetric tensors) and the induced homomorphism

$$\Gamma(\pi_2) \otimes_\Lambda \mathbb{Z} \to \pi_2 \otimes_\Lambda \pi_2$$

is injective. Now ξ is the homotopy class

$$\varphi_*([\partial D_1^4]) = (\alpha \vee \beta)_*([\partial D_1^4]) \in \Gamma(\pi_2(M)) \otimes_\Lambda \mathbb{Z}.$$

This class induces an intersection form λ_M^Λ which is compatible with the splitting of the second homology Λ-module

$$H_2(M; \Lambda) \cong H_2(P; \Lambda) \oplus K_2(f, \Lambda).$$

From the description of the top-dimensional obstruction given in [1] and [12], it follows that ξ corresponds in $\pi_2 \otimes_\Lambda \pi_2$ to the difference of Λ-forms

$$\lambda_M^\Lambda|_{K_2(f,\Lambda)} - (\lambda_M^{\mathbb{Z}}|_{K_2(f,\mathbb{Z})} \otimes_{\mathbb{Z}} \Lambda)$$

which is trivial by hypothesis. Now the injectivity of the homomorphism from $\Gamma(\pi_2) \otimes_\Lambda \mathbb{Z}$ into $\pi_2 \otimes_\Lambda \pi_2$ implies that ξ is trivial as requested. So φ extends to a map from $P\#M'$ to M, also denoted by φ. By construction, the extended map $\varphi : P\#M' \to M$ is of degree one, and induces isomorphisms on π_i for any $i \leq 2$. Hence φ is a homotopy equivalence by the Whitehead theorem. This completes the proof.

3. An alternative proof

We show that Theorem 1 can be obtained as a relevant consequence of Theorem 10.3 of [10] (corrected version as given in [23] and [24]). The goal of Section 10.3 in [10] is to determine when a closed topological 4-manifold M can be expressed as a connected sum $M'\#P$, where M' is a closed simply-connected topological 4-manifold. The hypothesis are given in terms of intersection and self-intersection forms on $\pi_2(M)$. If M is homeomorphic to a connected sum $M'\#P$, then there is an isomorphism

$$\pi_2(M) \cong (\pi_2(M') \otimes \Lambda) \oplus \pi_2(P),$$

where $\Lambda = \mathbb{Z}[\pi_1(M)]$ as usual. Intersection numbers on the first summand are given by

$$\lambda_M^\Lambda(x \otimes a, y \otimes b) = \lambda_{M'}^{\mathbb{Z}}(x, y) a\bar{b},$$

where $^-$ denotes the canonical anti-automorphism of Λ. Similarly, we have

$$\mu_M^\Lambda(x \otimes a) = \mu_{M'}^{\mathbb{Z}}(x)a\bar{a}.$$

Abstracting this, Freedman and Quinn said in [10] that a Λ-homomorphism from $\pi_2(M') \otimes \Lambda$ to $\pi_2(M)$ *preserves* λ *and* μ if the intersection numbers of images are given by the expressions written above. Since λ in $\pi_2(M')$ is non-singular, the homomorphism is an injection onto a direct summand of $\pi_2(M)$.

The following is Theorem 10.3 (part 1) of [10].

Theorem 4. *Let M' be a closed simply-connected topological 4-manifold, and suppose that $\pi_1 = \pi_1(M)$ is good. Let*

$$\pi_2(M') \otimes \Lambda \to \pi_2(M)$$

be a Λ-monomorphism which preserves λ and μ. If either the second Stiefel-Whitney class w_2 is zero on $\pi_2(M)$ or w_2 does not vanish on the subspace of $\pi_2(M)$, perpendicular to the image, then there is a decomposition $M \underset{\mathrm{TOP}}{\cong} M'\#P$ inducing the given decomposition of π_2. If $w_2 \neq 0$ does vanish on the perpendicular subspace, then exactly one of M or $*M$ decomposes.

