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The Classification of Generalized Quadrangles with Two Translation Points

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Abstract. Suppose $\mathcal{S}$ is a finite generalized quadrangle (GQ) of order (s,t), $s \neq 1 \neq t$, and suppose that L is a line of $\mathcal{S}$. A symmetry about L is an automorphism of the GQ which fixes every line of $\mathcal{S}$ meeting L (including L). A line is called an axis of symmetry if there is a full group of symmetries of size s about this line, and a point of a generalized quadrangle is a translation point if every line through it is an axis of symmetry. A GQ with a translation point is often called a translation generalized quadrangle. In the present paper, we classify the generalized quadrangles with at least two distinct translation points. In order to obtain the main result, we prove many more general theorems which are useful for the theory of span-symmetric generalized quadrangles (these are the GQ's with non-concurrent axes of symmetry), and using earlier results of the author, we give more general versions of our main theorem.

As a by-product of the proof of our main result, we will show that for any span-symmetric generalized quadrangle $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t$, s and t are powers of the same prime, and if $s \neq t$ and s is odd, then $\mathcal{S}$ always contains at least $s+1$ classical subquadrangles of order s.

In an addendum we obtain an explicit construction of some classes of spreads for the point-line duals of the Kantor flock generalized quadrangles as a second by-product of the proof of our main result.

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1. Introduction and statement of the main results

The main examples of finite generalized quadrangles (GQ's) are essentially of five types: (1) they are inside a projective space $\mathbf{PG}(n,q)$ over the Galois field $\mathbf{GF}(q)$, and these are the so-called 'classical examples', (2) they are the point-line duals of the classical examples (the dual $H(4,q^2)^D$ of the classical GQ $H(4,q^2)$ is never embedded (in the usual sense) in a projective space), (3) they are of order $(s-1,s+1)$ or, dually, of order $(s+1,s-1)$, and the examples of this type all are in some way connected to ovals or hyperovals of $\mathbf{PG}(2,q)$, (4) they arise as translation generalized quadrangles (from generalized ovals or generalized ovoids, see below), and (5) they arise from flocks of the quadratic cone in $\mathbf{PG}(3,q)$ (see further).

The main examples of GQ's which were discovered in the past fifteen years all are of type (4) and (5), and these GQ's all have a common property: the duals of the examples of type (5) all have at least one axis of symmetry (see below), and the examples of type (4) even have a point through which every line is an axis of symmetry, and in this case, this property characterizes this class of examples. This is a first motivation for the present paper: it is a step towards a classification of the generalized quadrangles with axes of symmetry. Aiming eventually at such a classification – a project which we started in [41] – our attention in [43] was drawn to the span-symmetric generalized quadrangles: these are the GQ's which have non-concurrent axes of symmetry. In [43] we completely classified the span-symmetric generalized quadrangles of order s, $s>1$, by proving that every span-symmetric generalized quadrangle of order s is classical, i.e. isomorphic to the GQ $\mathcal{Q}(4,s)$ which arises from a nonsingular parabolic quadric in $\mathbf{PG}(4,q)$. For span-symmetric generalized quadrangles of order (s,t) with $s\neq t$, $s\neq 1\neq t$, however, a similar result cannot hold (see below). Let $\mathcal{K}$ be the quadratic cone with equation $X_0X_1=X_2^2$ of $\mathbf{PG}(3,q)$, q odd. Then the q planes π_t with equation $tX_0-mt^\sigma X_1+X_3=0$, $t\in\mathbf{GF}(q)$, m a given non-square in $\mathbf{GF}(q)$ and σ a given field automorphism of $\mathbf{GF}(q)$, define a flock $\mathcal{F}$ of $\mathcal{K}$, and the GQ which arises from $\mathcal{F}$ is the Kantor (flock) generalized quadrangle. If σ is the identity, the flock $\mathcal{F}$ is a linear flock and then the GQ is classical. Recently, S. E. Payne noticed that the dual Kantor flock generalized quadrangles are span-symmetric, and this infinite class of generalized quadrangles contains nonclassical examples. Moreover, every nonclassical dual Kantor flock GQ even contains a line L for which every line which meets L is an axis of symmetry! Kantor [12, 21] however gave a partial classification theorem of span-symmetric generalized quadrangles by proving that for a span-symmetric generalized quadrangle of order (s,t), $s\neq 1\neq t$, necessarily $t=s$ or $t=s^2$. Together with [43], this paper contributes to a classification of span-symmetric generalized quadrangles, and we will for instance show the following improvement of Kantor's theorem.

Theorem 1. *Let $\mathcal{S}$ be a span-symmetric generalized quadrangle of order (s,t), $s\neq 1\neq t$. Then $s=t$ or $t=s^2$, and s and t are powers of the same prime.*

Moreover, in the case where s is odd, we obtain the following strong theorem.

Theorem 2. *Suppose $\mathcal{S}$ is a span-symmetric generalized quadrangle of order (s,t), $s\neq 1\neq t$, where $s\neq t$ and s is odd. Then $\mathcal{S}$ contains at least $s+1$ subquadrangles isomorphic to the classical GQ $\mathcal{Q}(4,s)$.*

Finally, another main goal of this paper is to state elementary combinatorial and group theoretical conditions for a GQ $\mathcal{S}$ such that $\mathcal{S}$ essentially arises from a flock, see also J. A. Thas [29], [31], [33], [32], [36] and K. Thas [45].

Our main result reads as follows:

Theorem 3. *Suppose $\mathcal{S}$ is a generalized quadrangle of order (s,t), $s \neq 1 \neq t$, with two distinct collinear translation points. Then we have the following:*

(i) *$s = t$, s is a prime power and $\mathcal{S} \cong \mathcal{Q}(4,s)$;*

(ii) *$t = s^2$, s is even, s is a prime power and $\mathcal{S} \cong \mathcal{Q}(5,s)$;*

(iii) *$t = s^2$, $s = q^n$ with q odd, where $\mathbf{GF}(q)$ is the kernel of the TGQ $\mathcal{S} = \mathcal{S}^{(\infty)}$ with (∞) an arbitrary translation point of $\mathcal{S}$, $q \geq 4n^2 - 8n + 2$ and $\mathcal{S}$ is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ where $\mathcal{F}$ is a Kantor flock;*

(iv) *$t = s^2$, $s = q^n$ with q odd, where $\mathbf{GF}(q)$ is the kernel of the TGQ $\mathcal{S} = \mathcal{S}^{(\infty)}$ with (∞) an arbitrary translation point of $\mathcal{S}$, $q < 4n^2 - 8n + 2$ and $\mathcal{S}$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ for some flock $\mathcal{F}$.*

If a thick GQ $\mathcal{S}$ has two non-collinear translation points, then $\mathcal{S}$ is always of classical type, i.e. isomorphic to one of $\mathcal{Q}(4,s)$, $\mathcal{Q}(5,s)$.

For s even the classification theorem is complete.

We emphasize that this theorem is rather remarkable, since we start from some very easy combinatorial and group theoretical properties, while flock generalized quadrangles are concretely described using so called *q-clans*, *4-gonal families* and finite fields.

We give some variations and weakenings of the hypotheses of the main result in Section 14.

2. Notation and some basics

2.1. Introduction to generalized quadrangles

A (finite) *generalized quadrangle (GQ) of order* (s,t) is an incidence structure $\mathcal{S} = (P, B, I)$ in which P and B are disjoint (nonempty) sets of objects called *points* and *lines* respectively, and for which I is a symmetric point-line incidence relation satisfying the following axioms.

(GQ1) Each point is incident with $t+1$ lines ($t \geq 1$) and two distinct points are incident with at most one line.

(GQ2) Each line is incident with $s+1$ points ($s \geq 1$) and two distinct lines are incident with at most one point.

(GQ3) If p is a point and L is a line not incident with p, then there is a unique point-line pair (q, M) such that $pIMIqIL$.

If $s = t$, then $\mathcal{S}$ is also said to be *of order s*.

Generalized quadrangles were introduced by J. Tits [48] in his celebrated work on triality, in order to understand the Chevalley groups of rank 2, as a subclass of a larger class of incidence structures, namely the *generalized polygons*, see [30] for a detailed overview without proofs, and [51] for an extensive analysis of the subject.

The main results, up to 1983, on finite generalized quadrangles are contained in the monograph *Finite Generalized Quadrangles* [21] (denoted **FGQ**) by S. E. Payne and J. A. Thas. A survey of some 'new' developments on this subject in the period 1984–1992, can be found in the article *Recent developments in the theory of finite generalized quadrangles* [28]. It is also worthwile mentioning [35].

Let $\mathcal{S} = (P, B, I)$ be a (finite) generalized quadrangle of order (s, t), $s \neq 1 \neq t$. Then $|P| = (s+1)(st+1)$ and $|B| = (t+1)(st+1)$. Also, $s \leq t^2$ and, dually, $t \leq s^2$, and $s+t$ divides $st(s+1)(t+1)$.

There is a point-line duality for GQ's of order (s, t) for which in any definition or theorem the words "point" and "line" are interchanged and also the parameters. Normally, we assume without further notice that the dual of a given theorem or definition has also been given.

Let p and q be (not necessarily distinct) points of the GQ $\mathcal{S}$; we write $p \sim q$ and say that p and q are *collinear*, provided that there is some line L such that $pILIq$ (so $p \not\sim q$ means that p and q are not collinear). Dually, for $L, M \in B$, we write $L \sim M$ or $L \not\sim M$ according as L and M are *concurrent* or *non-concurrent*. If $p \neq q \sim p$, the line incident with both is denoted by pq, and if $L \sim M \neq L$, the point which is incident with both is sometimes denoted by $L \cap M$.

For $p \in P$, put $p^\perp = \{q \in P \parallel q \sim p\}$ (note that $p \in p^\perp$). For a pair of distinct points $\{p, q\}$, the *trace* of $\{p, q\}$ is defined as $p^\perp \cap q^\perp$, and we denote this set by $\{p, q\}^\perp$. Then $|\{p, q\}^\perp| = s + 1$ or $t + 1$, according as $p \sim q$ or $p \not\sim q$. More general, if $A \subset P$, $A^\perp$ is defined by $A^\perp = \bigcap\{p^\perp \parallel p \in A\}$. For $p \neq q$, the *span* of the pair $\{p, q\}$ is $sp(p, q) = \{p, q\}^{\perp\perp} = \{r \in P \parallel r \in s^\perp$ for all $s \in \{p, q\}^\perp\}$. When $p \not\sim q$, then $\{p, q\}^{\perp\perp}$ is also called *the hyperbolic line* defined by p and q, and $|\{p, q\}^{\perp\perp}| = s + 1$ or $2 \leq |\{p, q\}^{\perp\perp}| \leq t + 1$ according as $p \sim q$ or $p \not\sim q$.

A *triad* of points (respectively lines) is a triple of pairwise non-collinear points (respectively pairwise disjoint lines). Given a triad T, a *center* of T is just an element of $T^\perp$. If $p \sim q$, $p \neq q$, or if $p \not\sim q$ and $|\{p, q\}^{\perp\perp}| = t + 1$, we say that the pair $\{p, q\}$ is *regular*. The point p is *regular* provided $\{p, q\}$ is regular for every $q \in P \setminus \{p\}$. Regularity for lines is defined dually. One easily proves that either $s = 1$ or $t \leq s$ if $\mathcal{S}$ has a regular pair of non-collinear points.

A GQ of order (s, t) is called *thick* if s and t are both different from 1. A *flag* of a GQ is an incident point-line pair, and an *anti-flag* is a nonincident point-line pair.

A *subquadrangle*, or also a *subGQ*, $\mathcal{S}' = (P', B', I')$ of a GQ $\mathcal{S} = (P, B, I)$ is a GQ for which $P' \subseteq P$, $B' \subseteq B$, and where I' is the restriction of I to $(P' \times B') \cup (B' \times P')$.

Notation. If (p, L) is a nonincident point-line pair of a GQ $\mathcal{S}$, then the unique line which is incident with p and which meets L will be denoted by $[p, L]$.

2.2. The classical generalized quadrangles

Consider a non-singular quadric of Witt index 2, that is, of projective index 1, in $\mathbf{PG}(3, q)$, $\mathbf{PG}(4, q)$, $\mathbf{PG}(5, q)$, respectively. The points and lines of the quadric form a generalized quadrangle which is denoted by $\mathcal{Q}(3, q)$, $\mathcal{Q}(4, q)$, $\mathcal{Q}(5, q)$, respectively, of order $(q, 1)$, (q, q), (q, q^2), respectively. Next, let $\mathcal{H}$ be a nonsingular Hermitian variety in $\mathbf{PG}(3, q^2)$, respectively $\mathbf{PG}(4, q^2)$. The points and lines of $\mathcal{H}$ form a generalized quadrangle $H(3, q^2)$, respectively

$H(4,q^2)$, which has order (q^2,q), respectively (q^2,q^3). The points of $\mathbf{PG}(3,q)$ together with the totally isotropic lines with respect to a symplectic polarity form a GQ $W(q)$ of order q.

The generalized quadrangles defined in this paragraph are the so-called *classical generalized quadrangles* as defined by Tits in [5], see also Chapter 3 of FGQ. Sometimes we will say that a GQ is classical if it is isomorphic to a classical GQ.

3. Introduction to span-symmetric generalized quadrangles

3.1. Span-symmetric generalized quadrangles

Definition. A *grid with parameters* $s+1, 2$ (respectively dual grid with parameters $2, t+1$) is a GQ of order $(s,1)$ (respectively of order $(1,t)$).

Suppose L is a line of a GQ $\mathcal{S}$ of order (s,t), $s,t \neq 1$. A *symmetry about* L is an automorphism of the GQ which fixes every line of $L^{\perp}$. The line L is called an axis of symmetry if there is a full group H of symmetries of size s about L. In such a case, if $M \in L^{\perp} \setminus \{L\}$, then H acts regularly on the points of M not incident with L.

Dually, one defines the notion *center of symmetry*. Any axis, respectively center, of symmetry in the GQ $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t$, is a regular line, respectively point.

A point of a GQ through which every line is an axis of symmetry is called a *translation point*.

Remark 4. Every line of the classical example $\mathcal{Q}(4,s)$ is an axis of symmetry [21].

Suppose $\mathcal{S}$ is a GQ of order (s,t), $s,t \neq 1$, and suppose L and M are distinct non-concurrent axes of symmetry; then it is easy to see by transitivity that every line of $\{L,M\}^{\perp\perp}$ is an axis of symmetry, and $\mathcal{S}$ is called a *span-symmetric generalized quadrangle (SPGQ) with base-span* $\{L,M\}^{\perp\perp}$.

