Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 43 (2002), No. 2, 411-422.

Covering the Unit Square by Squares

Janusz Januszewski

Instytut Matematyki i Fizyki ATR
ul. Kaliskiego 7, 85-796 Bydgoszcz, Poland
e-mail: januszew@mail.atr.bydgoszcz.pl

Abstract. Every sequence of squares with total area greater than or equal to 2 permits a covering of the unit square.

MSC 2000: 52C15
Keywords: covering, square

Let (S_n) be a finite or infinite sequence of squares and let R be a rectangle. We say that (S_n) *permits a covering* of R (or that R *can be covered* by (S_n)) if there exist rigid motions σ_n such that $R \subset \bigcup \sigma_n S_n$. The covering is *parallel* if all the squares $\sigma_n S_n$ have sides parallel to a side of R.

Over thirty years ago Moon and Moser [3] showed, that any sequence of squares with total area not smaller than 3 permits a parallel covering of the unit square. The number 3 cannot be lessened for parallel covering, as the example of three squares of side lengths smaller than 1 shows. The natural question is how much this number can be lessened if we reject the condition of parallelism.

Obviously, the unit square cannot by covered by two squares of side lengths smaller than 1. The conjecture of M. Bognár is that any sequence of squares with total area 2 permits a covering of the unit square (see Problem 108 (iv) in [4]). We show that the conjecture is right.

We start with few lemmas. Denote by $|S|$ the area of S and denote by $[b_1, b_2] \times [c_1, c_2]$ the set of the points (x, y) such that $b_1 \leq x \leq b_2$ and $c_1 \leq y \leq c_2$. Assume that $I = [0,1] \times [0,1]$ and that $R = [0,p] \times [0,q]$, where $p \geq q > 0$. Finally assume, that a_n denote the side length of a square S_n, for $n = 1, 2, \dots$ and that $a_1 \geq a_2 \geq \dots$

Lemma 1. *If the total area of (S_n) is not smaller than $(q + a_1)p + a_1 q$ or if it is not smaller than $(q + a_1)p + q^2$, then (S_n) permits a covering of R.*

Proof. We cover R by squares from (S_n) according to the method described in the proof of Theorem 7 of [3] and in the proof of Theorem 1 of [2]. Here is a short description of our method.

If $a_1 < q$, we put the squares in layers, whose bases are parallel to the base of R. We put the squares, side by side, from the left side of R to the right one along the base of the layer. The first layer has the same base as R (comp. Figure 1). If a square covers a point on the right side of R, the new layer is started directly above the last square in the preceding layer.

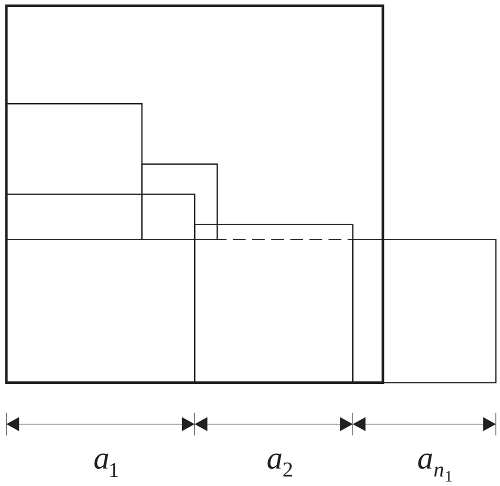

Figure 1

Assume now that $a_1 \geq q$. If $a_n \geq q$ for $n = 1, 2, \ldots$, then R can be covered by (S_n). In the opposite case denote by k the greatest integer such that $a_k \geq q$. We put the squares $S_1, \ldots, S_k$, side by side, in a layer, whose base is the base of R. We put them from the left side of R to the right one along the base. If the union $\sigma_1 S_1 \cup \ldots \cup \sigma_k S_k$ does not cover R, the squares $S_{k+1}, S_{k+2}, \ldots$ are used for the covering of the rectangle $[a_1 + \ldots + a_k, p] \times [0, q]$ by the method described in the previous paragraph.