We recall briefly the definition of $*M$ as given in [10], Section 10.4. Let M be a closed connected topological 4-manifold with good fundamental group. If $w_2 : \pi_2(M) \to \mathbb{Z}_2$ is trivial, then $*M = M$. If w_2 is nontrivial, then $*M$ is the closed topological 4-manifold with a homeomorphism $(*M)\#\mathbb{C}P^2 \cong M\#(*\mathbb{C}P^2)$ which preserves the decompositions of π_2. Here $*\mathbb{C}P^2$ is the unique non-smoothable simply-connected closed 4-manifold with integral intersection form (1) and Kirby-Siebenmann invariant ks $= 1$ (also called the *fake* $\mathbb{C}P^2$). Note that if $*M$ is not homeomorphic to M, then it has the opposite Kirby-Siebenmann invariant. However, there is a canonical homotopy equivalence from $*M$ to M.

Now we sketch how Theorem 1 can be derived from Theorem 4. Applying the sum-stable version of Theorem 4, we see that there is a connected sum decomposition (without any restriction on the fundamental group)

$$M\#r(\mathbb{S}^2 \times \mathbb{S}^2) \underset{\mathrm{TOP}}{\cong} M'\#Q,$$

for some non-negative integer r. Here $\pi_2(M')$ represents $K_2(f, \Lambda)$. Then we have a degree one map

$$f\#\,\mathrm{id} : M\#r(\mathbb{S}^2 \times \mathbb{S}^2) \cong M'\#Q \to P\#r(\mathbb{S}^2 \times \mathbb{S}^2).$$

Attaching 3-cells to kill off $\pi_2(M')$ (use the procedure described in [21]) produces a map

$$g : (M' \cup \{\text{ 3-cells }\})\#Q \to P\#r(\mathbb{S}^2 \times \mathbb{S}^2).$$

However, $M' \cup \{$ 3-cells $\}$ can be chosen to be homotopy equivalent to the standard 4-sphere $\mathbb{S}^4$. Hence we get a map

$$g' : Q \to P\#r(\mathbb{S}^2 \times \mathbb{S}^2).$$

This map induces isomorphisms on π_1 and π_2, and it is of degree one. So g' is a homotopy equivalence by the Whitehead theorem. Therefore, $M\#r(\mathbb{S}^2 \times \mathbb{S}^2)$ is homotopy equivalent to a connected sum $M'\#P\#r(\mathbb{S}^2 \times \mathbb{S}^2)$. This homotopy equivalence preserves the subspace spanned by the last summand. Then attaching $2r$ 3-cells to kill off $\pi_2(r(\mathbb{S}^2 \times \mathbb{S}^2)) \cong \oplus_{2r}\mathbb{Z}$ produces a homotopy equivalence from M to $M'\#P$ as required. This completes the proof.

4. Splitting results

Let M^4 be a closed connected orientable topological 4-manifold. This section is devoted to construct a degree one map $f : M \to P$ where the manifold P is very simple (that is, minimal) and in some sense depends on the fundamental group $\pi_1(M)$. This permits to apply Theorem 1 for many interesting classes of closed topological 4-manifolds. First, we state precisely our concept of minimality on P. Let π be a group, and let $\mathcal{M}(\pi)$ denote the class of closed connected orientable topological 4-manifolds M such that $\pi_1(M) \cong \pi$. A manifold $P \in \mathcal{M}(\pi)$ is said to be *minimal* if for every manifold $M \in \mathcal{M}(\pi)$ there exists a degree one map from M to P. If P is minimal, then it is unique, up to homotopy equivalence. In fact, suppose that there are maps $f : M \to P$ and $g : P \to M$ of degree one which induce isomorphisms on fundamental groups, i.e. $f_* : \pi_1(M) \cong \pi_1(P) \cong \pi$ and $g_* : \pi_1(P) \cong \pi_1(M) \cong \pi$. Then we have the split exact sequences

$$0 \longrightarrow K_2(f, \Lambda) \longrightarrow H_2(M; \Lambda) \xrightarrow{f_*^\Lambda} H_2(P; \Lambda) \longrightarrow 0$$

and

$$0 \longrightarrow K_2(g, \Lambda) \longrightarrow H_2(P; \Lambda) \xrightarrow{g_*^\Lambda} H_2(M; \Lambda) \longrightarrow 0$$

where $\Lambda = \mathbb{Z}[\pi]$, up to obvious identification. Then it follows that $k_2(f, \Lambda) \cong K_2(g, \Lambda) \cong 0$. Thus f is a homotopy equivalence by the Whitehead theorem.