Let $\mathcal{S}$ be a span-symmetric GQ of order (s,t), $s,t \neq 1$, with base-span $\{L,M\}^{\perp\perp}$. Throughout this paper, we will continuously use the following notations.

First of all, the base-span will always be denoted by $\mathcal{L}$. The group which is generated by all the symmetries about the lines of $\mathcal{L}$ is G, and sometimes we will call this group the *base-group*. This group clearly acts 2-transitively on the lines of $\mathcal{L}$, and fixes every line of $\mathcal{L}^{\perp}$ (see for instance [21]). The set of all the points which are on lines of $\{L,M\}^{\perp\perp}$ is denoted by Ω.[1] We will refer to $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^{\perp}, I')$, with I' being the restriction of I to $(\Omega \times (\mathcal{L} \cup \mathcal{L}^{\perp})) \cup ((\mathcal{L} \cup \mathcal{L}^{\perp}) \times \Omega)$, as being *the base-grid*.

The following remarkable theorem states that the order of an SPGQ is essentially known.

Result 5. (W. M. Kantor [12], see 10.7.4 of FGQ) *Suppose $\mathcal{S}$ is a span-symmetric generalized quadrangle of order (s,t), $s,t \neq 1$. Then $t \in \{s, s^2\}$.*

Finally, the following result solves a twenty year old conjecture.

Result 6. (K. Thas [43]) *A span-symmetric generalized quadrangle of order s, $s \neq 1$ is always isomorphic to $\mathcal{Q}(4,s)$.*

[1] Of course, Ω is also the set of points on the lines of $\{L,M\}^{\perp}$; we have that $|\{L,M\}^{\perp}| = |\{L,M\}^{\perp\perp}| = s+1$.

Using the 4-*gonal bases* which correspond to SPGQ's of order s (see [21, 10.7]), the following theorem is the group theoretical analogue of Result 6.

Result 7. (K. Thas [43]) *A group is isomorphic to* $\mathbf{SL}_2(s)$ *for some* s *if and only if it contains a* 4*-gonal basis.*

3.2. Split BN-pairs of rank 1 and SPGQ's

A group with a *split BN-pair of rank* 1 (see e.g. [24, 49]) is a permutation group (X, H) which satisfies the following properties[2].

(BN1) H acts 2-transitively on X;

(BN2) for every $x \in X$ there holds that the stabilizer of x in H has a normal subgroup H_x which acts regularly on $X \setminus \{x\}$.

The elements of X are the *points* of the split BN-pair of rank 1, and for any x, the group H_x will be called a *root group*[3]. An element of the group H which is generated by all the root groups is a *transvection*, and the group H is the *transvection group.* If X is a finite set, then the split BN-pair of rank 1 also is called *finite.* It is clear that the tranvection group acts 2-transitively on the set of points of X.

The following theorem classifies all finite split BN-pairs of rank 1 without using the classification of the finite simple groups, see E. E. Shult [25] and C. Hering, W. M. Kantor and G. M. Seitz [10].

Result 8. ([25, 10]) *Suppose* (X, H) *is a group with a finite split BN-pair of rank* 1*, and suppose* $|X| = s + 1$*, with* $s < \infty$*. Then* H *must always be one of the following (up to isomorphism):*

(1) *a sharply* 2*-transitive group on* X*;*

(2) $\mathbf{PSL}_2(s)$*;*

(3) *the Ree group* $\mathbf{R}(\sqrt[3]{s})$ *with* $\sqrt[3]{s}$ *an odd power of* 3*;*

(4) *the Suzuki group* $\mathbf{Sz}(\sqrt{s})$ *with* $\sqrt{s}$ *an odd power of* 2*;*

(5) *the unitary group* $\mathbf{PSU}_3(\sqrt[3]{s^2})$.[4] □

Every root group has order s. In the first case, (X, H) is a 2-transitive *Frobenius group*, and it is a known theorem (see e.g. [7]) that $s + 1$ is the power of a prime; in all of the other cases, s is the power of a prime. Further, we have that $|\mathbf{PSL}_2(s)| = (s+1)s(s-1)$ or $(s+1)s(s-1)/2$, according as s is even or odd, and the group acts (sharply) 3-transitive on X if and only if s is even; in the other cases, we have that $|\mathbf{R}(\sqrt[3]{s})| = (s+1)s(\sqrt[3]{s}-1)$, $|\mathbf{Sz}(\sqrt{s})| = (s+1)s(\sqrt{s}-1)$, and $|\mathbf{PSU}_3(\sqrt[3]{s^2})| = \frac{(s+1)s(\sqrt[3]{s^2}-1)}{(3,\sqrt[3]{s}+1)}$ (by (a, b) we denote the greatest common divisor of a and b, $a, b \in \mathbb{N}$).

(For references on the orders of these groups, see [7, 11].)

Remark 9. The root groups of $\mathbf{PSL}_2(s)$ and of the sharply 2-transitive groups are the only ones to be (all) abelian.

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[2]Split BN-pairs of rank 1 are essentially the same objects as *Moufang sets* as introduced by Tits in [50].

[3]Note that here 'H_x' is not a notation for the stabilizer of x in H.

[4]Note that we use the projective notation (see e.g. [11]) for the unitary group.

3.3. SPGQ's, split BN-pairs of rank 1 and a lemma concerning the order of the base-group of an SPGQ

Lemma 10. (K. Thas [43], see also 10.7 of FGQ) *Suppose $\mathcal{S}$ is a span-symmetric GQ of order (s,t), $s,t \neq 1$, with base-span $\mathcal{L}$ and base-group G. Then $(\mathcal{L}, G)$ is a finite split BN-pair of rank 1.*

Proof. See K. Thas [43]. □

Combining Result 8 and Lemma 10, we obtain the following theorem.

Theorem 11. *Suppose $\mathcal{S}$ is a span-symmetric generalized quadrangle of order (s,t), $s \neq 1 \neq t$, with base-span $\mathcal{L}$ and base-group G. If N is the kernel of the action of the group G on the set $\mathcal{L}$, then G/N acts as a sharply 2-transitive group on $\mathcal{L}$, or is isomorphic to one of the following list:*

(1) $\mathbf{PSL}_2(s)$;
(2) $\mathbf{R}(\sqrt[3]{s})$;
(3) $\mathbf{Sz}(\sqrt{s})$;
(4) $\mathbf{PSU}_3(\sqrt[3]{s^2})$. □

Lemma 12. *Let $\mathcal{S}$ be an SPGQ of order (s,t), $s \neq 1 \neq t$, with base-grid $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^\perp, I')$ and base-group G. Put $\mathcal{L} = \{U_0, \ldots, U_s\}$ and suppose G_i is the group of symmetries about U_i for all i. If p is a point which is not an element of Ω, and U is a line through p which meets Ω in a certain point qIU_k, then every point on U which is different from q is a point of the G-orbit which contains p.*

Proof. The group G_k acts transitively on the points of U different from q. □

Lemma 13. *Suppose $\mathcal{S}$ is an SPGQ of order (s,t), $s \neq 1 \neq t$, with base-grid Γ and base-group G. Then G has size at least $s^3 - s$.*

Proof. If $s = t$, then we already know by Section 3 that G has order $s^3 - s$, so suppose $s \neq t$.

Set $\mathcal{L} = \{U_0, \ldots, U_s\}$ and suppose G_i is the group of symmetries about U_i for all i. Suppose p is a point of $\mathcal{S}$ not incident with a line of $\mathcal{L}$, and consider the following $s+1$ lines $M_i := [p, U_i]$. If Λ is the G-orbit which contains p, then by Lemma 12 there holds that every point of M_i not in Ω is also a point of Λ. Now fix the line M_0, and consider an arbitrary point $q \neq p$ on M_0 which is not on a line of $\mathcal{L}$. Then again every point of $[q, U_i]$ not in Ω is a point of Λ. Hence we have the following inequality:

$$|\Lambda| \geq 1 + (s+1)(s-1) + (s-1)^2 s, \tag{1}$$

from which it follows that $|\Lambda| \geq s^3 - s^2 + s$.

Now fix a line U of $\mathcal{L}^\perp$. Every line of $\mathcal{S}$ which meets this line and which contains a point of Λ is completely contained in $\Lambda \cup \Omega$ by Lemma 12. Also, G acts transitively on the points of U. Suppose k is the number of lines through a (= every) point of U which are completely contained in Λ. If we count in two ways the number of point-line pairs (u, M) for which $u \in \Lambda, M \sim U$ and uIM, then there follows that

$$\begin{aligned} k(s+1)s &= |\Lambda| \geq s(s^2-s+1) \\ &\Longrightarrow k \geq \frac{s^2-s+1}{s+1} \\ &= s-2+\frac{3}{s+1}, \end{aligned} \tag{2}$$

and hence, since $k \in \mathbb{N}$, we have that

$$k \geq s-1. \tag{3}$$

Thus $|\Lambda| \geq s^3-s$ and so also $|G|$. $\square$

4. Elation generalized quadrangles and translation generalized quadrangles

A *whorl about the point* p of $\mathcal{S}$ is a collineation of $\mathcal{S}$ which fixes each line through p. An *elation about the point* p is a whorl about p that fixes no point of $P \setminus p^\perp$. By definition, the identical permutation is an elation (about every point). If p is a point of the GQ $\mathcal{S}$, for which there exists a group of elations G about p which acts regularly on the points of $P \setminus p^\perp$, then $\mathcal{S}$ is said to be an *elation generalized quadrangle (EGQ)* with *elation point* or *base point* p and *elation group* G, and we sometimes write $(\mathcal{S}^{(p)}, G)$ for $\mathcal{S}$. If a GQ $(\mathcal{S}^{(p)}, G)$ is an EGQ with elation point p, and if any line incident with p is an axis of symmetry, then we say that $\mathcal{S}$ is a *translation generalized quadrangle (TGQ)* with *base point* p and *translation group* G. In such a case, G is uniquely defined; G is generated by all symmetries about every line incident with p, or, respectively, G is the set of all elations about p, see FGQ. Note that the base point of a TGQ is always a translation point. TGQ's were introduced by J. A. Thas in [26] for the case $s = t$.

Result 14. (FGQ, 8.3.1) *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s, t), $s, t \geq 1$. Suppose each line through some point p is an axis of symmetry, and let G be the group generated by the symmetries about the lines through p. Then G is elementary abelian and $(\mathcal{S}^{(p)}, G)$ is a TGQ.*

For the case $s = t$, we have the following result of [21], see also [41, 45] for several shorter proofs.

Result 15. (FGQ, 11.3.5) *Let $\mathcal{S} = (P, B, I)$ be a GQ of order s, with $s \neq 1$. Suppose that there are at least three axes of symmetry through a point p, and let G be the group generated by the symmetries about these lines. Then G is elementary abelian and $(\mathcal{S}^{(p)}, G)$ is a TGQ.*

Result 16. (FGQ, 8.2.3, 8.3.2 and 8.5.2) *Suppose $(\mathcal{S}^{(x)}, G)$ is an EGQ of order (s, t), $s \neq 1 \neq t$. Then $(\mathcal{S}^{(x)}, G)$ is a TGQ if and only if G is an (elementary) abelian group. Also in such a case there is a prime p and there are natural numbers n and k, where k is odd, such that either $s = t$ or $s = p^{nk}$ and $t = p^{n(k+1)}$. It follows that G is a p-group.*

Finally, the following result is a recent result which will appear to be very usefull in the sequel.

Result 17. (J. A. Thas [37]) *Suppose $\mathcal{S}^{(p)}$ is a TGQ of order (s, s^2) with s even, which contains at least two classical subGQ's of order s containing the point p. Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5, s)$.*

Note. If $\mathcal{S}$ is the point-line dual of a TGQ $\mathcal{S}^D$ with base point p, where p corresponds to L in $\mathcal{S}$, then we also say that $\mathcal{S}$ is a TGQ, and L is called the *base line* of the TGQ (L is a *translation line*).

5. The cases $s = 2$, $s = 3$ and $s = 4$

Suppose $\mathcal{S}$ is a GQ of order $(2, t)$, $t \geq 2$. Then by the divisibility condition of Section 2, there easily follows that $t \in \{2, 4\}$. There is a unique GQ of order 2, respectively $(2, 4)$, namely the classical $\mathcal{Q}(4, 2)$, respectively $\mathcal{Q}(5, 2)$, see [21, 5.2.3], respectively [21, 5.3.2].

Next, suppose $\mathcal{S}$ is of order $(3, t)$, $t \neq 1$, and suppose $\mathcal{S}$ is an SPQG for some base-span $\mathcal{L}$. Then by Result 5, $t \in \{3, 9\}$. There is a unique GQ (up to duality) of order 3, respectively $(3, 9)$, namely the classical $\mathcal{Q}(4, 3)$, respectively $\mathcal{Q}(5, 3)$, see [21, 6.2.1], respectively [21, 6.2.3], and also [20], [6]. Since the GQ $W(3)$, which is isomorphic to the dual $\mathcal{Q}(4, 3)^D$ of $\mathcal{Q}(4, 3)$ (see Chapter 5 of FGQ), contains no regular lines, and since an axis of symmetry is regular, we have the following easy corollary.

Theorem 18. *Any SPGQ of order (s, t) with $t \neq 1$ and $s \in \{2, 3\}$ is classical.* □

The following theorem classifies all TGQ's of order $(4, 16)$.

Result 19. (M. Lavrauw and T. Penttila [14]) *Any TGQ of order $(4, 16)$ is isomorphic to $\mathcal{Q}(5, 4)$.*

Finally, any GQ of order 4 is isomorphic to $\mathcal{Q}(4, 4) \cong W(4)$ by [21, 6.3.1].

In the following we will sometimes suppose that $s \neq 2, 3, 4$ if this seems convenient.

6. The nonsemi-regular case

Suppose that $\mathcal{S}$ is a span-symmetric generalized quadrangle of order (s, t), $s, t \neq 1$, with base-span $\{L, M\}^{\perp\perp} = \mathcal{L}$, base-group G and base-grid $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^{\perp}, I')$. Furthermore, put $\mathcal{L} = \{U_0, \ldots, U_s\}$ and suppose G_i is the group of symmetries about U_i for all i. Since the case $s = t$ is completely settled by [43], we suppose that $s \neq t$ for convenience. Thus, by Result 5, we have that $t = s^2$.

In this section, it is our aim to exclude the case where G does not act semi-regularly on the points of $\mathcal{S} \setminus \Omega$ and with s odd. In the even case, we will start from a slightly different situation. First, we recall an interesting fixed elements theorem for SPGQ's.