From the proof of Theorem 1 of [2] we conclude that I can be covered by (S_n) by our method. $\square$

Lemma 2. *If $a_1 < (\sqrt{2}+1)q$ and if the total area of (S_n) is not smaller than $(\sqrt{2}+1)pq+q^2$, then (S_n) permits a covering of R.*

Proof. We use for the covering the method described in the proof of Lemma 1. The proof if similar to the proof of Lemma 1. $\square$

Lemma 3. *If $a_1 \geq (\sqrt{2}+1)q$ and if the total area of (S_n) is not smaller than $\frac{1}{2}(p+q)^2+q^2$, then (S_n) permits a covering of R.*

Proof. The motion σ_1 is defined so that the sides of $\sigma_1 S_1$ contain the points $(0, 0)$, $(0, q)$, $(a_1\sqrt{2} - q, 0)$, $(a_1\sqrt{2} - q, q)$ (comp. Figure 2).

If $a_1 \geq \frac{1}{2}\sqrt{2}(p+q)$, then $R \subset \sigma_1 S_1$. So, in the further considerations we assume that $a_1 < \frac{1}{2}\sqrt{2}(p+q)$.

Case 1, when $a_2 < (\sqrt{2}+1)q$.
The squares $S_2, S_3, \ldots$ are used for the covering of the rectangle $[a_1\sqrt{2}-q, p] \times [0, q]$ by the method described in the proof of Lemma 1. By Lemma 2 we see that if (S_n) does not permit a covering of R, then

$$\sum |S_n| < a_1^2 + (\sqrt{2}+1)(p - a_1\sqrt{2}+q)q + q^2.$$

Taking $\frac{1}{2}\sqrt{2}(p+q)$ in place of a_1 we have $\sum |S_n| < \frac{1}{2}(p+q)^2 + q^2$, which is again a contradiction.

Case 2, when $a_2 \geq (\sqrt{2}+1)q$.
If $a_n \geq (\sqrt{2}+1)q$ for $n = 1, 2, \ldots$, then R can be covered by (S_n). In the opposite case denote by k the greatest integer such that $a_k \geq (\sqrt{2}+1)q$. The motions $\sigma_2, \ldots, \sigma_k$ are defined so that the sides of $\sigma_i S_i$ contain the points

$$((a_1 + \ldots + a_{i-1})\sqrt{2} - (i-1)q, 0),\ ((a_1 + \ldots + a_{i-1})\sqrt{2} - (i-1)q, q),$$
$$((a_1 + \ldots + a_i)\sqrt{2} - iq, 0),\ ((a_1 + \ldots + a_i)\sqrt{2} - iq, q),$$

for $i = 2, \ldots, k$ (see Fig. 2). If R is not contained in $\sigma_1 S_1 \cup \ldots \cup \sigma_k S_k$, the squares $S_{k+1}, S_{k+2}, \ldots$ are used for the covering of the rectangle $[(a_1 + \ldots + a_k)\sqrt{2} - kq, p] \times [0, q]$ by the method from Lemma 1.

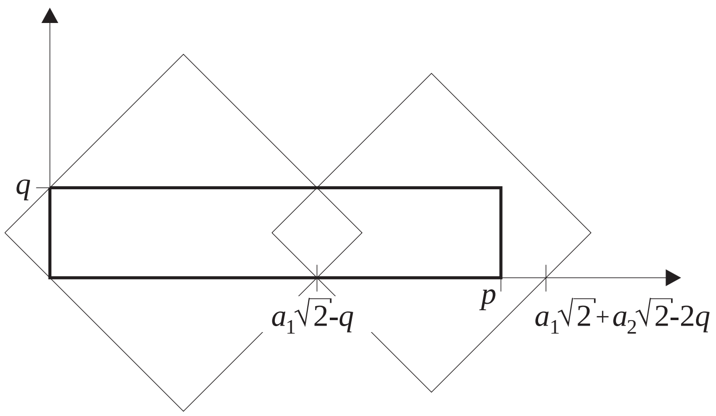

Figure 2

It is easy to show that $|S_1| + \ldots + |S_k|$ is not greater than the area of a square whose sides contain the points

$$(0, 0),\ (0, q),\ ((a_1 + \ldots + a_k)\sqrt{2} - kq, 0),\ ((a_1 + \ldots + a_k)\sqrt{2} - kq, q).$$

By Case 1 we see that R can be covered by (S_n). $\square$

Obviously, Lemmas 1–3 are still true if R is an arbitrary rectangle of sides of lengths p and q.

Theorem. *Every sequence of squares with total area greater than or equal to 2 permits a covering of the unit square.*

Proof. Let (S_n) be a sequence of squares with total area not smaller than 2. Denote by a_n the side length of S_n for $n = 1, 2, \dots$ Without loss of generality we can assume, that $a_1 \geq a_2 \geq \dots$.