4.1. Free groups. Let M be a closed connected orientable topological 4-manifold with fundamental group $\pi_1(M) \cong *_p\mathbb{Z}$ (the free product of p factors $\mathbb{Z}$). Define P to be the connected sum of p factors $\mathbb{S}^1 \times \mathbb{S}^3$, i.e. $P = p(\mathbb{S}^1 \times \mathbb{S}^3)$. Following [3], we construct a degree one map $f : M \to p(\mathbb{S}^1 \times \mathbb{S}^3)$. Choosing an isomorphism from $\pi = \pi_1(M)$ onto $*_p\mathbb{Z}$ yields a basis $(e_1, \ldots, e_p)$ of $H_1(M; \mathbb{Z})$. Let $(u_1, \ldots, u_p)$ be the dual basis in $H^1(M; \mathbb{Z}) \cong \mathrm{Hom}(H_1(M; \mathbb{Z}), \mathbb{Z})$, and let $(v_1, \ldots, v_p)$ be the Poincaré dual basis in $H^3(M; \mathbb{Z})$ of $(e_1, \ldots, e_p)$, respectively. Then we have $u_i \cup v_j = \delta_{i\,j}\omega_M$, where $\omega_M \in H^4(M; \mathbb{Z})$ is the dual of the fundamental class $[M] \in H_4(M; \mathbb{Z})$. Since $H^1(M; \mathbb{Z}) \cong [M, \mathbb{S}^1]$ and $H^3(M; \mathbb{Z}) \cong [M, K(\mathbb{Z}, 3)]$, we have a map

$$\phi = \prod_{i=1}^{p}(u_i \times v_i) : M \to \prod_{i=1}^{p}(\mathbb{S}^1 \times K(\mathbb{Z}, 3)).$$

Since $K(\mathbb{Z}, 3)$ can be obtained from $\mathbb{S}^3$ by attaching cells of dimension ≥ 5, we can always assume that

$$\phi : M \to \prod_{i=1}^{p}(\mathbb{S}^1 \times \mathbb{S}^3).$$

As shown in [3], this map factorizes, up to homotopy, to a degree one map

$$f : M \to p(\mathbb{S}^1 \times \mathbb{S}^3),$$

i.e. there is a diagram

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, equal] & \prod_{i=1}^{p}(\mathbb{S}^1 \times \mathbb{S}^3) \\ M \arrow[r, "f"'] & p(\mathbb{S}^1 \times \mathbb{S}^3), \arrow[u] \end{tikzcd}$$

which commutes, up to homotopy.

Now we can apply Theorem 1 to obtain the following

Theorem 5. *Let M^4 be a closed connected orientable topological 4-manifold with $\pi_1 \cong *_p\mathbb{Z}$ for some $p > 0$. Let M' be the closed simply-connected 4-manifold obtained from M by killing the fundamental group. Then M is simple homotopy equivalent to the connected sum $M'\#p(\mathbb{S}^1 \times \mathbb{S}^3)$ if and only if*

$$\lambda_M^\Lambda \cong \lambda_M^{\mathbb{Z}} \otimes \Lambda (\cong \lambda_{M'}^{\mathbb{Z}} \otimes \Lambda).$$

Proof. Since π_1 is torsion free, the intersection forms on $H_2(M;\mathbb{Z})$ and $H_2(M';\mathbb{Z})$ are the same, i.e. $H_2(M;\mathbb{Z}) \cong H_2(M';\mathbb{Z})$ and $\lambda_M^{\mathbb{Z}} \cong \lambda_{M'}^{\mathbb{Z}}$ (see [2] for the proof). In particular, M determines uniquely M' by Freedman's classification theorem. The exact sequences

$$0 \longrightarrow K_2(f,\Lambda) \longrightarrow H_2(M;\Lambda) \xrightarrow{f_*^\Lambda} H_2(p(\mathbb{S}^1 \times \mathbb{S}^3);\Lambda) \cong 0$$

and

$$0 \longrightarrow K_2(f,\mathbb{Z}) \longrightarrow H_2(M;\mathbb{Z}) \xrightarrow{f_*^{\mathbb{Z}}} H_2(p(\mathbb{S}^1 \times \mathbb{S}^3);\mathbb{Z}) \cong 0$$

imply that $K_2(f,\Lambda) \cong H_2(M;\Lambda)$ and $K_2(f,\mathbb{Z}) \cong H_2(M;\mathbb{Z}) \cong H_2(M';\mathbb{Z})$. Now we can apply Theorem 1. Finally, we observe that in our case any homotopy equivalence is simple because the Whitehead group of $*_p\mathbb{Z}$ is trivial (see [22]). □