Result 20. (S. E. Payne [16], see also 10.7.1 of FGQ) *Let $\mathcal{S}$ be an SPGQ of order (s, t), $s \neq 1 \neq t$, with base-span $\mathcal{L}$ and base-group G. If $\theta \neq \mathbf{1}$ is an element of G, then the substructure $\mathcal{S}_\theta = (P_\theta, B_\theta, I_\theta)$ of elements fixed by θ must be given by one of the following:*

(i) *$P_\theta = \emptyset$ and B_θ is a partial spread*[5] *containing $\mathcal{L}^\perp$.*

(ii) *There is a line $L \in \mathcal{L}$ for which P_θ is the set of points incident with L, and $M \sim L$ for each $M \in B_\theta$ ($\mathcal{L}^\perp \subseteq B_\theta$).*

(iii) *B_θ consists of $\mathcal{L}^\perp$ together with a subset B' of $\mathcal{L}$; P_θ consists of those points incident with lines of B'.*

(iv) *$\mathcal{S}_\theta$ is a subGQ of order (s, t') with $s \le t' < t$. This forces $t' = s$ and $t = s^2$.*

Suppose $\theta \neq \mathbf{1}$ is an element of G which fixes a point q of $\mathcal{S} \setminus \Omega$. Then by Result 20 the fixed elements structure of θ is a subGQ $\mathcal{S}_\theta$ of order s. It is clear that $\mathcal{S}_\theta$ is also span-symmetric with respect to the same base-span. Hence $G_\theta := G/N_\theta$ with N_θ the kernel of the action of G on $\mathcal{S}_\theta$ (we can speak of "an action" since G fixes $\mathcal{S}_\theta$) has order $s^3 - s$ by Chapter 10 of [21]; G_θ is precisely the base-group corresponding to $\mathcal{L}$ seen as a base-span of $\mathcal{S}_\theta$. Since by [43] there holds that $\mathcal{S}_\theta \cong \mathcal{Q}(4, s)$, there holds that $G_\theta \cong \mathbf{SL}_2(s)$.

Next, let x be an arbitrary point of $\mathcal{S} \setminus \mathcal{S}_\theta$, and consider the set of points $V = x^\perp \cap \mathcal{S}_\theta$. Note that $|V| = t + 1 = s^2 + 1$ because $\mathcal{S}_\theta$ is a GQ of order s and hence every line of $\mathcal{S}$ meets $\mathcal{S}_\theta$, see Chapter 2 of [21]. Then x cannot be fixed by θ – otherwise $\theta = \mathbf{1}$, see also Chapter 2 of [21] – and $\{x, x^\theta\} \subseteq V^\perp$. The GQ $\mathcal{S}$ has order (s, s^2), and hence $|\{x, x^\theta\}^{\perp\perp}| = 2$, see Chapter 1 of FGQ. Thus, since N_θ acts semi-regularly on the points of $\mathcal{S}$ outside $\mathcal{S}_\theta$, there follows that N_θ has size 2 if N_θ is not trivial. So, N_θ is a normal subgroup of G of order 2 and N_θ is thus contained in the center of G.

We obtain the following pathological situation:

- $G_\theta = G/N_\theta \cong \mathbf{SL}_2(s)$;
- $|N_\theta| = 2$ and, as a normal subgroup of order 2 of G, N_θ is contained in the center $Z(G)$ of G;
- $|G| = 2(s^3 - s)$.

Lemma 21. *Suppose $\mathcal{S}$ is an SPGQ of order (s, t), $s \neq 1 \neq t$, with base-span $\mathcal{L}$ and base-group G, and suppose s is the power of a prime p. If s is the largest power of p which divides $|G|$, then the groups of symmetries about the lines of $\mathcal{L}$ are precisely the Sylow p-subgroups of G, and hence G is generated by its Sylow p-subgroups.*

Proof. Put $\mathcal{L} = \{U_0, \ldots, U_s\}$, and suppose G_i is the full group of symmetries about U_i, $i = 0, 1, \ldots, s$. Since the groups G_j all have order s, there follows that all these groups are Sylow p-subgroups in G. By Lemma 10, the set $\mathcal{T} = \{G_0, \ldots, G_s\}$ is a complete conjugacy class in G, implying that every Sylow p-subgroup of G is contained in $\mathcal{T}$, and hence G has exactly $s + 1$ Sylow p-subgroups. □

Definition and notation. If G is a group, then by G' we denote the *derived group of G*, see [13]. If G is a group for which $G = G'$, then G is called a *perfect group.*

Lemma 22. *If G does not act semi-regularly on $\mathcal{S} \setminus \Omega$, then G is perfect if $s > 3$ and s odd.*

[5] This is a set of mutually non-concurrent lines, see also the addendum.

Proof. Suppose G does not act semi-regularly on $\mathcal{S}\setminus\Omega$, and suppose $G \neq G'$. Then $(G/N_\theta)' = G'N_\theta/N_\theta = G/N_\theta$ (the group $\mathbf{SL}_2(s)$ is perfect if $s \neq 2, 3$, see [13]), and hence $G'N_\theta = G$, and G' is a subgroup of G of index 2. Since s is odd, there follows that G and G' have *exactly* the same Sylow p-subgroups, with s a power of the odd prime p. The group G is generated by its subgroups G_i, and since these are precisely the Sylow p-subgroups by Lemma 21, there follows that $G = G'$, a contradiction. Hence G is perfect. □

The following notions and results are taken from [1, 1.4C]. Suppose G and H are groups. Then H is called a *central extension* of G if there is a surjective homomorphism

$$\phi : H \longrightarrow G$$

for which $ker(\phi) \leq Z(H)$ ($ker(\phi)$ is the *kernel* of the homomorphism ϕ, $Z(H)$ is the *center* of H). Sometimes the pair (H, ϕ) is also called a central extension of G. A central extension $(\overline{G}, \xi)$ of a group G is called *universal*, if for any other central extension (H, ξ') of G there exists a unique homomorphism

$$\psi : \overline{G} \longrightarrow H$$

such that the diagram defined by

$$\overline{G} \xrightarrow{\psi} H \xrightarrow{\xi'} G$$

and

$$\overline{G} \xrightarrow{\xi} G$$

commutes. If a group G has a universal central extension $\overline{G}$, then $\overline{G}$ is known to be unique, up to isomorphism.

Result 23. ([1]) *A group G has a universal central extension if and only if it is perfect. The universal central extension of a group is always perfect if it exists.*

Using the preceding remarks and Result 23, it is possible to prove the following well-known result.

Result 24. ([1]) *Suppose G is a perfect group, and suppose $\overline{G}$ is its universal central extension. Furthermore, let H be a perfect group which is a central extension of G. Then there exists a subgroup N of the center $Z(\overline{G})$ of $\overline{G}$, such that*

$$\overline{G}/N \cong H.$$

Lemma 25. *G acts semi-regularly on $\mathcal{S} \setminus \Omega$ if s is odd.*

Proof. First suppose that $s = 3$. Then $\mathcal{S} \cong \mathcal{Q}(4, 3)$ by Section 5, and then it is well-known that $G \cong \mathbf{SL}_2(3)$, hence G acts semi-regularly on $\mathcal{S} \setminus \Omega$ by Lemma 13 and the fact that $|\mathbf{SL}_2(q)| = q^3 - q$ for arbitrary q.

Next suppose that G does not act semi-regularly on the points of $\mathcal{S} \setminus \Omega$, and suppose $\mathcal{S}$ is of order (s, t), $1 < s \neq 3, 9$ and s odd. We then know that $G/N_\theta \cong \mathbf{SL}_2(s)$ with s a power of an odd prime p and $s \neq 3, 9$, and where N_θ is a central subgroup of order 2. The group G is perfect by Lemma 22 and has size $2(s^3 - s)$, see above. It is a known fact that the universal central extension of $\mathbf{SL}_2(s)$ coincides with $\mathbf{SL}_2(s)$ if $s \neq 4, 9$, see [13], and this contradicts the fact that $|G| = 2(s^3 - s)$. Hence G *does* act semi-regularly on the points of $\mathcal{S} \setminus \Omega$.

Finally, suppose that $s = 9$. It is a well-known fact, see e.g. [13], that, if $\overline{\mathbf{PSL}_2(9)}$ is the univeral central extension of $\mathbf{PSL}_2(9)$, there are four possibilities for the central extension G of $\mathbf{PSL}_2(9)$;

- $G = \mathbf{PSL}_2(9)$;
- $G \cong \overline{\mathbf{PSL}_2(9)}/C_2$;
- $G \cong \overline{\mathbf{PSL}_2(9)}$;
- $G \cong \mathbf{SL}_2(9)$.

Suppose that G does not act semi-regularly on $\mathcal{S} \setminus \Omega$. Then by Lemma 22 G is a perfect group, $G_\theta = G/N_\theta \cong \mathbf{SL}_2(9)$, and $N_\theta \leq Z(G)$ as a normal subgroup of G of order 2. Hence G is a perfect central extension of $\mathbf{SL}_2(9)$, and $|G| = 4|\mathbf{PSL}_2(9)| = 2|\mathbf{SL}_2(9)|$. This is clearly impossible. Hence G acts semi-regularly on $\mathcal{S} \setminus \Omega$. □

Lemma 26. *Suppose $\mathcal{S}$ is a GQ of order (s,t), where $s \neq t$ and $s \neq 1 \neq t$, and suppose L is a line of which every point is a translation point. Furthermore, fix two non-concurrent lines U and V of $L^\perp$ and suppose that s is even. If G is the base-group which is defined by the base-span $\{U,V\}^{\perp\perp}$, and G does not act semi-regularly on the points of $\mathcal{S} \setminus \Omega$, then $\mathcal{S}$ is classical, i.e. isomorphic to $\mathcal{Q}(5,s)$.*

Proof. If G does not act semi-regularly on $\mathcal{S} \setminus \Omega$, then by the same argument as in the odd case, there follows that $\mathcal{S}_\theta$ is a classical subGQ of order s, with $\mathcal{S}_\theta$ as above. Since L contains at least one translation point, it follows easily that $\mathcal{S}$ contains at least two classical subGQ's which contain a translation point of $\mathcal{S}$. Hence by Result 17, we have that $\mathcal{S}$ is isomorphic to the classical $\mathcal{Q}(5,s)$. □

Note. It follows that this case does not occur since G acts semi-regularly on $\mathcal{S} \setminus \Omega$ if $\mathcal{S} \cong \mathcal{Q}(5,s)$ (see further on).

7. The sharply 2-transitive case

In this section it is our goal to exclude the case where G/N acts sharply 2-transitive on the lines of $\mathcal{L}$, if $s > 3$ (for notations, see below). Recall that N is the kernel of the action of G on the lines of $\mathcal{L}$.

Definition. Suppose $\mathcal{S}$ is a GQ of order (s, s^2), $s \neq 1$. Then for any triad of points $\{p,q,r\}$, $|\{p,q,r\}^\perp| = s+1$, see 1.2.4 of [21]. Evidently $|\{p,q,r\}^{\perp\perp}| \leq s+1$. We say that $\{p,q,r\}$ is 3-*regular* provided that $|\{p,q,r\}^{\perp\perp}| = s+1$. A point p is 3-*regular* if each triad of points containing p is 3-regular.

Result 27. (J. A. Thas [34], see 2.6.1 of FGQ) *Let $\{x,y,z\}$ be a 3-regular triad of the GQ $\mathcal{S} = (P,B,I)$ of order (s,s^2), $s \neq 1$, and let P' be the set of all points incident with lines of the form uv, with $u \in \{x,y,z\}^\perp = \mathbf{X}$ and $v \in \{x,y,z\}^{\perp\perp} = \mathbf{Y}$. If L is a line which is incident with no point of $\mathbf{X} \cup \mathbf{Y}$ and if k is the number of points in P' which are incident with L, then $k \in \{0,2\}$ if s is odd and $k \in \{1, s+1\}$ if s is even.*

Let $\{x, y, z\}$ be a 3-regular triad of the GQ $\mathcal{S} = (P, B, I)$ of order (s, s^2), $s \neq 1$ and s even. Let P' be the set of all points incident with lines of the form uv, with $u \in \{x, y, z\}^\perp = \mathbf{X}$ and $v \in \{x, y, z\}^{\perp\perp} = \mathbf{Y}$, and let B' be the set of lines L which are incident with at least two points of P'. Then J. A. Thas proves in [34] (see also [21, 2.6.2]) that, with I' the restriction of I to $(P' \times B') \cup (B' \times P')$, the geometry $\mathcal{S}' = (P', B', I')$ is a subGQ of $\mathcal{S}$ of order s. Moreover, (x, y) is a regular pair of points of $\mathcal{S}'$, with $\{x, y\}^{\perp'} = \{x, y, z\}^\perp$ and $\{x, y\}^{\perp'\perp'} = \{x, y, z\}^{\perp\perp}$ (with the meaning of " $'$ " being obvious).

Lemma 28. *Let $\mathcal{S}$ be an SPGQ of order (s, t), $s \neq 1 \neq t$, with base-span $\mathcal{L}$, base-grid $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^\perp, I')$ and base-group G. Furthermore, let N be the kernel of the action of G on $\mathcal{L}$. If G acts semi-regularly on $\mathcal{S} \setminus \Omega$, then G/N cannot act as a sharply 2-transitive group on the lines of $\mathcal{L}$ if $s > 3$.*

Proof. Suppose G/N acts as a sharply two-transitive group on the lines of $\mathcal{L}$. Then $|G/N| = (s+1)s$. Hence, since $|G| \geq s^3 - s$, we have that $|N| \geq s - 1$. Let q be an arbitrary point of $\mathcal{S} \setminus \Omega$, and define V as $V := q^\perp \cap \Omega$ (so $|V| = s + 1$). Then since G acts semi-regularly on the points of $\mathcal{S} \setminus \Omega$, there follows that $|N| = |q^N|$, and $q^N \subseteq V^\perp$. The GQ $\mathcal{S}$ is of order (s, s^2), $s \neq 1$, hence every triad of points has exactly $s + 1$ centers, see FGQ [21, 1.2.4]. So, we immediately have that $|N| = |q^N| \leq s + 1$.

Now suppose that $|N| \neq s - 1$, so that $|N| \in \{s, s + 1\}$. First suppose $|N| = s$. Then the order of G is $s^2(s + 1)$, and by the semi-regularity condition this must be a divisor of $|\mathcal{S} \setminus \Omega| = (s + 1)(s^3 - s)$, clearly a contradiction. Hence $|N| = s + 1$.

