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# On a Theorem of F. S. Macaulay on Colon Ideals*

*Dedicated to my teacher Prof. William Heinzer*

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In §86 of his monograph [4] of 1916, F. S. Macaulay proved the following

**Theorem 1.1.** *Let $R$ be the polynomial ring $k[X_1, X_2, \dots, X_n]$ over a field $k$. Let $I$ be a height $n$ ideal of $R$ generated by homogeneous polynomials $f_1, f_2, \dots, f_n$ of degree $d_1, d_2, \dots, d_n$ respectively. Set $M = (X_1, X_2, \dots, X_n)R$ and $\delta = d_1 + d_2 + \dots + d_n - n + 1$. Then for all integers $t = 0, 1, \dots, \delta$*

$$I : m^t = I + m^{\delta - t}.$$

P. Griffiths [2] used Macaulay's theorem in Hodge theory of smooth hypersurfaces in projective space. He also provided a proof of this theorem by using the local duality theorem in the complex case involving the Grothendieck residue symbol.

The objective of this note is to generalize Macaulay's theorem to Gorenstein standard graded algebras over a field.

A crucial result used in our proof is a theorem of Macaulay about Hilbert series of Gorenstein graded algebras. For a modern proof see the corollary 4.4.6 of [1]. This has been generalized for Gorenstein graded rings over Artin rings in [3]. We shall use the zero dimensional case of this.

**Theorem 1.2.** *Suppose that $S = \bigoplus_{n=0}^{\infty} S_n$ is a standard Gorenstein graded algebra over an Artin local ring $S_0$. Let $\delta = \max\{n | S_n \neq 0\}$. Then $\lambda(S_i) = \lambda(S_{\delta - i})$ for all $i = 1, 2, \dots, \delta$.*

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We recall certain standard facts from [1] about local cohomology. Let $R = \oplus_{n=0}^{\infty} R_n$ be a standard graded algebra over a field $k$. Let $M$ denote the irrelevant ideal $\oplus_{n=1}^{\infty} R_n$. The a-invariant of $R$ is defined by $a(R) = \max\{n \mid \left[H_M^d(R)\right]_n \neq 0\}$. If $R$ is Cohen-Macaulay and $f$ is a homogeneous degree $d$ nonzerodivisor in $R$ then $a(R/fR) = a(R) + d$. By Serre's theorem on cohomology of projective space the a-invariant of a polynomial ring over a field in $n$ variables is $-n$. This can also be seen by the Grothendieck-Serre formula.

**Theorem 1.3.** (Grothendieck-Serre) *Let $R = \oplus_{n=0}^{\infty} R_n$ be a $d$-dimensional finitely generated graded algebra over an artinian local ring $R_0$. Let $M$ denote the irrelevant ideal of $R$. Let $\lambda(M)$ denote length of an $R_0$ module $M$. Put $H(n) = \lambda(R_n)$ and $P(n)$ the Hilbert polynomial for the Hilbert function $H(n)$. Then for all integers $n$*

$$P(n) - H(n) = \sum_{i=0}^{d} (-1)^i \lambda\left(H_M^i(R)_n\right)$$

If $R$ is a polynomial ring over a field in $d$ variables then $P(n) = \binom{n+d-1}{d-1}$ and $P(-d+i) = H(-d+i)$ for all $i > 0$, but $P(-d) \neq H(-d)$. Hence $a(R) = -d$. Thus If $\mathbf{f} = f_1, f_2, \dots, f_d$ is a regular sequence with $\deg(f_i) = r_i$ for all $i = 1, 2, \dots, d$ then $a(R/\mathbf{f}) = r_1 + r_2 + \ldots + r_d - d$.

**Theorem 1.4.** *Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a standard Gorenstein graded algebra of dimension $d$ over a field $R_0 = k$. Let $M$ denote the maximal homogeneous ideal of $R$. Let $I = (f_1, f_2, \dots, f_d)$ be an $M$-primary ideal generated by forms $f_i$ of degree $r_i$ for $i = 1, 2, \dots, d$. Put $\delta = a(R) + r_1 + r_2 + \cdots + r_d + 1$. Then for $t = 0, 1, \dots, \delta$*

$$I : M^t = I + M^{\delta - t}$$

*Proof.* We know that $a(R/I) = a(R) + r_1 + \cdots + r_d$. As $R/I$ is Artin, $a(R/I) = \max\{n \mid (R/I)_n \neq 0\}$. Let $N$ denote the maximal homogeneous ideal of $R/I$. Thus we need to prove that

$$0 : N^t = N^{\delta - t}.$$

Apply induction on $t$. As $S$ is a zero-dimensional Gorenstein ring, $S = E_S(k)$ where $E$ denotes the injective hull of $k$ over $S$. Thus $\operatorname{Hom}(-, S)$ is an exact functor. Consider the exact sequence

$$0 \longrightarrow N \longrightarrow S \longrightarrow S/N \longrightarrow 0.$$

This gives the exact sequence

$$0 \longrightarrow \operatorname{Hom}_S(S/N, S) \longrightarrow \operatorname{Hom}_S(S, S) \longrightarrow \operatorname{Hom}_S(N, S) \longrightarrow 0.$$

By Matlis duality $\operatorname{Hom}_S(S, S) = S^*$ where $S^*$ denotes the completion of $S$ in the $N$-adic topology. As $S$ is Artin, it is complete in this topology. As $\operatorname{Hom}_S(S/N, S) = (0 : N)$ and $\lambda(\operatorname{Hom}_S(P, S)) = \lambda(P)$ for any finite length $S$-module $P$, we obtain $\lambda(0 : N) = \lambda(S/N) = 1$. But $N^{\delta-1} \subset (0 : N)$ and $N^{\delta-1} \neq 0$. Hence $\lambda(N^{\delta-1}) = \lambda(0 : N) = 1$. Therefore $N^{\delta-1} = (0 : N)$.

Assume that the result has been proved for $t - 1$. To prove it for $t$, consider the exact sequence,

$$0 \longrightarrow N^{t-1}/N^t \longrightarrow S/N^t \longrightarrow S/N^{t-1} \longrightarrow 0.$$

As $\mathrm{Hom}_S(-,S)$ is an exact functor, we get the exact sequence

$$0 \longrightarrow \mathrm{Hom}(S/N^{t-1},S) \longrightarrow \mathrm{Hom}(S/N^t,S) \longrightarrow \mathrm{Hom}(N^{t-1}/N^t,S) \longrightarrow 0.$$

Therefore $\lambda(0:N^t) = \lambda(0:N^{t-1}) + \lambda(N^{t-1}/N^t)$.

By induction, $\lambda(N^{\delta-t+1}) = \lambda(0:N^{t-1})$. Hence $\lambda(0:N^t) = \lambda(N^{\delta-t+1}) + \lambda(N^{t-1}/N^t)$. Since the Hilbert function of the Gorenstein graded ring $S$ is symmetric, $\lambda(S_i) = \lambda(S_{\delta-i-1})$ for each $i = 1,2,\dots,\delta-1$. Thus $\lambda(N^{t-1}/N^t) = \lambda(N^{\delta-t}/N^{\delta-t+1})$. Therefore $\lambda(0:N^t) = \lambda(N^{\delta-t+1}) + \lambda(N^{\delta-t}/N^{\delta-t+1})$. This proves $(0:N^t) = N^{\delta-t}$. □

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