The proof of Theorem 1 also shows that $M^{(3)} = M^4 \backslash \overset{\circ}{D}{}^4$ is homotopy equivalent to the wedge $(M')^{(3)} \vee (p(\mathbb{S}^1 \times \mathbb{S}^3))^{(3)}$. But $p(\mathbb{S}^1 \times \mathbb{S}^3)^{(3)}$ has the homotopy type of a bouquet $\vee_1^p(\mathbb{S}^1 \vee \mathbb{S}^3)$, and $(M')^{(3)} = M' \backslash \overset{\circ}{D}{}^4$ has the homotopy type of a wedge of q 2-spheres, where q is the rank of $H_2(M;\mathbb{Z}) \cong H_2(M';\mathbb{Z})$. So $M^{(3)}$ is homotopy equivalent to a bouquet of spheres of dimensions 1, 2, and 3.

This gives a simple alternative proof of the main theorem of [20]. In fact, we have

Theorem 6. *Let M^4 be a closed connected orientable topological 4-manifold whose fundamental group is a free group $*_p\mathbb{Z}$ of rank p. Then the punctured manifold obtained from M by removing an open 4-cell has the homotopy type of a bouquet $\vee_p \mathbb{S}^1 \vee_p \mathbb{S}^3 \vee_q \mathbb{S}^2$, where q is the rank of the second integral homology group of M.*

Finally, we observe that the group $*_p\mathbb{Z}$ is good for $p = 1$. So we can apply surgery theory to obtain the following characterization theorem (see also [18] and [19]).

Theorem 7. *Let M^4 be a closed connected orientable 4-manifold with $\pi_1 \cong \mathbb{Z}$. Let M' be the simply-connected manifold obtained from M by killing the fundamental group. Then M is topologically homeomorphic to the connected sum $M'\#(\mathbb{S}^1 \times \mathbb{S}^3)$ if and only if $\lambda_M^\Lambda \cong \lambda_M^{\mathbb{Z}} \otimes \Lambda$. In particular, M is topologically homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^3$ if the Euler characteristic of M vanishes.*

Note that Hambleton and Teichner have constructed in [13] an example of closed topological 4-manifold M with $\pi_1 \cong \mathbb{Z}$ which is not the connected sum of $\mathbb{S}^1 \times \mathbb{S}^3$ with a simply-connected 4-manifold. This means that the nonsingular hermitian form λ_M^Λ over the group ring $\Lambda = \mathbb{Z}[\mathbb{Z}]$ can not be extended from the integers, i.e. λ_M^Λ is not isomorphic to $\lambda_M^{\mathbb{Z}} \otimes_{\mathbb{Z}} \Lambda$.

4.2. Surface groups. Let M^4 be a closed connected oriented spin topological 4-manifold whose fundamental group is isomorphic to that of a closed connected aspherical surface F, i.e. $\pi_1(M) \cong \pi_1(F)$. Since F is aspherical, we have that $F \simeq K(\pi_1, 1)$, where $\simeq$ means *homotopy equivalent to*. Let $c_M : M \to F$ be a classifying map. Following [5], we construct a degree one map $f : M \to F \times \mathbb{S}^2$. By Lemma 2.1 of [5], there exists a map $j : F \to M$ such that the composite map

$$F \xrightarrow{j} M \xrightarrow{c_M} F$$

is homotopic to the identity map. Now define $u = j_*[F] \in H_2(M; \mathbb{Z})$. By Lemma 2.5 of [5], there exists a map $g : M \to \mathbb{S}^2$ such that $g^*(\omega_{\mathbb{S}^2}) = \mathrm{PD}^{-1}(u)$, where $\omega_{\mathbb{S}^2}$ generates $H^2(\mathbb{S}^2; \mathbb{Z})$, and PD denotes the Poincaré duality. Now the product map $f = c_M \times g : M \to F \times \mathbb{S}^2$ is proved to have degree one (see [5] for more details). If M is not spin, we have a degree one map from M to $F \underset{\sim}{\times} \mathbb{S}^2$ (the twisted $\mathbb{S}^2$-bundle over F). So Theorem 1 applies to give the following theorem (which is related to some results of [14] and [15]).