Now suppose that s is odd. Since G/N is supposed to be a sharply 2-transitive group on the lines of $\mathcal{L}$, there follows that $|G| = (s + 1)^2 s$. Suppose that Λ is a G-orbit in $\mathcal{S} \setminus \Omega$, so since G acts semi-regularly on $\mathcal{S} \setminus \Omega$ there holds that $|\Lambda| = |G|$. Consider an arbitrary point p in Λ. Then every point of $\mathbf{X} = p^N$ is collinear with every point of $\mathbf{Y} = p^\perp \cap \Omega$, and we denote the set of points which are on a line of the form uv with $u \in \mathbf{X}$ and $v \in \mathbf{Y}$ by $\mathbf{XY}$. It is clear that $\mathbf{XY} \setminus \mathbf{Y}$ is completely contained in Λ, and that the order of this set is $(s + 1)s^2$. Now take a point q of Λ outside $\mathbf{XY}$. The points of $\mathbf{X}$ and $\mathbf{Y}$ are the points of a dual grid with parameters $(s + 1, s + 1)$, and hence, if x, y and z are arbitrary distinct points of $\mathbf{Y}$ (or $\mathbf{X}$), there follows that the triad $\{x, y, z\}$ is 3-regular. Put $q^N = \{q = q^0, q^1, \ldots, q^s\}$. If q^i is an arbitrary point of q^N, then there follows that $|\mathbf{Y} \cap (q^i)^\perp| =: k_{q_i} \leq 2$. One notes that $k_{q_i} = k_{q_j} =: k$ for some constant k, and that $\mathbf{Y} \cap (q^i)^\perp = \mathbf{Y} \cap (q^j)^\perp$, for all i and j, by the action of N. If W is an arbitrary line through q which intersects Ω, then W does not contain a point of $\mathbf{X}$ since this would imply that q is not outside $\mathbf{XY}$. Applying Result 27 (and recalling the fact that s is odd), we count the number of points which are collinear with a point of q^N and contained in Λ, together with the points of $\mathbf{XY} \cap \Lambda$. One notes that every point of q^N is collinear with every point of $q^\perp \cap \Omega$, and also that q^N is skew to $\mathbf{XY}$. We obtain the following.

$$|\Lambda| = (s + 1)^2 s \geq (s + 1)s^2 + s + 1 + k(s + 1)(s - 1) + (s + 1)(s + 1 - k)(s - 3),$$

with $k \in \{0, 1, 2\}$, from which follows that $s < 4$, a contradiction. Hence this case is excluded.

Next, suppose that s is even. From the fact that $|N| = s+1$, there follows that $\mathcal{S}$ contains a 3-regular triad, and hence a subGQ $\mathcal{S}'$ of order s. There follows that $|\mathbf{XY}| = |\mathcal{S}'|$. Take a point q of Λ outside $\mathbf{XY}$. Then every line which is incident with q and which intersects Ω has exactly one point in common with $\mathcal{S}'$. Counting the number of points which are collinear with a point of q^N and contained in Λ, together with the points of $\mathbf{XY} \cap \Lambda$, we get the following inequality.

$$|\Lambda| = (s+1)^2 s \geq (s+1)s^2 + s + 1 + (s+1)^2(s-2),$$

and thus there holds that $2s+1 \geq s^2$, a contradiction if $s \geq 3$. The proof is complete. $\square$

8. Construction of subquadrangles

Suppose $\mathcal{S}$ is an SPGQ of order (s, s^2), $s \neq 1$, with base-span $\{U, V\}^{\perp\perp}$, base-group G and base-grid $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^{\perp}, I')$. Also, suppose that G acts semi-regularly on the points of $\mathcal{S} \setminus \Omega$ and that G has order $(s+1)s(s-1)$. Let Λ be an arbitrary G-orbit in $\mathcal{S} \setminus \Omega$, and fix a line W of $\mathcal{L}^{\perp}$. By the semi-regularity of G onto the points of $\mathcal{S} \setminus \Omega$, the fact that $|G| = (s+1)s(s-1)$ and that G acts transitively on the points of W, we have that any point on W is incident with exactly $s-1$ lines of $\mathcal{S}$ which are completely contained in Λ except for the point on W which is in Ω, and every point of Λ is incident with a line which meets W (recall that G is generated by groups of symmetries). Now define the following incidence structure $\mathcal{S}' = (P', B', I')$;

- Lines. The elements of B' are the lines of $\mathcal{S}'$ and they are essentially of two types:
    1. the lines of $\{U, V\}^{\perp} \cup \{U, V\}^{\perp\perp}$;
    2. the lines of $\mathcal{S}$ which contain a point of Λ and a point of Ω.
- Points. The elements of P' are the points of the incidence structure and they are just the points of $\Omega \cup \Lambda$.
- Incidence. Incidence I' is the 'induced incidence'.

Then by the previous lemmas and observations, any point of $\mathcal{S}'$ is incident with $s+1$ lines of $\mathcal{S}'$ and any line of $\mathcal{S}'$ is incident with $s+1$ points of the structure, and there are exactly $(s+1)(s^2+1)$ points and equally as many lines, and hence one can easily conclude that $\mathcal{S}'$ is a generalized quadrangle of order s (since it is a subgeometry of a GQ, it cannot contain triangles).

Remark 29. The GQ $\mathcal{S}'$ is isomorphic to the GQ $\mathcal{Q}(4, s)$ by K. Thas [43], since $\mathcal{S}'$ is clearly span-symmetric for the base-span $\mathcal{L}$.

Lemma 30. *Suppose that $\mathcal{S}$ is a span-symmetric generalized quadrangle of order (s, t), $s, t \neq 1$, with base-span $\{L, M\}^{\perp\perp} = \mathcal{L}$, base-group G and base-grid $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^{\perp}, I')$. Furthermore, let N be the kernel of the action of G on $\mathcal{L}$. If G/N does not act as a sharply 2-transitive group on the lines of $\mathcal{L}$ and G acts semi-regularly on the points of $\mathcal{S} \setminus \Omega$, then G is perfect.*

Proof. If G/N does not act as a sharply 2-transitive group on $\mathcal{L}$, then we proved in Theorem 11 that G/N is isomorphic to one of the following list: (a) $\mathbf{PSL}_2(s)$, (b) $\mathbf{R}(\sqrt[3]{s})$, (c) $\mathbf{Sz}(\sqrt{s})$, or

(d) $\mathbf{PSU}_3(\sqrt[3]{s^2})$. All these groups are perfect groups, see [13]. Hence, since G/N is a perfect group, there follows that

$$(G/N)' = (G/N) = G'N/N,$$

and thus $G'N = G$.

Since G acts semi-regularly on $\mathcal{S} \setminus \Omega$, there follows that $|G| = |G/N| \times |N| = (s^n - 1)(s+1)s/r \times |N|$, with $r \in \{1, 2, (3, \sqrt[3]{s}+1)\}$ and $n \in \{1, 2/3, 1/2, 1/3\}$, is a divisor of $|\mathcal{S} \setminus \Omega| = (s+1)(s^3 - s)$, where $r = 2$ if and only if s is even and $G/N \cong \mathbf{PSL}_2(s)$ and where $r = (3, \sqrt[3]{s}+1)$ if and only if $G/N \cong \mathbf{PSU}_3(\sqrt[3]{s^2})$.

Hence, we have that

$$r(s^2 - 1)/(s^n - 1) \equiv 0 \; mod \; |N|. \tag{4}$$

First suppose that s is odd, or that s is even and $G/N \not\cong \mathbf{PSL}_2(s)$.

If $r = (3, \sqrt[3]{s}+1)$ and $G/N \cong \mathbf{PSU}_3(\sqrt[3]{s^2})$, then s and $|N|$ have a nontrivial common divisor if and only if $r = 3$ and if 3 is a divisor of s, clearly in contradiction with $3 = (3, \sqrt[3]{s}+1)$. It follows now immediately from (4) that $|N|$ and s are coprime. Hence with $s = p^h$ for some odd prime p and $h \in \mathbb{N}_0$, s is the largest power of p which divides $|G|$. Thus the full groups of symmetries about the lines of $\mathcal{L}$ are exactly the Sylow p-subgroup of G by Lemma 21.

Now suppose $G \neq G'$. We know that $|N|$ and s are coprime, so since $|G'| = (|G| \times |G' \cap N|)/|N|$, there follows that $|G'| \equiv 0 \; mod \; s$ and G and G' have exactly the same Sylow p-subgroups. But $G' \leq G$ and G is generated by its Sylow p-subgroups, so $G = G'$, a contradiction. Hence G is perfect. Next, suppose that s is even and that $G/N \cong \mathbf{PSL}_2(s)$.

Then we know that $|G| = |G/N| \times |N| = |N| \times |\mathbf{PSL}_2(s)| = |N| \times (s^3 - s)$ is a divisor of $(s+1)(s^3 - s)$, and again there follows that s and $|N|$ are mutually coprime. In the same way as before, there follows now that G is a perfect group. □

Remark 31. For $s = 2$ or $s = 3$, $G/N \cong \mathbf{PSL}_2(2)$ or $\mathbf{PSL}_2(3)$ since $\mathcal{S}$ is classical, and in both cases G/N acts sharply 2-transitive on $\mathcal{L}$.

The following crucial result is taken from [43], but we repeat the proof here since that paper essentially only covers the case $s = t$ (although that specific theorem was also valid in general).

Lemma 32. (K. Thas [43]) *Suppose $\mathcal{S}$ is an SPGQ of order (s,t), $s \neq 1 \neq t$, with base-span $\mathcal{L}$ and base-group G. If N is the kernel of the action of G on the lines of $\mathcal{L}$, then N is in the center of G.*

Proof. Fix non-concurrent lines U and U' of $\mathcal{L}$, and suppose that N is the kernel of the action of G on the lines of $\{U, U'\}^{\perp\perp}$ (so N fixes every point of Ω). Then N is a normal subgroup of G. Let H be the full group of symmetries about an arbitrary line M of $\{U, U'\}^{\perp\perp}$. Then N and H normalize each other, and hence they commute. □

Lemma 33. *Suppose that $\mathcal{S}$ is a span-symmetric generalized quadrangle of order (s,t), $s, t \neq 1$, with base-span $\mathcal{L}$, base-group G and base-grid $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^\perp, I')$. Furthermore, let N be the kernel of the action of G on $\mathcal{L}$. If G/N does not act as a sharply 2-transitive group on $\mathcal{L}$, then G is a perfect group.*

Proof. By Section 6 there follows that G acts semi-regularly on the points of $\mathcal{S} \setminus \Omega$, and by Lemma 30 the lemma follows. □

Lemma 34. *If $\mathcal{S}$ is an SPGQ of order (s,t), $s \neq 1 \neq t$ and $s \neq t$, with base-group G and base-span $\mathcal{L}$, then G/N acts either as a $\mathbf{PSL}_2(s)$ or a sharply 2-transitive group on the lines of $\mathcal{L}$.*

Proof. Assume by way of contradiction that G/N does not act as a $\mathbf{PSL}_2(s)$ or a sharply 2-transitive group on the lines of $\mathcal{L}$. First of all, by Lemma 33, G is a perfect group, and since N is in the center of G, we have that the group G is a perfect central extension of the group G/N which acts on $\mathcal{L}$. Since G is perfect, there is a subgroup F of the center of the universal central extension $\overline{G/N}$ of G/N for which $\overline{G/N}/F \cong G$. We now look at the possible cases.

1. If $G/N \cong \mathbf{Sz}(\sqrt{s})$, then N must be trivial if $\sqrt{s} > 8$ since the Suzuki group has a trivial universal central extension (i.e. $\overline{G/N} \cong G/N$) if $\sqrt{s} > 8$ [13], an impossibility since the orders of G and $\mathbf{Sz}(\sqrt{s})$ are not the same. If $\sqrt{s} = 8$, then we have the following possibilities for the orders of any perfect central extension H of $\mathbf{Sz}(8)$.
   (a) $|H| = |\mathbf{Sz}(8)|$;
   (b) $|H| = 2|\mathbf{Sz}(8)|$;
   (c) $|H| = 4|\mathbf{Sz}(8)|$.

   None of these cases occurs since $|G| \geq (64)^3 - 64 = 262080$ and since $|\mathbf{Sz}(8)| = 29120$.
2. If $G/N \cong \mathbf{R}(\sqrt[3]{s})$, then we have exactly the same situation as in the preceding case [13], hence this case is excluded as well.
3. Finally, assume that $G/N \cong \mathbf{PSU}_3(\sqrt[3]{s^2})$. The universal extension of $\mathbf{PSU}_3(\sqrt[3]{s^2})$ is known to be $\mathbf{SU}_3(\sqrt[3]{s^2})$ [13], and also, we know that $|\mathbf{SU}_3(\sqrt[3]{s})| = (3, \sqrt[3]{s}+1)|\mathbf{PSU}_3(\sqrt[3]{s})| = (s+1)s(\sqrt[3]{s^2} - 1)$ [7, 11]. This provides us the contradiction since with $s > 1$ there follows that $s - 1 > \sqrt[3]{s^2} - 1$.

The assertion is proved. □

Lemma 35. *Suppose $\mathcal{S}$ is an SPGQ of order (s,t), $s \neq 1 \neq t$ and $s \neq t$, with base-group G and base-span $\mathcal{L}$. If s is odd, then G is always of order $s^3 - s$ and G acts semi-regularly on the points of $\mathcal{S} \setminus \Omega$.*

Proof. For $s = 3$ the case is already settled, so we can suppose that $s \neq 3$. Putting the results of the preceding sections together, we obtain the following properties if $s \neq 3$.

1. If G acts semi-regularly on $\mathcal{S} \setminus \Omega$, then G/N cannot act as a sharply 2-transitive group on $\mathcal{L}$ (where N is the kernel of the action of G on $\mathcal{L}$).
2. G acts semi-regularly on $\mathcal{S} \setminus \Omega$.
3. G acts sharply 2-transitive on the lines of $\mathcal{L}$ or as a $\mathbf{PSL}_2(s)$.
4. If $G/N \cong \mathbf{PSL}_2(s)$ and G acts semi-regularly on the points of $\mathcal{S} \setminus \Omega$, then G is a perfect group.
5. $|G| \geq s^3 - s$.
6. N is contained in the center of G.