If $a_1 \leq \frac{1}{2}$, by Lemma 1 we see that (S_n) permits a covering of I. Obviously, if $a_1 \geq 1$, then I can be covered by S_1. So, in the further considerations we assume that $\frac{1}{2} < a_1 < 1$.

The covering method depends on the size of squares in (S_n). Consider three cases.

Case 1, when $a_2 + a_3 < 1$.

Subcase 1a, when $a_1 + a_3 < 1$. We use for the covering the method from Lemma 1. Denote by S_{n_1} the last square lying in the first layer.

If $a_1 + a_2 < 1$, the total area of squares lying in the first layer is smaller than $a_1^2 + (1 - a_1)^2 + a_{n_1}^2$ (see Fig. 1). Obviously, $a_{n_1} < 1 - a_1$. Assume that I is not covered. By Lemma 1 we see that the total area of squares which are used for the covering of the rectangle $[0,1] \times [a_{n_1}, 1]$ is smaller than or equal to $1 - a_{n_1} + a_{n_1} + (1 - a_{n_1})a_{n_1}$. Consequently,

$$\sum |S_n| < a_1^2 + (1 - a_1)^2 + a_{n_1}^2 + 1 + (1 - a_{n_1})a_{n_1} = 2a_1^2 - 2a_1 + 2 + a_{n_1}.$$

Taking $1 - a_1$ in place of a_{n_1}, we have $\sum |S_n| < 2a_1^2 - 3a_1 + 3 < 2$, which is a contradiction.

If $a_1 + a_2 \geq 1$ and if I is not covered, by Lemma 1 we see that

$$\sum |S_n| < a_1^2 + a_2^2 + 1 - a_2 + a_3 + (1 - a_2)a_3.$$

Taking $1 - a_1$ in place of a_3 we obtain that $\sum |S_n| < a_1^2 + a_2^2 - 2a_1 - 2a_2 + a_1 a_2 + 3$. Denote by $h(a_1, a_2)$ the function on the right side of this formula. Let $T_h = \{(a_1, a_2);\ 1 - a_1 \leq a_2 \leq a_1 \leq 1\}$. We see that $h(a_1, a_2)$ does not have any maximum inside T_h. Consider h on the boundary of T_h. We have $h(1, a_2) = a_2^2 - a_2 + 2 \leq 2$, provided $0 \leq a_2 \leq 1$. Moreover, $h(a_1, a_1) = 3a_1^2 - 4a_1 + 3 \leq 2$ for $\frac{1}{2} \leq a_1 \leq 1$. Finally, $h(a_1, 1 - a_1) = a_1^2 - a_1 + 2 \leq 2$ provided $\frac{1}{2} \leq a_1 \leq 1$. Consequently, $\sum |S_n| < h(a_1, a_2) \leq 2$ for $(a_1, a_2) \in T_h$, which is a contradiction.

Subcase 1b, when $a_1 + a_3 \geq 1$. We put the three first squares so that

$$\sigma_1 S_1 = [0, a_1] \times [0, a_1],\ \sigma_2 S_2 = [1 - a_2, 1] \times [0, a_2],\ \sigma_3 S_3 = [1 - a_3, 1] \times [a_2, a_2 + a_3].$$

If $a_2 + a_3 \leq a_1$, the squares $S_4, S_5, \dots$ are used for the covering of the rectangle $[0,1] \times [1 - a_2 - a_3, 1]$ by the method from Lemma 1. Assume that I is not covered. By Lemma 1 we have

$$\sum |S_n| < a_1^2 + a_2^2 + a_3^2 + 1 - a_2 - a_3 + a_3 + (1 - a_2 - a_3)a_3. \tag{1}$$

If $a_2 \geq \frac{1}{2}$, taking $1-a_2$ in place of a_3 and 1 in place of a_1 in (1) we have $\sum |S_n| < 2a_2^2-3a_2+3 \leq 2$, a contradiction. If $a_2 < \frac{1}{2}$, replacing a_3 by a_2 in (1) we have $\sum |S_n| < 1 + a_1^2 < 2$, which is a contradiction.