Theorem 8. *Let M^4 be a closed connected oriented spin topological 4-manifold with $\pi_1(M) \cong \pi_1(F)$, where F is a closed aspherical surface. Then M is simple homotopy equivalent to a connected sum $M' \#(F \times \mathbb{S}^2)$ if and only if*

$$\lambda_M^\Lambda|_{K_2(f,\Lambda)} \cong \lambda_M^{\mathbb{Z}}|_{K_2(f,\mathbb{Z})} \otimes \Lambda.$$

In particular, if $\chi(M) = 2\chi(F)$, then M is simple homotopy equivalent to $F \times \mathbb{S}^2$.

Note that any homotopy equivalence is simple because the Whitehead group of $\pi_1(F)$ is trivial (see [22]).

If F is the torus, then the fundamental group $\pi_1 \cong \mathbb{Z} \oplus \mathbb{Z}$ is good so s-cobordisms are topologically products. This gives the following characterization:

Theorem 9. *Let M^4 be a closed connected oriented topological manifold with $\pi_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}$. Let M' be the simply-connected manifold obtained from M by killing the fundamental group. Then M is topologically homeomorphic to the connected sum of M' with an $\mathbb{S}^2$-bundle over the torus if and only if the homological condition of Theorem 8 holds. In particular, a closed connected orientable topological 4-manifold M is homeomorphic to an $\mathbb{S}^2$-bundle over the torus if and only if $\pi_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $\chi(M) = 0$.*

4.3. Asphericity. Let M^4 be a closed connected oriented topological 4-manifold having the fundamental group of a fixed closed aspherical 4-manifold X^4. Let $f : M \to X$ be the classifying map of the universal covering. As proved in [4], Proposition 2.3, the map f is of degree one if and only if the k-invariant $k_M^3 \in H^3(K(\pi_1, 1); \pi_2(M))$ vanishes. Observe that $K(\pi_1, 1) \simeq X$. Under this hypothesis, Theorem 1 gives the following consequence:

Theorem 10. *Let M^4 be a closed connected oriented topological 4-manifold whose fundamental group is that of a fixed closed connected aspherical 4-manifold X. Assume that $k_M^3 = 0$ (or equivalently, the classifying map $f : M \to X$ is of degree one). Then M is homotopy equivalent to a connected sum of a closed simply-connected 4-manifold M' with X if and only if*

$$\lambda_M^\Lambda|_{K_2(f,\Lambda)} \cong \lambda_M^{\mathbb{Z}}|_{K_2(f,\mathbb{Z})} \otimes \Lambda.$$