By (1), (2) and (3), we know that $G/N \cong \mathbf{PSL}_2(s)$, and by (4) we can conclude that G is a perfect group. Since N is contained in the center of G, this means that G is a perfect central extension of $\mathbf{PSL}_2(s)$. By definition, this also means that G is a central quotient of the universal central extension of $\mathbf{PSL}_2(s)$, which we denote by $\overline{\mathbf{PSL}_2(s)}$.

First suppose $s \neq 9$. Because of (5), G does not coincide with $\mathbf{PSL}_2(s)$, and if $s \neq 9, 4$, then it is known that $\overline{\mathbf{PSL}_2(s)} = \mathbf{SL}_2(s)$, and $|\mathbf{SL}_2(s)| = s^3 - s$, see [13]. From (5) it follows now immediately that $G \cong \mathbf{SL}_2(s)$, and hence the order of G is precisely $s^3 - s$.

Now put $s = 9$. We already know that G is a perfect central extension of $\mathbf{PSL}_2(9)$. We recall from Lemma 30 that, since $G/N \cong \mathbf{PSL}_2(9)$, $|N|$ is a divisor of $2(s+1) = 20$, and from previous observations we can assume that $|N| < s+1$. It follows directly that $|N| \in \{2, 4, 5\}$. Recalling the possible perfect central extensions of $\mathbf{PSL}_2(9)$, it is clear that $|N| \notin \{4, 5\}$, and hence $|N| = 2$. Thus $G \cong \mathbf{SL}_2(9)$. Thus, we can conclude that $|G| = 9^3 - 9$. □

Lemma 36. *If s is even and if every line which meets some line $L \in \{U, V\}^\perp$ is an axis of symmetry, then G is always of order $s^3 - s$ and G acts semi-regularly on the points of $\mathcal{S} \setminus \Omega$.*

Proof. In this case, we also have the properties (1)–(5). And as in the proof of Lemma 35, there follows that G is a perfect central extension of $\mathbf{PSL}_2(s)$. Since $\overline{\mathbf{PSL}_2(s)} = \mathbf{SL}_2(s) \cong \mathbf{PSL}_2(s)$ if s is even and $s \neq 4$, there follows by (5) that $G \cong \mathbf{SL}_2(s)$, hence $|G| = s^3 - s$.

Now put $s = 4$. Then by Result 19, $\mathcal{S} \cong \mathcal{Q}(5, 4)$ and the assertion becomes trivial. □

Theorem 37. *Suppose $\mathcal{S}$ is a span-symmetric generalized quadrangle with base-span $\{U, V\}^{\perp\perp}$ and of order (s, t), $s \neq 1 \neq t$ and $s \neq t$. Furthermore, suppose G acts semi-regularly on $\mathcal{S} \setminus \Omega$ and that G has order $s^3 - s$. Then there exist $s + 1$ subquadrangles of order s which are all isomorphic to $\mathcal{Q}(4, s)$, and such that they mutually intersect exactly in the points and lines of $\{U, V\}^\perp \cup \{U, V\}^{\perp\perp}$.*

Proof. From each G-orbit in $\mathcal{S} \setminus \Omega$ there arises a GQ of order s which is classical by Remark 29, i.e. isomorphic to $\mathcal{Q}(4, s)$. There are exactly $s + 1$ such distinct G-orbits, and G fixes each orbit. □

9. Proof of Theorem 1 and Theorem 2

In this section we prove two strong results on (general) span-symmetric GQ's. The first result is an improvement of Result 5 of Kantor.

9.1. Proof of Theorem 1

If $\mathcal{S}$ is an SPQG of order (s, t), $s \neq 1 \neq t$, then $s = t$ or $t = s^2$ by Result 5. If $s = t$, then by Result 6, $\mathcal{S} \cong \mathcal{Q}(4, s)$, and hence s is a prime power. Now suppose $t = s^2$. Then by Section 7, Lemma 35 and Lemma 36 the theorem follows. □

9.2. Proof of Theorem 2

Suppose $\mathcal{S}$ is an SPGQ with base-span $\{U,V\}^{\perp\perp}$ and of order (s,t), $s,t \neq 1$ and $s \neq t$, s odd. Then G acts semi-regularly on $\mathcal{S} \setminus \Omega$ and G has order $s^3 - s$. Consequently, there exist $s+1$ subquadrangles of order s which are all isomorphic to $\mathcal{Q}(4,s)$, and such that they mutually intersect exactly in the points and lines of $\{U,V\}^{\perp} \cup \{U,V\}^{\perp\perp}$. □

Appendix. In [46], we have established the even case of Theorem 2.

10. Flock generalized quadrangles and property (G)

10.1. Property (G)

Let $\mathcal{S}$ be a generalized quadrangle of order (s,s^2), $s \neq 1$. Let x_1, y_1 be distinct collinear points. We say that the pair $\{x_1,y_1\}$ has *Property (G)*, or that $\mathcal{S}$ has *Property (G) at* $\{x_1,y_1\}$, if every triad $\{x_1,x_2,x_3\}$ of points for which $y_1 \in \{x_1,x_2,x_3\}^{\perp}$ is 3-regular. The GQ $\mathcal{S}$ has *Property (G) at the line* L, or the line L has Property (G), if each pair of points $\{x,y\}$, $x \neq y$ and $xILIy$, has Property (G). If (x,L) is a flag, then we say that $\mathcal{S}$ has *Property (G) at* (x,L), or that (x,L) has Property (G), if every pair $\{x,y\}$, $x \neq y$ and yIL, has Property (G). Property (G) was introduced by S. E. Payne [18] in connection with generalized quadrangles of order (q^2,q) arising from flocks of quadratic cones in $\mathbf{PG}(3,q)$, see below.

10.2. Flock generalized quadrangles, 4-gonal families and *q*-clans

Suppose $(\mathcal{S}^{(p)},G)$ is an EGQ of order (s,t), $s,t \neq 1$, with elation point p and elation group G, and let q be a point of $P \setminus p^{\perp}$. Let $L_0, L_1, \ldots, L_t$ be the lines incident with p, and define r_i and M_i by $L_i I r_i I M_i I q$, $0 \leq i \leq t$. Put $H_i = \{\theta \in G \parallel M_i^{\theta} = M_i\}$, $H_i^* = \{\theta \in G \parallel r_i^{\theta} = r_i\}$, and $\mathcal{J} = \{H_i \parallel 0 \leq i \leq t\}$. Also, set $\mathcal{J}^* = \{H_i^* \parallel 0 \leq i \leq t\}$. Then $|G| = s^2 t$ and $\mathcal{J}$ is a set of $t+1$ subgroups of G, each of order s. Also, for each i, H_i^* is a subgroup of G of order st containing H_i as a subgroup. Moreover, the following two conditions are satisfied:

(K1) $H_i H_j \cap H_k = \mathbf{1}$ for distinct i,j and k;

(K2) $H_i^* \cap H_j = \mathbf{1}$ for distinct i and j.

Conversely, if G is a group of order $s^2 t$ and $\mathcal{J}$ (respectively $\mathcal{J}^*$) is a set of $t+1$ (respectively $t+1$) subgroups H_i (respectively H_i^*) of G of order s (respectively of order st), and if the conditions (K1) and (K2) are satisfied, then the H_i^* are uniquely defined by the H_i, and $(\mathcal{J},\mathcal{J}^*)$ is said to be a *4-gonal family of type* (s,t) *in* G.

Let $(\mathcal{J},\mathcal{J}^*)$ be a 4-gonal family of type (s,t) in the group G of order s^2t, $s,t > 1$. Define an incidence structure $\mathcal{S}(G,\mathcal{J})$ as follows.

- Points of $\mathcal{S}(G,\mathcal{J})$ are of three kinds:
  - (i) elements of G;
  - (ii) right cosets $H_i^* g$, $g \in G$, $i \in \{0,\ldots,t\}$;
  - (iii) a symbol (∞).

- LINES are of two kinds:
  (a) right cosets $H_i g$, $g \in G$, $i \in \{0, \dots, t\}$;
  (b) symbols $[H_i]$, $i \in \{0, \dots, t\}$.
- INCIDENCE. A point g of type (i) is incident with each line $H_i g$, $0 \leq i \leq t$. A point $H_i^* g$ of type (ii) is incident with $[H_i]$ and with each line $H_i h$ contained in $H_i^* g$. The point (∞) is incident with each line $[H_i]$ of type (b). There are no further incidences.

It is straightforward to check that the incidence structure $\mathcal{S}(G, \mathcal{J})$ is a GQ of order (s,t). Moreover, if we start with an EGQ $(\mathcal{S}^{(p)}, G)$ to obtain the family $\mathcal{J}$ as above, then we have that $(\mathcal{S}^{(p)}, G) \cong \mathcal{S}(G, \mathcal{J})$; for any $h \in G$ let us define θ_h by $g^{\theta_h} = gh$, $(H_i g)^{\theta_h} = H_i gh$, $(H_i^* g)^{\theta_h} = H_i^* gh$, $[H_i]^{\theta_h} = [H_i]$, $(\infty)^{\theta_h} = (\infty)$, with $g \in G$, $H_i \in \mathcal{J}$, $H_i^* \in \mathcal{J}^*$. Then θ_h is an automorphism of $\mathcal{S}(G, \mathcal{J})$ which fixes the point (∞) and all lines of type (b). If $G' = \{\theta_h \parallel h \in G\}$, then clearly $G' \cong G$ and G' acts regularly on the points of type (1). Hence, a group of order $s^2 t$ admitting a 4-gonal family can be represented as a general elation group of a suitable elation generalized quadrangle. This was first noted by Kantor [12].

Let $\mathbb{F} = \mathbf{GF}(q)$, q any prime power, and put $G = \{(\alpha, c, \beta) \parallel \alpha, \beta \in \mathbb{F}^2, c \in \mathbb{F}\}$. Define a binary operation on G by $(\alpha, c, \beta)(\alpha', c', \beta') = (\alpha + \alpha', c + c' + \beta\alpha'^T, \beta + \beta')$. This makes G into a group whose center is $C = \{(0, c, 0) \in G \parallel c \in \mathbb{F}\}$. Let $\mathcal{C} = \{A_u : u \in \mathbb{F}\}$ be a set of q distinct upper triangular 2×2-matrices over $\mathbb{F}$. Then $\mathcal{C}$ is called a *q-clan* provided $A_u - A_r$ is *anisotropic* whenever $u \neq r$, i.e. $\alpha(A_u - A_r)\alpha^T = 0$ has only the trivial solution $\alpha = (0,0)$. For $A_u \in \mathcal{C}$, put $K_u = A_u + A_u^T$. Let $A_u = \begin{pmatrix} x_u & y_u \\ 0 & z_u \end{pmatrix}$, $x_u, y_u, z_u, u \in \mathbb{F}$. For q odd, $\mathcal{C}$ is a q-clan if and only if

$$-\det(K_u - K_r) = (y_u - y_r)^2 - 4(x_u - x_r)(z_u - z_r) \tag{5}$$

is a nonsquare of $\mathbb{F}$ whenever $r, u \in \mathbb{F}$, $r \neq u$. For q even, $\mathcal{C}$ is a q-clan if and only if

$$y_u \neq y_r \text{ and } tr((x_u + x_r)(z_u + z_r)(y_u + y_r)^{-2}) = 1 \tag{6}$$

whenever $r, u \in \mathbb{F}$, $r \neq u$.

Now we can define a family of subgroups of G by $A(u) = \{(\alpha, \alpha A_u \alpha^T, \alpha K_u) \in G \parallel \alpha \in \mathbb{F}^2\}, u \in \mathbb{F}$, and $A(\infty) = \{(0, 0, \beta) \in G \parallel \beta \in \mathbb{F}^2\}$. Then put $\mathcal{J} = \{A(u) \parallel u \in \mathbb{F} \cup \{\infty\}\}$ and $\mathcal{J}^* = \{A^*(u) \parallel u \in \mathbb{F} \cup \{\infty\}\}$, with $A^*(u) = A(u)C$. So $A^*(u) = \{(\alpha, c, \alpha K_u) \in G \parallel \alpha \in \mathbb{F}^2, c \in \mathbb{F}\}, u \in \mathbb{F}$,[6] and $A^*(\infty) = \{(0, c, \beta) \parallel \beta \in \mathbb{F}^2\}$. With $G, A(u), A^*(u), \mathcal{J}$ and $\mathcal{J}^*$ as above, the following important theorem is a combination of results of S. E. Payne and W. M. Kantor.

Result 38. (S. E. Payne [15], W. M. Kantor [12]) *The pair $(\mathcal{J}, \mathcal{J}^*)$ is a 4-gonal family for G if and only if $\mathcal{C}$ is a q-clan.*

Let $\mathcal{F}$ be a *flock* of a quadratic cone $\mathcal{K}$ with vertex v of $\mathbf{PG}(3, q)$, that is, a partition of $\mathcal{K} \setminus \{v\}$ into q disjoint (irreducible) conics.

[6] Note that in fact $A^*(u) = \{(\alpha, \alpha A_u \alpha^T + c, \alpha K_u) \in G \parallel \alpha \in \mathbb{F}^2, c \in \mathbb{F}\}$ but this yields of course the same group.

In his celebrated paper on flock geometry [27], J. A. Thas showed in an algebraic way that (5) and (6) are exactly the conditions for the planes

$$x_u X_0 + z_u X_1 + y_u X_2 + X_3 = 0$$

of $\mathbf{PG}(3,q)$ to define a flock of the quadratic cone K with equation $X_0X_1 = X_2^2$. Hence we have the following theorem.

Result 39. (J. A. Thas [27]) *To any flock of the quadratic cone of* $\mathbf{PG}(3,q)$ *corresponds a GQ of order* (q^2,q).

The following important theorem on flock GQ's is due to Payne.

Result 40. (S. E. Payne [18]) *Any flock GQ satisfies Property (G) at its point* (∞).

Now we come to the main theorem of the masterful sequence of papers [29], [31], [33], [32], [36] of J. A. Thas; it is a converse of the previous theorem and the solution of a longstanding conjecture.

Result 41. (J. A. Thas [33]) *Let* $\mathcal{S} = (P,B,I)$ *be a GQ of order* (q^2,q), $q > 1$, *and assume that* $\mathcal{S}$ *satisfies Property (G) at the flag* (x,L). *If* q *is odd then* $\mathcal{S}$ *is the dual of a flock GQ. If* q *is even and all ovoids* $\mathcal{O}_z$ *(see Section* 5 *of* [33]*) are elliptic quadrics, then we have the same conclusion.*

Remark 42. The point (∞) of a flock GQ $\mathcal{S}(\mathcal{F})$ always is a center of symmetry.