If $a_2 + a_3 > a_1$, the squares $S_4, S_5, \dots$ are used for the covering of the rectangle $[0,1] \times [a_1, 1]$ by the method from Lemma 1. If I is not covered, by Lemma 1 we have

$$\sum |S_n| < a_1^2 + a_2^2 + a_3^2 + 1 - a_1 + a_3 + (1-a_1)a_3. \tag{2}$$

If $a_2 \geq \frac{1}{2}$, taking $1 - a_2$ in place of a_3 in (2) we obtain that

$$\sum |S_n| < a_1^2 + 2a_2^2 - 2a_1 - 4a_2 + a_1a_2 + 4.$$

Denote by $k(a_1, a_2)$ the function on the right side of this formula. Let $T_k = \{(a_1, a_2);\ \frac{1}{2} \leq a_2 \leq a_1 \leq 1\}$. We see that $k(a_1, a_2)$ does not have any maximum inside T_k. Consider k on the boundary of T_k. We have $k(a_1, \frac{1}{2}) = a_1^2 - \frac{3}{2}a_1 + \frac{5}{2}$. This value is not greater than 2 provided $\frac{1}{2} \leq a_1 \leq 1$. Moreover, $k(1, a_2) = 2a_2^2 - 3a_2 + 3 \leq 2$ for $\frac{1}{2} \leq a_2 \leq 1$. Finally, $k(a_1, a_1) = 4a_1^2 - 6a_1 + 4 \leq 2$ provided $\frac{1}{2} \leq a_1 \leq 1$. Consequently, $\sum |S_n| < k(a_1, a_2) \leq 2$ for $(a_1, a_2) \in T_k$, which is a contradiction. If $a_2 < \frac{1}{2}$, replacing a_3 by a_2 in (2) we have

$$\sum |S_n| < a_1^2 + 2a_2^2 - a_1 + 2a_2 - a_1a_2 + 1.$$

This value is also not greater than 2 provided $0 < a_2 < \frac{1}{2}$ and $\frac{1}{2} < a_1 < 1$, which is a contradiction.

Case 2, when $a_2 \leq 0.87\sqrt{2} - 0.37\sqrt{2}a_1$ *and when* $a_2 + a_3 \geq 1$.

Observe, that the second condition of the case implies that $a_2 \geq \frac{1}{2}$.

First assume that $a_3 \leq (\sqrt{2}+1)(1-a_1)$ or that $2a_1 + a_2\sqrt{2} + a_3\sqrt{2} \leq 3$. We put the three first squares so that:

$$\sigma_1 S_1 = [0, a_1] \times [0, a_1],\ \sigma_2 S_2 = [1-a_2, 1] \times [0, a_2],\ \sigma_3 S_3 = [1-a_3, 1] \times [1-a_3, 1].$$

The remaining squares are used for the covering of the rectangle $[0, 1-a_3] \times [a_1, 1]$ by the method from Lemma 1. Assume that I is not covered. By Lemma 1 we have

$$\sum |S_n| < a_1^2 + a_2^2 + a_3^2 + (1 - a_1 + a_3)(1-a_3) + (1-a_1)^2. \tag{3}$$

If $a_3 \leq (\sqrt{2}+1)(1-a_1)$, taking $(\sqrt{2}+1)(1-a_1)$ in place of a_3 and a_1 in place of a_2 in (3) we have $\sum |S_n| < (2-\sqrt{2})a_1^2 + (\sqrt{2}-2)a_1 + 2 \leq 2$, a contradiction. If $2a_1 + a_2\sqrt{2} + a_3\sqrt{2} \leq 3$, from $a_2 + a_3 \geq 1$ we see that $a_1 \leq \frac{1}{2}(3 - a_2\sqrt{2} - a_3\sqrt{2}) \leq \frac{1}{2}(3-\sqrt{2})$. Moreover, $a_3 \leq \frac{1}{2}\sqrt{2}(3 - 2a_1 - a_2\sqrt{2}) \leq \frac{1}{2}\sqrt{2}(3 - 2a_3 - a_3\sqrt{2})$, and consequently $a_3 \leq \frac{3}{2}(\sqrt{2}-1)$. Taking $\frac{3}{2}(\sqrt{2}-1)$ in place of a_3 and a_1 in place of a_2 in (3) we have $\sum |S_n| < 3a_1^2 + a_1(\frac{3}{2}\sqrt{2} - \frac{9}{2}) + 2$. This value is not greater than 2 for $\frac{1}{2} < a_1 \leq \frac{1}{2}(3-\sqrt{2})$, which is a contradiction.

Assume now that

$$a_3 > (\sqrt{2}+1)(1-a_1) \tag{4}$$

and that

$$2a_1 + a_2\sqrt{2} + a_3\sqrt{2} > 3. \tag{5}$$

By $a_1 \geq a_3 > (\sqrt{2}+1)(1-a_1)$ we conclude that $a_1 > \frac{1}{2}\sqrt{2}$. Consider four subcases.