4.4. Four-manifolds with $\boldsymbol{\pi_2} \cong \mathbf{0}$. Let M^4 be a closed connected orientable topological 4-manifold with $\pi = \pi_1(M)$ torsion free and infinite, and $\pi_2(M) \cong 0$. By Specker's lemma (see for example [15]), it follows that π is isomorphic to a free product of factors $\mathbb{Z}$ with groups having one end (of course, some factors may be absent). Assume for example that $\pi \cong \rho * (*_p\mathbb{Z})$, $p > 0$, where the group ρ has one end. By surgery we can kill off the part $*_p\mathbb{Z}$. This yields a closed connected aspherical 4-manifold Y such that $\pi_1(Y) \cong \rho$. This claim can be proved by an iterative procedure on p. So for simplicity we assume $p = 1$. Let N be the compact 5-manifold obtained from $M \times I$ ($I = [0, 1]$) by attaching a 2-handle along the loop which generates the factor $\mathbb{Z}$ in $\pi_1(M) \cong \rho * \mathbb{Z}$. Then N has vanishing second homotopy group since $\pi_2(M) \cong 0$. The other end of the cobordism N is the closed connected 4-manifold Y obtained from M by surgery along the generator of the factor $\mathbb{Z}$. So N is also obtained from $Y \times I$ by attaching a 3-handle. This implies that both N and Y have fundamental groups isomorphic to ρ. We look now at the homology and cohomology exact sequences (with compact supports) of the covering spaces $Y_\rho \subset N_\rho$ with covering group ρ (to simplify notation we suppress coefficients in the (co)homology modules, and write $\mathbb{ZZ}$ for the group ring of the integers). Then $H_2(N, M)$ is a free $\mathbb{ZZ}[\rho]$-module on one generator; it injects into $H_1(M)$ as a direct summand. Dually the map from $H^1(M)$ to $H^2(N, M)$ is a split epimorphism of modules. So the map from $H^2(N, M)$ to $H^2(N)$ (and hence that from $H^2(N, Y)$ to $H^2(Y)$) is zero. By duality, the map from $H_3(N, Y)$ to $H_2(Y)$ is zero. Since $H_3(N) \cong 0$, it follows that $H_2(Y)$ vanishes as $\mathbb{Z}[\rho]$-module. But we took coefficients in the group ring of the fundamental group of Y, so the module $H_2(Y)$ is isomorphic to $\pi_2(Y)$. This implies that $\pi_2(Y) \cong 0$. Then Y is aspherical since $\pi_1(Y) \cong \rho$ has one end and $\pi_2(Y) \cong 0$. Moreover, we have the isomorphism $H_2(M) \cong H_2(Y)$ (between integral second homology groups), and $\lambda_M^{\mathbb{Z}} \cong \lambda_Y^{\mathbb{Z}}$ (see [2]). Since $k_M^3 \in H^3(K(\pi, 1); \pi_2(M))$ vanishes (in fact, the cohomology group is trivial as $\pi_2(M) \cong 0$), we have a degree one map $f_1 : M \to Y$. By the procedure described in 4.1, we can construct a degree one map $f_2 : M \to p(\mathbb{S}^1 \times \mathbb{S}^3)$. So there is a degree one map $f = f_1 \# f_2 : M \to Y \# p(\mathbb{S}^1 \times \mathbb{S}^3)$ which induces an isomorphism on the fundamental group. The exact sequences

$$0 \longrightarrow K_2(f, \Lambda) \longrightarrow H_2(M; \Lambda) \cong \pi_2(M) \cong 0$$

and

$$0 \to K_2(f, \mathbb{Z}) \to H_2(M; \mathbb{Z}) \underset{\cong}{\xrightarrow{f_*^{\mathbb{Z}}}} H_2(Y \# p(\mathbb{S}^1 \times \mathbb{S}^3)) \cong H_2(Y) \to 0$$

imply that $K_2(f, \Lambda) \cong K_2(f, \mathbb{Z}) \cong 0$. So the homological condition of Theorem 1 is trivially verified. Then M is homotopy equivalent to the connected sum of the closed connected aspherical 4-manifold Y with copies of $\mathbb{S}^1 \times \mathbb{S}^3$. This proves Theorem 2. If π is good, then M is topologically homeomorphic to a connected sum of closed aspherical 4-manifolds and factors $\mathbb{S}^1 \times \mathbb{S}^3$. This gives evidence for the validity of Hillman's conjecture stated in Section 1.

References

[1] Bauer, S.: *The homotopy type of a 4-manifold with finite fundamental group.* Lect. Notes in Math. **1361**, Springer Verlag, Berlin–Heidelberg–New York 1988, 1–6. Zbl 0667.57007

[2] Cavicchioli, A.; Hegenbarth, F.: *On the intersection forms of closed 4-manifolds.* Publ. Mat. **36** (1992), 73–83. Zbl 0779.57012

[3] Cavicchioli, A.; Hegenbarth, F.: *On 4-manifolds with free fundamental group.* Forum Math. **6** (1994), 415–429. Zbl 0822.57015

[4] Cavicchioli, A.; Hegenbarth, F.: *The homotopy classification of 4-manifolds having the fundamental group of an aspherical 4-manifold.* Osaka J. Math. **37** (2000), 859–871. Zbl 0979.55007

[5] Cavicchioli, A.; Hegenbarth, F.; Repovš, D.: *Four-manifolds with surface fundamental groups.* Trans. Amer. Math. Soc. **349** (1997), 4007–4019. Zbl 0887.57026