11. Translation generalized quadrangles and generalized ovoids

Suppose $H = \mathbf{PG}(2n+m-1,q)$ is the finite projective $(2n+m-1)$-space over $\mathbf{GF}(q)$, and let H be embedded in a $\mathbf{PG}(2n+m,q)$, say H'. Now define a set $\mathcal{O} = \mathcal{O}(n,m,q)$ of subspaces as follows: $\mathcal{O}$ is a set of q^m+1 $(n-1)$-dimensional subspaces, denoted $\mathbf{PG}(n-1)^{(i)}$, of H, every three of which generate a $\mathbf{PG}(3n-1,q)$ and such that the following condition is satisfied: for every $i = 0,1,\ldots,q^m$ there is a subspace $\mathbf{PG}(n+m-1,q)^i$ of H of dimension $n+m-1$, which contains $\mathbf{PG}(n-1,q)^{(i)}$ and which is disjoint from any $\mathbf{PG}(n-1,q)^{(j)}$ if $j \neq i$. If $\mathcal{O}$ satisfies all these conditions, then $\mathcal{O}$ is called a *generalized ovoid*, or an *egg*. The spaces $\mathbf{PG}(n+m-1,q)^{(i)}$ are the *tangent spaces* of the egg, or just the *tangents*.

Now let $\mathcal{O}(n,m,q)$ be an egg of $H = \mathbf{PG}(2n+m-1,q)$, and define a point-line incidence structure $T(n,m,q)$ as follows.

- The POINTS are of three types.
  1. A symbol (∞).
  2. The subspaces $\mathbf{PG}(n+m,q)$ of H' which intersect H in a $\mathbf{PG}(n+m-1,q)^{(i)}$.
  3. The points of $H' \setminus H$.
- The LINES are of two types.
  (a) The elements of the egg $\mathcal{O}(n,m,q)$.

(b) The subspaces $\mathbf{PG}(n,q)$ of $\mathbf{PG}(2n+m,q)$ which intersect H in an element of the egg.

- INCIDENCE is defined as follows: the point (∞) is incident with all the lines of type (a) and with no other lines; the points of type (2) are incident with the unique line of type (a) contained in it and with all the lines of type (b) which they contain (as subspaces), and finally, a point of type (3) is incident with the lines of type (b) that contain it.

Then J. A. Thas [26] for some particular cases, and S. E. Payne and J. A. Thas [21] for the general case, prove that $T(n,m,q)$ is a TGQ of order (q^n,q^m), and that, conversely, any TGQ can be seen in this way. Hence, *the study of translation generalized quadrangles is equivalent to the study of the generalized ovoids.*

If $n \neq m$, then by [21, 8.7.2] the q^m+1 tangent spaces of $\mathcal{O}(n,m,q)$ form an egg $\mathcal{O}^*(n,m,q)$ in the dual space of $\mathbf{PG}(2n+m-1,q)$. So in addition to $T(n,m,q)$ there arises a TGQ $T(\mathcal{O}^*)$, also denoted $T^*(n,m,q)$, or $T^*(\mathcal{O})$. The TGQ $T^*(\mathcal{O})$ is called the *translation dual* of the TGQ $T(\mathcal{O})$.

Each TGQ $\mathcal{S}$ of order $(s,s^{\frac{a+1}{a}})$, with translation point (∞), where a is odd and $s \neq 1$, has a *kernel* $\mathbb{K}$, which is a field with a multiplicative group isomorphic to the group of all collineations of $\mathcal{S}$ fixing the point (∞), and any given point not collinear with (∞), linewise. We have $|\mathbb{K}| \leq s$, see [21]. The field $\mathbf{GF}(q)$ is a subfield of $\mathbb{K}$ if and only if $\mathcal{S}$ is of type $T(n,m,q)$, see [21]. The TGQ $\mathcal{S}$ is isomorphic to a $T_3(\mathcal{O})$ of Tits with $\mathcal{O}$ an ovoid in $\mathbf{PG}(3,s)$ if and only if $|\mathbb{K}| = s$. The TGQ $T(\mathcal{O})$ and its translation dual $T(\mathcal{O}^*)$ have isomorphic kernels.

Now suppose that $\mathcal{S}^{(p)} = (P,B,I)$ is a TGQ of order (q^n,q^{2n}), $q \neq 1$; then we can see $\mathcal{S}$ as a $T(n,2n,q)$ associated to a generalized ovoid $\mathcal{O}(n,2n,q)$ in $\mathbf{PG}(4n-1,q)$. Suppose $\mathcal{S}'$ is a subGQ of order q of $\mathcal{S}$ which contains the point p. Then $\mathcal{S}'$ is also a TGQ, and in the projective model $\mathcal{S}'$ of Thas $\mathcal{S}'$ is defined by an $\mathcal{O}(n,n,q) =: \mathcal{O}' \subset \mathcal{O}(n,2n,q)$ which is contained in a $\mathbf{PG}(3n-1,q) \subset \mathbf{PG}(4n-1,q)$. $\mathcal{O}(n,n,q)$ is sometimes called a *generalized oval* of $\mathbf{PG}(3n-1,q)$. Now let $\mathcal{S}'$ be classical, i.e. isomorphic to $\mathcal{Q}(4,q)$ ($\mathcal{Q}(4,q)$ is the only classical GQ of order q having regular lines, up to isomorphism, see FGQ), and $\mathcal{O}'$ is said to be a *generalized conic* of $\mathbf{PG}(3n-1,q)$.

A TGQ $T(\mathcal{O})$ of order (q^n,q^m), $n \neq m$, is called *good* for an element $\pi \in \mathcal{O}$ if for every two distinct elements π' and π'' of $\mathcal{O} \setminus \{\pi\}$ the $(3n-1)$-space $\pi\pi'\pi''$ contains exactly q^n+1 elements of $\mathcal{O}$ (and is disjoint with the other elements). If the egg $\mathcal{O}$ contains a good element, then the egg is subconsequently called *good*, and for a good egg $\mathcal{O}(n,m,q)$ there holds that $m = 2n$.

Result 43. (J. A. Thas [29]) *If the TGQ $\mathcal{S}^{(\infty)}$ contains a good element π, then its translation dual satisfies Property (G) for the corresponding flag $((\infty)',\pi')$.*

Result 44. (J. A. Thas [37]) *Suppose $\mathcal{S} = T(\mathcal{O})$ is a TGQ of order (s,s^2) with s even, with $\mathcal{O}$ a generalized ovoid which is good at an element π. Then $\mathcal{S}$ is classical if and only if $\mathcal{O}$ contains at least one generalized conic.*

12. An important observation and proof of Theorem 3

12.1. TGQ's arising from flocks

We recall two classes of TGQ's for which the translation dual is the point-line dual of a flock GQ.

(1) If $\mathcal{S}(\mathcal{F})$ is the classical GQ $H(3,q^2)$, then it is a TGQ with base line L, L any line of $\mathcal{S}(\mathcal{F})$. The dual $\mathcal{S}(\mathcal{F})^D$ of $\mathcal{S}(\mathcal{F})$ is isomorphic to $T_3(\mathcal{O})$, $\mathcal{O}$ an elliptic quadric of $\mathbf{PG}(3,q)$. Hence the kernel $\mathbb{K}$ is the field $\mathbf{GF}(q)$. Also, $\mathcal{S}(\mathcal{F})^D$ is isomorphic to its translation dual $(\mathcal{S}(\mathcal{F})^D)^*$.

(2) Let $\mathcal{K}$ be the quadratic cone with equation $X_0X_1 = X_2^2$ of $\mathbf{PG}(3,q)$, q odd. Then the q planes π_t with equation

$$tX_0 - mt^\sigma X_1 + X_3 = 0,$$

$t \in \mathbf{GF}(q)$, m a given non-square in $\mathbf{GF}(q)$ and σ a given automorphism of $\mathbf{GF}(q)$, define a flock $\mathcal{F}$ of $\mathcal{K}$; see [27]. All the planes π_t contain the exterior point $(0,0,1,0)$ of $\mathbb{K}$. This flock is *linear*; that is, all the planes π_t contain a common line, if and only if $\sigma = \mathbf{1}$. Conversely, every nonlinear flock $\mathcal{F}$ of $\mathbb{K}$ for which the planes of the q conics share a common point, is of the type just described, see [27]. The corresponding GQ $\mathcal{S}(\mathcal{F})$ was first discovered by W. M. Kantor, and is therefore called the *Kantor (flock) generalized quadrangle.* The kernel $\mathbb{K}$ is the fixed field of σ, see [23]. The described quadrangle is a TGQ for some baseline, and the following was shown by Payne in [18].

Result 45. (S. E. Payne [18]) *Suppose a TGQ $\mathcal{S} = T(\mathcal{O})$ is the point-line dual of a GQ $\mathcal{S}(\mathcal{F})$ which arises from a Kantor flock $\mathcal{F}$. Then $T(\mathcal{O})$ is isomorphic to its translation dual $T^*(\mathcal{O})$.*

Recently, Bader, Lunardon and Pinneri [2], relying heavily on results of J. A. Thas and H. Van Maldeghem [39], proved that a TGQ which is the point-line dual of a flock GQ is isomorphic to its translation dual if and only if it is a dual Kantor flock GQ.[7]

Remark 46. There are two other types of TGQ's which arise from flocks, namely the *Roman generalized quadrangles* and the *sporadic semifield flock generalized quadrangle of Penttila-Williams*, see [22]. For both types the kernel is isomorphic to $\mathbf{GF}(3)$.

The following very interesting theorem classifies TGQ's arising from flocks in the odd case.

Result 47. (Blokhuis, Lavrauw and Ball [4]) *Let $T(\mathcal{O})$ be a TGQ of order (q^n, q^{2n}), where $\mathbf{GF}(q)$ is the kernel, and suppose $T(\mathcal{O})$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$, with the additional condition that $q \geq 4n^2 - 8n + 2$ with q odd. Then $T(\mathcal{O})$ is isomorphic to the point-line dual of a Kantor flock GQ.*

[7] A much easier proof of that theorem is contained in [47].

12.2. Kantor flock generalized quadrangles as SPGQ's

It was a longstanding conjecture that every SPGQ of order s, $s > 1$, is isomorphic to the classical GQ $\mathcal{Q}(4,s)$ (and then s is the power of a prime). S. E. Payne published a crucial paper on SPGQ's (*Span-symmetric generalized quadrangles*, The Geometric Vein, Springer, New York/Berlin (1981), 231–242) which contained a 'proof' of this conjecture, but later Payne noted that the proof was invalid. We solved the conjecture in [43] (see Result 6 and Result 7). It is the goal of this paragraph to show that there is no such conjecture for the case $s \neq t$ $(s,t > 1)$; although it was conjectured in 1983 by Payne that every SPGQ of order (s,s^2) is isomorphic to $\mathcal{Q}(5,s)$, see Problem 26 of [17], we will indicate here that any member $\mathcal{S}$ of the dual Kantor flock GQ's (a class which contains nonclassical examples) has a line L such that, for any (!) two distinct and non-concurrent lines $U, V \in L^\perp$, $\mathcal{S}$ is an SPGQ with base-span $\{U,V\}^{\perp\perp}$. It was Payne who noted this (implicitely) in *A garden of generalized quadrangles*, Algebras, Groups, Geom. **3** (1985), 323–354.

We will start with some notions and results.

A (finite) *net of order* $k(\geq 2)$ *and degree* $r(\geq 2)$ is an incidence structure $\mathcal{N} = (P,B,I)$ satisfying the following properties:

(N1) each point is incident with r lines and two distinct points are incident with at most one line;

(N2) each line is incident with k points and two distinct lines are incident with at most one point;

(N3) if p is a point and L a line not incident with p, then there is a unique line M incident with p and not concurrent with L.

A net of order k and degree r has k^2 points and kr lines. For more on nets, see e.g. [3].

Result 48. (FGQ, 1.3.1]) *Let p be a regular point of a GQ $\mathcal{S} = (P,B,I)$ of order (s,t), $s \neq 1 \neq t$. Then the incidence structure with pointset $p^\perp \setminus \{p\}$, with lineset the set of spans $\{q,r\}^{\perp\perp}$, where q and r are non-collinear points of $p^\perp \setminus \{p\}$, and with the natural incidence, is the dual of a net of order s and degree $t+1$. If in particular $s = t$, there arises a dual affine plane of order s. Also, in the case $s = t$, the incidence structure π_p with pointset $p^\perp$, with lineset the set of spans $\{q,r\}^{\perp\perp}$, where q and r are different points in $p^\perp$, and with the natural incidence, is a projective plane of order s.*

The following theorem is taken from [44] and implies that a net which arises from a regular point in a thick GQ cannot contain proper subnets of the same degree and different from an affine plane:

Result 49. (K. Thas [44]) *Suppose $\mathcal{S} = (P,B,I)$ is a GQ of order (s,t), $s,t \neq 1$, with a regular point p. Let $\mathcal{N}_p$ be the net which arises from p, and suppose $\mathcal{N}'_p$ is a subnet of the same degree as $\mathcal{N}_p$. Then we have the following possibilities.*

1. *$\mathcal{N}'_p$ coincides with $\mathcal{N}_p$;*
2. *$\mathcal{N}'_p$ is an affine plane of order t and $s = t^2$; also, from $\mathcal{N}'_p$ there arises a proper subquadrangle of $\mathcal{S}$ of order t having p as a regular point.*

If, conversely, $\mathcal{S}$ has a proper subquadrangle containing the point p and of order (s', t) with $s' \neq 1$, then it is of order t, and hence $s = t^2$. Also, there arises a proper subnet of $\mathcal{N}_p$ which is an affine plane of order t.

In [19], S. E. Payne notes the following (we use the standard notation as earlier mentioned). Suppose $\mathcal{S} = \mathcal{S}(G, \mathcal{J})$ is a 4-gonal representation of the Kantor flock GQ of order (q^2, q), q odd, with the convention that A_0 is the zero matrix (see [15, 19]).

Payne shows that $\mathcal{S}$ is a TGQ for the line $[A(\infty)]$ (so each point of $[A(\infty)]$ is a center of symmetry). Next, for each $r \neq 0 \in GF(q)$, Payne defines the following automorphism of the group G:

$$\theta_r : (\alpha, c, \beta) \longrightarrow (\alpha + \beta K_r^{-1}, c + \frac{1}{4}\beta A_r^{-1}\beta^T, \beta). \tag{7}$$

The automorphism θ_r has the properties that it maps $A(0)$ onto itself, $A(\infty)$ onto $A(r)$ while $A(-r)$ is mapped onto $A(\infty)$, and finally, $A(t)$ is mapped onto $A(rt/(r+t))$ $(r \neq -t)$. Also, he proves that the collineation of $\mathcal{S}$ induced by θ_r is a symmetry about $A^*(0)$, from which it follows that $A^*(0)$ is a center of symmetry. There is a major corollary.