Subcase 2a, when $a_2 \leq \frac{1}{2}\sqrt{2}$ *and when* $a_4 < (\sqrt{2}+1)(1-a_1)$. From $a_2 \leq \frac{1}{2}\sqrt{2}$ we see that $a_1 + a_2\sqrt{2} - 1 \leq a_1$. From (5) it follows that $3 - a_1 - a_2\sqrt{2} - a_3\sqrt{2} < a_1$. Obviously, $3 - a_1 - a_2\sqrt{2} - a_3\sqrt{2} \geq 3 - a_1 - 2a_2\sqrt{2} > 0$ (see Figure 3).

We put the three first squares so that $\sigma_1 S_1 = [0, a_1] \times [0, a_1]$, that the sides of $\sigma_2 S_2$ contain the points $(a_1, 0)$, $(1, 0)$, $(a_1, a_1 + a_2\sqrt{2} - 1)$, $(1, a_1 + a_2\sqrt{2} - 1)$ and that the sides of $\sigma_3 S_3$ contain the points $(1, a_1 + a_2\sqrt{2} - 1)$, $(3 - a_1 - a_2\sqrt{2} - a_3\sqrt{2}, 1)$, $(1, 1)$ (see Figure 3). The remaining squares are used for the covering of the rectangle $[0, 3 - a_1 - a_2\sqrt{2} - a_3\sqrt{2}] \times [a_1, 1]$ by the method from Lemma 1.

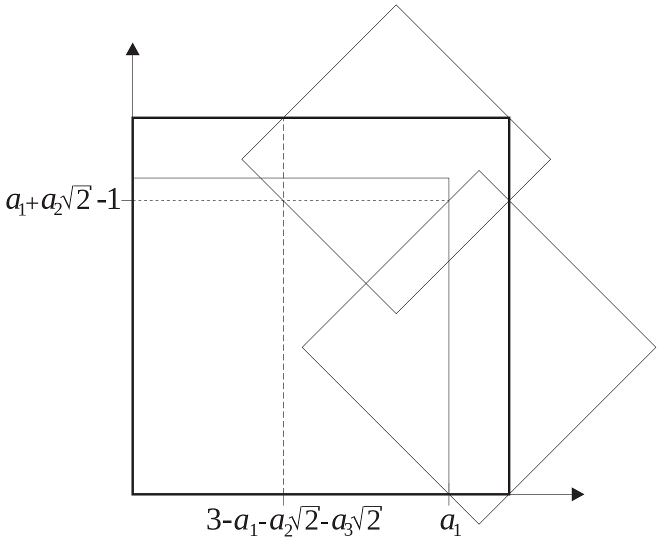

Figure 3

Assume that I is not covered. By Lemma 2 we see that

$$\sum |S_n| < a_1^2 + a_2^2 + a_3^2 + (\sqrt{2}+1)(3 - a_1 - a_2\sqrt{2} - a_3\sqrt{2})(1 - a_1) + (1 - a_1)^2.$$

By applying the method of finding the absolute maximum of the function of three variables (similarly as in Case 1 for the function of two variables), we show that this value not greater than 2 provided $(\sqrt{2}+1)(1-a_1) \leq a_3 \leq a_2 \leq \frac{1}{2}\sqrt{2} \leq a_1 \leq 1$, which is a contradiction.

Subcase 2b, when $a_2 \leq \frac{1}{2}\sqrt{2}$ *and when* $a_4 \geq (\sqrt{2}+1)(1-a_1)$. We put the three first squares as in Subcase 2a. The squares $S_4, S_5, \ldots$ are used for the covering of the rectangle $[0, 3 - a_1 - a_2\sqrt{2} - a_3\sqrt{2}] \times [1 - a_1, 1]$ by the method described in Lemma 3. By Lemma 3 we see that if I is not covered, then

$$\sum |S_n| < a_1^2 + a_2^2 + a_3^2 + \frac{1}{2}(4 - 2a_1 - a_2\sqrt{2} - a_3\sqrt{2})^2 + (1 - a_1)^2.$$

By applying the method of finding the absolute maximum of the function of three variables we show that this value is smaller than or equal to 2 provided $\frac{1}{2} \le a_2 \le \frac{1}{2}\sqrt{2} \le a_1 \le 1$ and $1 - a_2 \le a_3 \le a_2$, which is a contradiction.