[6] Cavicchioli, A.; Hegenbarth, F.; Spaggiari, F.: *A splitting theorem for homotopy equivalent smooth 4-manifolds.* Rendiconti di Mat. Ser. VII **17** (1997), 523–539. Zbl 0890.57032

[7] Cochran, T.; Habegger N.: *On the homotopy theory of simply connected four-manifolds.* Topology **29** (1990), 419–440. Zbl 0713.55005

[8] Freedman, M. H.: *The topology of four-dimensional manifolds.* J. Differential Geometry **17** (1982), 357–453. Zbl 0528.57011

[9] Freedman, M. H.: *Poincaré transversality and four-dimensional surgery.* Topology **27** (1988), 171–175. Zbl 0654.57007

[10] Freedman, M. H.; Quinn, F. S.: *Topology of 4-Manifolds.* Princeton Univ. Press, Princeton, N. J., 1990. Zbl 0705.57001

[11] Freedman, M. H.; Taylor, L.: *Γ-Splitting 4-manifolds.* Topology **16** (1977), 181–184. Zbl 0363.57004

[12] Hambleton, I.; Kreck, M.: *On the classification of topological 4-manifolds with finite fundamental group.* Math. Ann. **280** (1988), 85–104. Zbl 0616.57009

[13] Hambleton, I.; Teichner, P.: *A non-extended hermitian form over* $\mathbb{Z}[\mathbb{Z}]$. Manuscripta Math. **93**(4) (1997), 435–442. Zbl 0890.57034

[14] Hillman, J. A.: *On 4-manifolds homotopy equivalent to surface bundles over surfaces.* Topology Appl. **40** (1991), 275–286. Zbl 0754.57014

[15] Hillman, J. A.: *The Algebraic Characterization of Geometric 4-Manifolds.* London Math. Soc. Lect. Note Ser. **198**, Cambridge Univ. Press, Cambridge 1994. Zbl 0812.57001

[16] Hillman, J. A.: *Free products and 4-dimensional connected sums.* Bull. London Math. Soc. **27** (1995), 387–391. Zbl 0834.57011

[17] Kawauchi, A.: *Splitting a 4-manifold with infinite cyclic fundamental group.* Osaka J. Math. **31** (1994), 489–495. Zbl 0849.57018

[18] Kawauchi, A.: *Torsion linking forms on surface-knots and exact 4-manifolds.* Proc. Conf. Knots in Hellas 1998. World Scientific. Ser. Knots Everything **24** (2000), 208–228. Zbl 0969.57020

[19] Kawauchi, A.: *Algebraic characterization of an exact* 4*-manifold with infinite cyclic first homology.* Atti Sem. Mat. Fis. Univ. Modena **48** (2000), 405–424.

[20] Matumoto, T.; Katanaga, A.: *On* 4*-dimensional closed manifolds with free fundamental groups.* Hiroshima Math. J. **25** (1995), 367–370. Zbl 0921.57007

[21] Milnor, J.: *A procedure for killing the homotopy groups of differentiable manifolds.* Proc. Sympos. Pure Math. **3** (1961), 39–55. Zbl 0118.18601

[22] Milnor, J.: *Whitehead torsion.* Bull. Amer. Math. Soc. **72** (1966), 358–428. Zbl 0147.23104

[23] Stong, R.: *Uniqueness of π_1-negligible embeddings in* 4*-manifolds: a correction to theorem 10.5 of Freedman and Quinn.* Topology **32** (1993), 677–699. Zbl 0811.57021

[24] Stong, R.: *Uniqueness of connected sum decompositions in dimension 4.* Topology and its Appl. **56** (1994), 277–291. Zbl 0814.57016

[25] Stong, R.: *A structure theorem and a splitting theorem for simply connected smooth* 4*-manifolds.* Math. Research Letters **2** (1995), 497–503. Zbl 0858.57024

[26] Wall, C. T. C.: *Surgery on Compact Manifolds.* Academic Press, London - New York 1970. Zbl 0219.57024

[27] Whitehead, J. H. C.: *A certain exact sequence.* Ann. of Math. (2) **52** (1950), 51–110. Zbl 0037.26101

Received April 13, 2001