Theorem 50. *Every point of $(\infty)^\perp$ is a center of symmetry.*

Proof. First of all, since (∞) is a center of symmetry there follows that (∞) is also regular, and hence there is a net $\mathcal{N}_{(\infty)}$ associated to it. Next, we know that $A^*(0)$ is a center of symmetry, and also every point of the line $[A(\infty)]$. By Result 49, there follows that the net which is generated by all the centers of symmetry of $(\infty)^\perp$ coincides with $\mathcal{N}_{(\infty)}$, and hence every point of $(\infty)^\perp$ is a center of symmetry. □

Note that the dual of Theorem 50 is in fact the following.

Theorem 51. *Let $\mathcal{S}(\mathcal{F})$ be a Kantor flock GQ of order (q^2, q), $q > 1$. Then the dual $\mathcal{S}(\mathcal{F})^D$ contains a line L such that every line of $L^\perp$ is an axis of symmetry, i.e. every point of L is a translation point.*

This observation is one of the main motivations of the paper; our main result is a generalization of the converse of Theorem 51: *what are the GQ's which have two distinct translation points?*

Note. If a flock GQ is not classical, then it is well-known that the automorphism group of the GQ fixes (∞).

12.3. Proof of Theorem 3

A *panel* of a generalized quadrangle $\mathcal{S} = (P, B, I)$ is an element (p, L, q) of $P \times B \times P$ for which $pILIq$ and $p \neq q$. Dually, one defines *dual panels.* If (p, L, q) is a panel of the GQ $\mathcal{S}$, then a *(p, L, q)-collineation* of $\mathcal{S}$ is a whorl about p, L and q. A panel (p, L, q) of a GQ of order (s, t), $s, t \neq 1$, is called *Moufang* if there is a group of (p, L, q)-collineations of size s. A line L is *Moufang* if every panel of the form (p, L, q) is Moufang. A GQ is *half Moufang* if every panel or every dual panel is Moufang, and a GQ is a *Moufang GQ* if every panel and every dual panel is Moufang.

We are now ready to prove Theorem 3. For completeness' sake, we repeat the statement of the theorem.

Theorem 52. *Suppose $\mathcal{S}$ is a generalized quadrangle of order (s,t), $s \neq 1 \neq t$, with two distinct collinear translation points. Then we have the following:*

(i) *$s = t$, s is a prime power and $\mathcal{S} \cong \mathcal{Q}(4,s)$;*

(ii) *$t = s^2$, s is even, s is a prime power and $\mathcal{S} \cong \mathcal{Q}(5,s)$;*

(iii) *$t = s^2$, $s = q^n$ with q odd, where $\mathbf{GF}(q)$ is the kernel of the TGQ $\mathcal{S} = \mathcal{S}^{(\infty)}$ with (∞) an arbitrary translation point of $\mathcal{S}$, $q \geq 4n^2 - 8n + 2$ and $\mathcal{S}$ is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ where $\mathcal{F}$ is a Kantor flock;*

(iv) *$t = s^2$, $s = q^n$ with q odd, where $\mathbf{GF}(q)$ is the kernel of the TGQ $\mathcal{S} = \mathcal{S}^{(\infty)}$ with (∞) an arbitrary translation point of $\mathcal{S}$, $q < 4n^2 - 8n + 2$ and $\mathcal{S}$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ for some flock $\mathcal{F}$.*

If a thick GQ $\mathcal{S}$ has two non-collinear translation points, then $\mathcal{S}$ is always of classical type, i.e. isomorphic to one of $\mathcal{Q}(4,s)$, $\mathcal{Q}(5,s)$.

Proof. If $\mathcal{S}$ contains two non-collinear translation points, then it is clear that every point of $\mathcal{S}$ is a translation point, and hence that every line of $\mathcal{S}$ is an axis of symmetry. Hence every line is Moufang, and $\mathcal{S}$ is half Moufang. By J. A. Thas, S. E. Payne and H. Van Maldeghem [40], there follows that $\mathcal{S}$ is Moufang, and by Fong and Seitz [8, 9] $\mathcal{S}$ is classical, i.e. $\mathcal{S}$ is isomorphic to one of $\mathcal{Q}(4,s)$,$\mathcal{Q}(5,s)$ since these are the only classical GQ's with regular lines, see Chapter 5 of [21].

For $s = t$, the statement follows immediately from Result 6. If $s \neq t$, then $t = s^2$ by Result 5.

Suppose s is even. First suppose that the translation points are collinear. Fix two non-concurrent lines U and V of $L^{\perp}$. If G is the base-group which is defined by the axes of symmetry U and V, and G does not act semi-regularly on the points of $\mathcal{S} \setminus \Omega$, then $\mathcal{S}$ is classical, i.e. isomorphic to $\mathcal{Q}(5,s)$ by Lemma 26. Thus we can suppose that G acts semi-regularly on the points of $\mathcal{S} \setminus \Omega$. By Lemma 36 there then follows that $G \cong \mathbf{SL}_2(s)$, and hence that $|G| = s^3 - s$. Hence, by Theorem 37 we know that $\mathcal{S}$ contains more than 2 classical subGQ's, and then by Result 17 there follows that $\mathcal{S} \cong \mathcal{Q}(5,s)$.

Now let s be odd. Now suppose that $\mathcal{S}$ contains two collinear translation points u and v. Then clearly by transitivity, every point of $L := uv$ is a translation point, and hence every line of $L^{\perp}$ is an axis of symmetry. If we fix some base-span $\mathcal{L}$ for which $L \in \mathcal{L}^{\perp}$, then by Theorem 37 there are $s+1$ classical subGQ's of order s which mutually intersect precisely in the base-grid $(\Omega, \mathcal{L} \cup \mathcal{L}^{\perp}, I')$. Now fix an arbitrary translation point $(\infty)IL$, and consider the TGQ $T(\mathcal{O})$ with base-point (∞). Then clearly $\mathcal{O}$ is good at its element π which corresponds to L. Hence, the translation dual $T^*(\mathcal{O})$ of $T(\mathcal{O})$ satisfies Property (G) for the flag $((\infty)', L')$ by Result 43, where $(\infty)'$ corresponds to (∞), and L' to L. If we now apply Result 41, then there follows that $T^*(\mathcal{O})$ is the point-line dual of a flock generalized quadrangle $\mathcal{S}(\mathcal{F})$. The theorem now follows from Result 47 and the fact that the dual Kantor flock quadrangles are isomorphic to their translation duals. □

Remark 53. A span-symmetric GQ with base-span (U, U') defines a split BN-pair of rank 1, where the root groups are the the full groups of symmetries about the lines of $\{U, U'\}^{\perp\perp}$. Suppose $\mathcal{S}$ is a thick GQ with two distinct collinear translation points p and q, and $pq = L$. Suppose M and N are arbitrary non-concurrent lines of $L^{\perp}$. Since $L \cap M$ and $L \cap N$ are translation points of the GQ, there follows that the groups of symmetries about M and N are abelian groups (the translation groups corresponding to $L \cap M$ and $L \cap N$ are abelian), and hence every full group of symmetries about a line of $\{L, M\}^{\perp\perp}$ is abelian, since they are all isomorphic. As we mentioned before in Remark 9, the only transvection groups of a finite split BN-pair of rank 1 with abelian rootgroups of order s are isomorphic to a $\mathbf{PSL}_2(s)$, or a sharply 2-transitive group. We did not use this fact since it was one of our aims to obtain Theorem 2.

Appendix. Recently [47], we proved that every TGQ of order (q, q^2), $q > 1$ and q odd, which is the translation dual of the point-line dual of a flock GQ, has a line of translation points. By Theorem 3, this mean that, in the odd case, *the generalized quadrangles with two (collinear) translation points are precisely those generalized quadrangles which are the translation dual of the point-line dual of a (semifield) flock GQ.* In particular, besides the dual Kantor GQ's, the theorem applies to the Roman GQ's and the Penttila-Williams GQ. In [47] it is shown that this observation has important corollaries for the theory of semifield flock GQ's, and several well-known problems are completely solved in that paper.

13. A complete classification for the even case

The following theorem completely classifies all generalized quadrangles of order (s, t), $s \neq 1 \neq t$ and s even, with two distinct translation points.

Theorem 54. *Let $\mathcal{S}$ be a generalized quadrangle of order (s, t), $s \neq 1 \neq t$ and s even, with two distinct translation points. Then $\mathcal{S}$ is classical, i.e. isomorphic to $\mathcal{Q}(4, s)$ or $\mathcal{Q}(5, s)$.*

Proof. Immediate from Theorem 3. □

14. Generalizations of the main result

14.1. A useful lemma

In this section it is our aim to generalize Theorem 3 in various ways. We first recall a theorem which is an improvement of Result 14 for the case $s \neq t$.

Result 55. (K. Thas [41, 45]) *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s, t), $s \neq t$ and $s \neq 1 \neq t$. Suppose that there are at least $t - s + 3$ axes of symmetry through a point p, and let G be the group generated by the symmetries about these lines. Then G is elementary abelian and $(\mathcal{S}^{(p)}, G)$ is a TGQ.*

Lemma 56. *Suppose $\mathcal{S}$ is a GQ of order (s, t), $s \neq 1 \neq t$, and suppose L, p and q are such that L is a regular line, pIL is a point which is incident with at least $s+1$ axes of symmetry different from L, and qIL is a point different from p which is incident with at least one axis of symmetry which is not L. Then every point of L is a translation point.*

Proof. Suppose $\mathcal{N}_L$ is the net which corresponds to the regular line L by the dual of Result 48, and suppose $\mathcal{N}'_L$ is the subnet of $\mathcal{N}_L$ of the same degree which is generated by the lines which correspond to the axes of symmetry meeting L in the quadrangle. Then by Result 49 there only are two possibilities:

(a) $\mathcal{N}'_L$ is an affine plane of order s;

(b) $\mathcal{N}'_L = \mathcal{N}_L$.

Since we supposed that there is a point on L which is incident with $s+1$ axes of symmetry different from L, the second possibility holds. It is clear that there follows that every line of $L^\perp \setminus \{L\}$ is an axis of symmetry, and by Result 55, L is also an axis of symmetry. Hence, $\mathcal{S}$ is a translation generalized quadrangle for every point on L by Section 4. □

14.2. Regularity revisited

The following lemma is a generalization of [21, 1.3.6 (iv)], which states the dual of the lemma for $s = t$.

Lemma 57.

1. *Suppose $\mathcal{S} = (P, B, I)$ is a GQ of order (s,t), $s \neq 1 \neq t$, and let L be a line such that every line of $B \setminus L^\perp$ is regular. Then every line of $\mathcal{S}$ is regular.*
2. *If L is a line such that every line of $L^\perp \setminus \{L\}$ is regular, then L is also regular.*

Proof. Suppose N is an arbitrary line of $B \setminus L^\perp$. Then since N is a regular line there follows that $\{L, N\}$ is a regular pair. Hence L is a regular line. Now suppose $L' \neq L$ is a line of $L^\perp$. Let M be a line of $L^\perp$, $N \not\sim M$. Then since L is regular there follows that $\{L', M\}$ is regular. Now suppose that M' is arbitrary in $B \setminus L^\perp$. Then M' is regular, and hence $\{L', M'\}$ is regular. Hence L' is regular, and then also every line of $\mathcal{S}$. This proves part (1). Part (2) immediately follows from the proof of part (1). □

14.3. Classifications for generalized quadrangles

Following K. Thas [42], if p is a point of a GQ $\mathcal{S} = (P, B, I)$ such that there is a group of whorls about p which acts transitively on $P \setminus p^\perp$, then we call p a *center of transitivity*.

Result 58. (K. Thas [42]) *Suppose $\mathcal{S}$ is a GQ of order (s,t), $s, t \neq 1$. Then $\mathcal{S}$ is isomorphic to either $\mathcal{Q}(4,s)$ or $\mathcal{Q}(5,s)$ if and only if $\mathcal{S}$ contains a center of transitivity p, a collineation θ of $\mathcal{S}$ for which $p^\theta \not\sim p$, and a regular line.*

Theorem 59. *Suppose $\mathcal{S}$ is a generalized quadrangle of order (s,t), $s \neq 1 \neq t$, and suppose that one of the following conditions is satisfied.*

1. *There is a regular line L and there are points p and q such that pIL is a point which is incident with at least $s+1$ axes of symmetry different from L, and qIL is a point different from p which is incident with at least one axis of symmetry which is not L.*

2. *There is a regular line L and there are points p and q such that pIL is a point which is incident with at least $s+1$ regular lines $L_0, \ldots, L_s$ different from L, for which there are lines $M_0, \ldots, M_s$ and points $p_0, \ldots, p_s$ such that $p_i I L_i \sim M_i \not{I} p_i$ for all i, and such that there is a group of whorls about p_i which fixes M_i and which acts transitively on the points of M_i which are not on L_i, and qIL is a point different from p which is incident with at least one regular U line which is not L, for which there is a line M and a point u such that $uIU \sim M \not{I} u$, and such that there is a group of whorls about u which fixes M and which acts transitively on the points of M which are not incident with U.*
3. *$\mathcal{S}$ has two distinct centers of transitivity p and q and a regular line which is contained in $(pq)^\perp \setminus \{pq\}$ if p and q are collinear.*
4. *s is even and $\mathcal{S}$ contains a regular line and an elation point p which is not fixed by the group of automorphisms of $\mathcal{S}$.*
5. *s is odd and $\mathcal{S}$ contains two distinct regular lines and an elation point p which is not fixed by the group of automorphisms of $\mathcal{S}$.*

Then we have the following possibilities:

(i) *$s = t$, s is a prime power and $\mathcal{S} \cong \mathcal{Q}(4, s)$;*

(ii) *$t = s^2$, s is even, s is a prime power and $\mathcal{S} \cong \mathcal{Q}(5, s)$;*

(iii) *$t = s^2$, $s = q^n$, where $\mathbf{GF}(q)$ is the kernel of $\mathcal{S} = \mathcal{S}^{(\infty)}$ with (∞) an arbitrary translation point of $\mathcal{S}$ and q is odd, $q \geq 4n^2 - 8n + 2$ and $\mathcal{S}$ is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ where $\mathcal{F}$ is a Kantor flock;*

(iv) *$t = s^2$, $s = q^n$, where $\mathbf{GF}(q)$ is the kernel of $\mathcal{S} = \mathcal{S}^{(\infty)}$ with (∞) an arbitrary translation point of $\mathcal{S}$ with q odd, $q < 4n^2 - 8n + 2$ and $\mathcal{S}$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ for some flock $\mathcal{F}$.*

Proof. If (1) is satisfied, then by Lemma 56 there follows that L is a line of translation points, and the statement follows by Theorem 3. Case (2) immediately follows from (1) and Theorem 2.7 of K. Thas [42]. Next suppose that we are in case (3). If p and q are non-collinear, then $\mathcal{S}$ is classical by Result 58. Suppose $p \sim q$; then clearly every point of pq is a center of transitivity. Since there is a regular line M in $(pq)^\perp \setminus \{pq\}$, every line of $(pq)^\perp \setminus \{pq\}$ is regular, and by Lemma 57 pq is also regular. By Theorem 2.7 of [42] this implies that every line of $(pq)^\perp \setminus \{pq\}$ is an axis of symmetry, and by Result 55 we can conclude that pq is also an axis of symmetry. The statement follows from Theorem 3 since pq is a line of translation points. Now suppose we are in case (4) of the theorem. If p is mapped onto a point not collinear with p by some automorphism of $\mathcal{S}$, then the statement follows from (3). Hence suppose that p is mapped onto a point $x \sim p \neq x$ by some automorphism of $\mathcal{S}$. Then px is a line of which every point is an elation point, and if the regular line is incident with one of these elation points, then by Theorem 4.2 of [42] there follows that every point is a translation point. Now suppose that there is a regular line in $\mathcal{S} \setminus (px)^\perp$. Then by transitivity every line of $\mathcal{S} \setminus (px)^\perp$ is a regular line, and hence Lemma 57 implies that every line of $\mathcal{S}$ is regular. Now Theorem 4.2 of [42] applies. The proof of case (5) of the theorem is similar by using Theorem 4.1 of [42]. □

As a direct corollary of the theorem, the following theorem is a characterization of the classical GQ's $\mathcal{Q}(4, s)$ and $\mathcal{Q}(5, s)$ with s even.