Subcase 2c, when $a_2 > \frac{1}{2}\sqrt{2}$ and when $a_3 \le \frac{3}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}a_1 - a_2$. We put the three first squares as in Fig. 4, i.e. so that $\sigma_1 S_1 = [0, a_1] \times [0, a_1]$, so that the sides of $\sigma_2 S_2$ contain the points $(a_1, 0)$, $(1, 0)$, $(a_1, a_1 + a_2\sqrt{2} - 1)$, $(1, a_1 + a_2\sqrt{2} - 1)$ and so that the sides of $\sigma_3 S_3$ contain the points $(0, a_1)$, $(0, 1)$, $(a_1 + a_3\sqrt{2} - 1, a_1)$, $(a_1 + a_3\sqrt{2} - 1, 1)$.

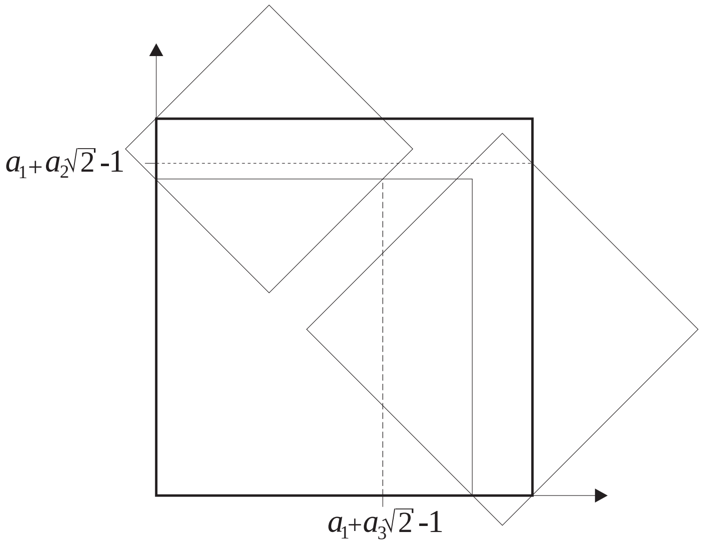

Figure 4

If $a_4 < (\sqrt{2} + 1)(1 - a_1)$, the squares $S_4, S_5, \ldots$ are used for the covering of the rectangle $[a_1 + a_3\sqrt{2} - 1, 1] \times [a_1, 1]$ by the method from Lemma 1.

Assume that I is not covered. By Lemma 2 we see that if (S_n) does not permit a covering of I,

$$\sum |S_n| < a_1^2 + a_2^2 + a_3^2 + (\sqrt{2} + 1)(2 - a_1 - a_3\sqrt{2})(1 - a_1) + (1 - a_1)^2.$$

By applying the method of finding the absolute maximum of the function of three variables it can be shown that this value is not greater than 2 provided $\frac{1}{2}\sqrt{2} \le a_2 \le a_1 \le 1$, $a_2 \le 0.87\sqrt{2} - 0.37\sqrt{2}a_1$ and $(\sqrt{2} + 1)(1 - a_1) \le a_3 \le \frac{3}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}a_1 - a_2$, which is a contradiction.

If $a_4 \ge (\sqrt{2}+1)(1-a_1)$, the squares $S_4, S_5, \ldots$ are used for the covering of the rectangle $[a_1 + a_3\sqrt{2} - 1, 1] \times [a_1, 1]$ by the method described in Lemma 3. Assume that I is not covered by our method. By Lemma 3 we see that

$$\sum |S_n| < a_1^2 + a_2^2 + a_3^2 + \frac{1}{2}(3 - 2a_1 - a_3\sqrt{2})^2 + (1 - a_1)^2.$$

It can be shown that, under the assumptions of the subcase, this value is not greater than 2, which is a contradiction.

Subcase 2d, when $a_2 > \frac{1}{2}\sqrt{2}$ *and when* $a_3 > \frac{3}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}a_1 - a_2$. We put the three first squares as in Subcase 2c. From $a_2 \geq a_3 > \frac{3}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}a_1 - a_2$ we have $a_2 > \frac{3}{4}\sqrt{2} - \frac{1}{4}\sqrt{2}a_1$. Consequently, we see that $a_2\sqrt{2} - 1 - (\frac{1}{2} - \frac{1}{2}a_1) > 0$, i.e. that there are points in $\sigma_2 S_2$ with second coordinate greater than 1 (see Fig. 5). Moreover, the second condition of the subcase implies that $a_1 + a_3\sqrt{2} - 1 \geq 2 - a_2\sqrt{2}$ (see Fig. 5).