Theorem 60. *Suppose $\mathcal{S}$ is a generalized quadrangle of order (s,t), where $s \neq 1 \neq t$ and s even. Then $\mathcal{S}$ is isomorphic to one of $\mathcal{Q}(4,s)$,$\mathcal{Q}(5,s)$ if and only if $\mathcal{S}$ contains a regular line and an elation point p which is not fixed by the group of automorphisms of $\mathcal{S}$.* □

A. Addendum: Spreads in span-symmetric generalized quadrangles and dual Kantor flock generalized quadrangles

A *spread* of a generalized quadrangle $\mathcal{S}$ is a set $\mathbf{T}$ of mutually non-concurrent lines such that every point of $\mathcal{S}$ is incident with a (necessarily unique) line of $\mathbf{T}$. If the GQ is of order (s,t), then $|\mathbf{T}| = st+1$, and, conversely, a set of $st+1$ mutually non-concurrent lines of a GQ of order (s,t) is a spread. Dually, one defines *ovoids* of generalized quadrangles.

Two spreads $\mathbf{T}$ and $\mathbf{T}'$ of a GQ $\mathcal{S}$ are said to be *isomorphic* if there is an automorphism of $\mathcal{S}$ which maps $\mathbf{T}$ onto $\mathbf{T}'$.

In this final part of the paper, we describe a concrete construction of spreads in SPGQ's, and the construction applies to the dual Kantor GQ's. We note that for the dual Kantor GQ's, other constructions (some in a more general context) are known, see e.g. [38].

A.1. Spreads in span-symmetric generalized quadrangles

Theorem 61. *Suppose $\mathcal{S}$ is a span-symmetric generalized quadrangle of order (s,t), $s \neq 1 \neq t$ and $t \neq s$, with base-span $\mathcal{L}$ and base-group G. If $|G| = s^3 - s$, then $\mathcal{S}$ contains at least $2(s^2 - s)$ distinct spreads.*

Proof. Since $s \neq t$, we know by Result 5 that $t = s^2$. Suppose Λ is an arbitrary G orbit in $\mathcal{S} \setminus \Omega$, where Ω is the set of points incident with the lines of $\mathcal{L}$. Then by Section 8, $\Lambda \cup \Omega$ is the set of points of a subGQ $\mathcal{S}'$ of order s, which is classical by the main result of [43], and there are $s+1$ subGQ's which arise in this way, see Section 8. Let L be an arbitrary line of $\mathcal{S}$ which is not contained in $\mathcal{S}'$, and which does not intersect Ω. Then by Chapter 2 of FGQ, there follows that L intersects $\mathcal{S}'$ in exactly one point ($\mathcal{S}'$ is a "geometric hyperplane" of $\mathcal{S}$). Since $|\Lambda| = s^3 - s$, there follows that the action of G on the points of $\mathcal{S} \setminus \Omega$ is semi-regular, and any of the $s+1$ subGQ's which contain Ω, see above, is fixed by G. Now consider the lineset L^G. Then $|L^G| = s^3 - s$ and no two distinct lines of L^G intersect since the action of G on Λ is regular. Now put $\mathbf{T} = \mathcal{L} \cup L^G$ and $\mathbf{T}' = \mathcal{L}^{\perp} \cup L^G$. Then $\mathbf{T}$ and $\mathbf{T}'$ clearly are spreads of $\mathcal{S}$. The theorem now follows from the fact that through a fixed point of Λ there are $s^2 - s$ choices for L. □

Note. It may be clear to the reader that exactly the same $2(s^2 - s)$ spreads are obtained if one considers another G-orbit in $\mathcal{S} \setminus \Omega$.

As a direct consequence of Theorem 3 and Theorem 61, we obtain the following theorem which is valid for any SPGQ of order (s,t), $s \neq 1 \neq t$, with $s \neq t$ and s odd.

Theorem 62. *Suppose $\mathcal{S}$ is an SPGQ of order (s,t), $s \neq 1 \neq t$, with $s \neq t$ and s odd. Then $\mathcal{S}$ contains at least $2(s^2 - s)$ distinct spreads.*

Proof. Immediate. □

Remark 63. If $\mathbf{T}$ is such a spread of $\mathcal{S}$, then the group of automorphisms of $\mathcal{S}$ which fix $\mathbf{T}$ contains a group isomorphic to $\mathbf{SL}_2(s)$.

A.2. Spreads in the dual Kantor flock generalized quadrangles

In the remains of this section, we use the following notations: $\mathcal{S}$ is the point-line dual of a flock generalized quadrangle $\mathcal{S}^D = \mathcal{S}(\mathcal{F})$ with $\mathcal{F}$ a Kantor flock, and where $\mathcal{S}^D$ is a GQ of order (q^2, q), q an odd prime power. Also, if $\mathcal{S}$ is not classical, i.e. not isomorphic to $\mathcal{Q}(5, q)$, then L will be the (unique) line of translation points of $\mathcal{S}$, see Theorem 3. If $\mathcal{S} \cong \mathcal{Q}(5, q)$, then L is arbitrary.

For every pair (M, N) of non-concurrent lines (which are axes of symmetry) in $L^\perp$, we know for the dual Kantor flock GQ's that the base-group G corresponding to the base-span $\{M, N\}^{\perp\perp}$ is isomorphic to $\mathbf{SL}_2(q)$, and hence, since $|\mathbf{SL}_2(q)| = q^3 - q$, there holds by Theorem 61 that there are $2(q^2 - q)$ distinct spreads, all containing $\{M, N\}^\perp$ or $\{M, N\}^{\perp\perp}$, and such that G stabilizes all these spreads and fixes L. The class of spreads which arise with this construction method and containing a span of the form $\{U, V\}^{\perp\perp}$ with $U \neq V$ lines of $L^\perp$, will be denoted by $\mathfrak{T}$. The class of spreads which arise with this construction method but not containing such a line span is denoted by $\mathfrak{T}'$. Since an element of $\mathfrak{T} \cup \mathfrak{T}'$ is uniquely defined by some line span $\mathcal{L} = \{M, N\}^{\perp\perp}$, $M \neq N$ and $M, N \in L^\perp$, respectively $\mathcal{L} = \{M, N\}^\perp$, $M \neq N$ and $M, N \in L^\perp$, and a line V which is not contained in a $\mathcal{Q}(4, q)$ which contains $\mathcal{L}$, we will sometimes denote a spread $\mathbf{T}$ of $\mathfrak{T} \cup \mathfrak{T}'$ by $\mathbf{T} = \mathbf{T}(\mathcal{L}, V)$. Hence,

$$\mathbf{T} = \mathbf{T}(\mathcal{L}, V) = \{\mathcal{L}\} \cup \{V^\theta \parallel \theta \in G\},$$

where G is the base-group corresponding to $\mathcal{L}$, respectively $\mathcal{L}^\perp$.

Observation 64. *If $\mathcal{S}$ is nonclassical, then no spread of $\mathfrak{T}$ is isomorphic to a spread of $\mathfrak{T}'$.*

Proof. Suppose $\mathbf{T} \in \mathfrak{T}$ and $\mathbf{T}' \in \mathfrak{T}'$. If $\mathcal{S}$ is nonclassical, then it is clear from Theorem 3 that no line of $\mathcal{S} \setminus L^\perp$ is an axis of symmetry, hence $\mathbf{T}$ and $\mathbf{T}'$ can never be isomorphic. □

Remark 65. If $\mathcal{S} \cong \mathcal{Q}(5, q)$, then every element of $\mathfrak{T}$ is isomorphic to every element of $\mathfrak{T}'$ since the group of automorphisms of $\mathcal{S}$ acts transitively on the pairs of non-concurrent lines, see e.g. Chapter 9 of [21].

A spread $\mathbf{T}$ of a GQ is called *locally Hermitian* if there is a line $M \in \mathbf{T}$ such that for any $N \in \mathbf{T}$, $N \neq M$, the pair $\{M, N\}$ is regular, and such that $\{M, N\}^{\perp\perp} \subseteq \mathbf{T}$. In that case, the spread is *locally Hermitian w.r.t. M*. [8]

Lemma 66. *Suppose $\mathbf{T}$ is a locally Hermitian spread w.r.t. a line M of a GQ $\mathcal{S}$ of order (s, t), $s \neq 1 \neq t$. For any $N \in \mathbf{T} \setminus \{M\}$, the set $(\mathbf{T} \setminus \{M, N\}^{\perp\perp}) \cup \{M, N\}^\perp$ is also a spread of $\mathcal{S}$ which is not locally Hermitian.*

[8] In [41], we called such a spread *semi-regular*.

Proof. The fact that $\mathbf{T}' = (\mathbf{T} \setminus \{M,N\}^{\perp\perp}) \cup \{M,N\}^{\perp}$ is a spread is well-known and easy to prove. Suppose that $U \in \{M,N\}^{\perp}$ and $U' \in \mathbf{T} \setminus \{M,N\}^{\perp}$ are arbitrary. If $\{U,U'\}^{\perp\perp} \not\subseteq \mathbf{T}'$, then clearly $\mathbf{T}'$ is not locally Hermitian. Suppose $\{U,U'\}^{\perp\perp} \subseteq \mathbf{T}'$. It is clear that $\{U,U'\}^{\perp} \cap \{M,U'\}^{\perp} \neq \emptyset$, a contradiction since this implies that $\mathbf{T}'$ has distinct concurrent lines. $\square$

Observation 67. *Any spread of $\mathfrak{T}'$ is locally Hermitian. No spread of $\mathfrak{T}$ is locally Hermitian.*

Proof. Suppose $\mathbf{T} = \mathbf{T}(\mathcal{L}, U)$ is an element of $\mathfrak{T}'$. Recall that $L \in \mathcal{L}^{\perp}$. Let M be an arbitrary line of $\mathcal{L}$. If $N \in \mathcal{L}$, $M \neq N$, then $\{M,N\}^{\perp\perp} = \mathcal{L} \subseteq \mathbf{T}$. Next suppose that $N \in \mathbf{T} \setminus \mathcal{L}$. If G is the base-group corresponding to the base-span $\mathcal{L}$, then the group G_M of symmetries about M is contained in G. This implies that any line of $\{M,N\}^{\perp\perp}$ is also a line of $\mathbf{T}$. Hence $\mathbf{T}$ is locally Hermitian.

The fact that no spread of $\mathfrak{T}$ is locally Hermitian follows from Lemma 66 and the definitions of $\mathfrak{T}$ and $\mathfrak{T}'$. $\square$

Observation 68. *For each $\mathbf{T} \in \mathfrak{T} \cup \mathfrak{T}'$ the group of automorphisms of $\mathcal{S}$ which stabilize $\mathbf{T}$ contains a subgroup isomorphic to $\mathbf{SL}_2(q)$.*

Proof. Immediate by the definition of $\mathfrak{T}$ and $\mathfrak{T}'$. $\square$

Observation 69. $|\mathfrak{T}'| = q^6 - q^5 = |\mathfrak{T}|$. *Also,* $\mathfrak{T} \cap \mathfrak{T}' = \emptyset$.

Proof. It is trivial that $\mathfrak{T} \cap \mathfrak{T}' = \emptyset$ since every element of $\mathfrak{T}'$ contains lines which intersect L. Now consider two not necessarily distinct elements of $\mathfrak{T}'$, say $\mathbf{T} = \mathbf{T}(\mathcal{L}, U)$ and $\mathbf{T}' = \mathbf{T}(\mathcal{L}', U')$ (so $L \in \mathcal{L}^{\perp}, \mathcal{L}'^{\perp}$). If $\mathcal{L} \neq \mathcal{L}'$, then clearly $\mathbf{T} \neq \mathbf{T}'$. If we now count the number k of pairs $(\mathcal{L}'', \mathbf{T}'')$, where $\mathcal{L}''$ is a line span for which $L \in \mathcal{L}''^{\perp}$ and where $\mathbf{T}''$ is of the form $\mathbf{T}(\mathcal{L}'', U'')$ for some line U'' (so $\mathbf{T}'' \in \mathfrak{T}'$), then by Theorem 61 there follows that $k = q^4(q^2 - q)$, and k is exactly the number of elements of $\mathfrak{T}'$. The fact that $|\mathfrak{T}| = |\mathfrak{T}'|$ follows from the definition of $\mathfrak{T}$ and $\mathfrak{T}'$, and the proof of Observation 67 (no element of $\mathfrak{T}$ contains two distinct regular line spans which both contain L). $\square$

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