If $a_1 + a_2\sqrt{2} \geq 2$, then I is covered by $\sigma_1 S_1 \cup \sigma_2 S_2 \cup \sigma_3 S_3$. In the opposite case, the squares $S_4, S_5, \ldots$ are used for the covering of the square $[a_1 + a_2\sqrt{2} - 1, 1] \times [a_1 + a_2\sqrt{2} - 1, 1]$ by the method from Lemma 1.

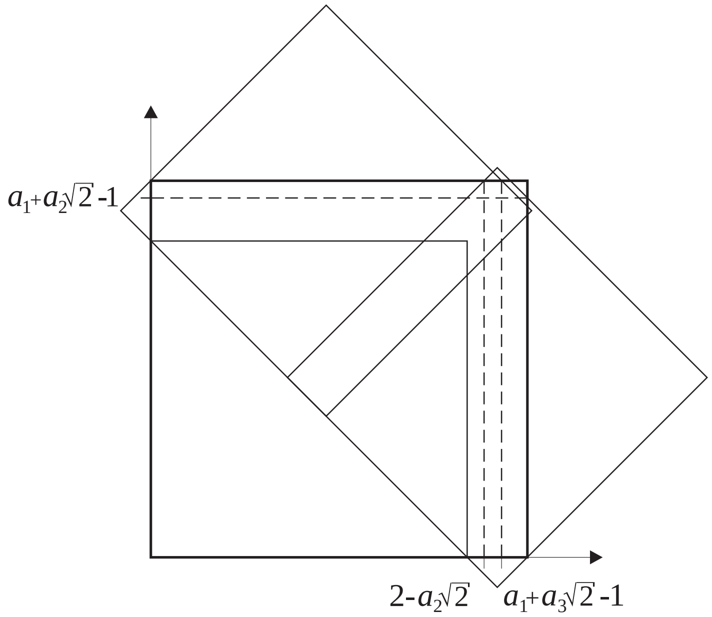

Figure 5

Assume that I is not covered. By Lemma 1 we deduce that

$$|S_4| + |S_5| + \ldots < 3\big(2 - a_1 - a_2\sqrt{2}\big)^2.$$

Consequently,

$$\sum |S_n| < a_1^2 + a_2^2 + a_3^2 + 3\big(2 - a_1 - a_2\sqrt{2}\big)^2.$$

It is easy to check that, under the assumptions of the subcase, this value is not greater than 2, which is a contradiction.

Case 3, when $a_2 > 0.87\sqrt{2} - 0.37\sqrt{2}a_1$.

For simplicity of the notation, we will write a instead of a_1 and b instead of a_2. From $a \geq b$ we conclude that $a > \frac{0.87\sqrt{2}}{1+0.37\sqrt{2}} = 0.8077\ldots$ Moreover, $b > \frac{1}{2}\sqrt{2}$.

We put the two first squares as in Fig. 6, i.e. so that $\sigma_1 S_1 = [0, a] \times [0, a]$ and that the sides of $\sigma_2 S_2$ contain the points $(a, 0)$, $(1, 0)$, $(1, 1)$.

We show that the motion σ_2 is achievable. From Fig. 6 we see that to show this, it is sufficient to show that $u < b$.

We see that $u = \frac{b-w}{1-a}$ and $w = \sqrt{1-u^2}$. Hence, $u = \frac{b-\sqrt{1-u^2}}{1-a}$, and consequently

$$u = \frac{b - ab + \sqrt{2 + a^2 - 2a - b^2}}{a^2 - 2a + 2} \tag{6}$$

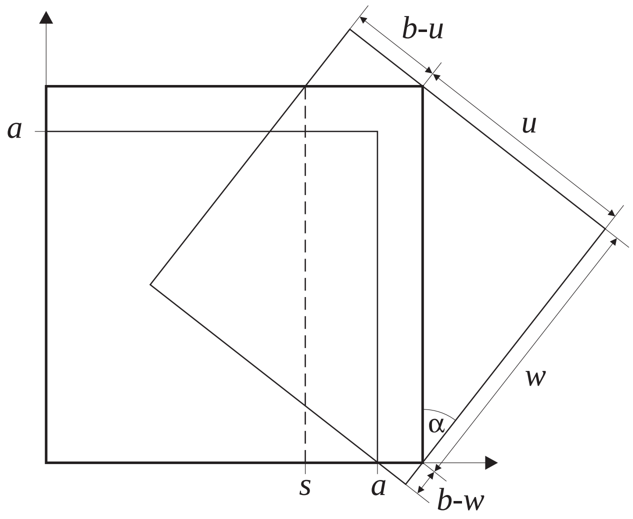

Figure 6

A simple computation shows that the condition $0.87\sqrt{2} - 0.37\sqrt{2}a > \frac{1}{\sqrt{1+a^2}}$ is equivalent to the inequality

$$(a-1)(0.2738a^3 - 1.0138a^2 + 0.7738a - 0.5138) > 0 \tag{7}$$

for $0.8077\ldots < a < 1$. The value in the first parentheses is negative. Denote the function in the second parentheses by $h(a)$. We see that $h'(a)$ is negative in the considered interval. Consequently, $h(a) \le h(0.8077\ldots) < 0$ and (7) holds. Hence, the condition $b > 0.87\sqrt{2} - 0.37\sqrt{2}a$ implies that $b > \frac{1}{\sqrt{1+a^2}}$. It is easy to show that $u < b$ provided $b > \frac{1}{\sqrt{1+a^2}}$. Thus, the covering as in Fig. 6 is always possible.

Denote by $(s, 1)$, where $s \neq 1$, the common point of the boundaries of I and $\sigma_2 S_2$. From Fig. 6 we see that $w = \frac{b-u}{1-s}$ and $\frac{b-w}{1-a} = u$. Hence,

$$s = \frac{ua}{b - u + au}. \tag{8}$$

By (6) and (8) we have

$$s = \frac{ab - a^2 b + a\sqrt{2 + a^2 - 2a - b^2}}{b + (a-1)\sqrt{2 + a^2 - 2a - b^2}}. \tag{9}$$

The squares $S_3, S_4, \dots$ are used for the covering of the rectangle $[0,s]\times[a,1]$.
If $a_3 < (\sqrt{2}+1)(1-a)$, we use the method from Lemma 1 for the covering of $[0,s]\times[a,1]$. Assume that I is not covered. By Lemma 2 we see that

$$\sum |S_n| < a^2 + b^2 + (\sqrt{2}+1)(1-a)s + (1-a)^2.$$

From $s<1$ we have $\sum |S_n| < 2a^2 + (\sqrt{2}+1)(1-a) + (1-a)^2 \le 2$, which is a contradiction.
If $a_3 \ge (\sqrt{2}+1)(1-a)$, we use the method described in Lemma 3 for the covering of $[0,s]\times[a,1]$. Consequently, if (S_n) does not permit a covering of I, then

$$\sum |S_n| < a^2 + b^2 + \frac{1}{2}(s+1-a)^2 + (1-a)^2. \tag{10}$$

In order to obtain a contradiction, we show that the value on the right side of (10) is not greater than 2. We have

$$s+1-a = \frac{b - a^2 b + (3a - a^2 - 1)\sqrt{2+a^2-2a-b^2}}{b + (a-1)\sqrt{2+a^2-2a-b^2}}.$$

From $3a - a^2 - 1 \le a$ we see that in order to complete the proof it is sufficient to show that

$$a^2 + b^2 + \frac{1}{2}\left[\frac{b - a^2 b + a\sqrt{2+a^2-2a-b^2}}{b + (a-1)\sqrt{2+a^2-2a-b^2}}\right]^2 + (1-a)^2 \le 2. \tag{11}$$

In order to get this inequality it is convenient to multiply (11) by

$$2\left[b + (a-1)\sqrt{2+a^2-2a-b^2}\right]^2.$$

Consequently, (11) is equivalent to

$$\begin{aligned}&4a^6 - 20a^5 + 43a^4 - 46a^3 + 20a^2 + 4a - 4 + \\ &+ 5b^2 - a^4b^2 + 4a^3b^2 - 2a^2b^4 + 5a^2b^2 + 4ab^4 - 16ab^2 + \\ &\sqrt{2+a^2-2a-b^2}\left(6a^3b - 16a^2b + 4ab^3 + 6ab - 4b^3 + 4b\right) \le 0.\end{aligned} \tag{12}$$

for $b \le a \le 1$ and for $b \ge 0.87\sqrt{2} - 0.37\sqrt{2}a$.

The value in the parentheses is positive, as is easy to check.

For simplicity of further computations consider three subcases.

Subcase 3a, when $b \ge 0.8$. Observe that $2 + a^2 - 2a - b^2 < 0.392$ for $(a,b) \in D$, where

$$D = \{(a,b);\ 0.8 \le b \le a \le 1,\ b \ge 0.87\sqrt{2} - 0.37\sqrt{2}a\}.$$

Consequently, replacing $\sqrt{2+a^2-2a-b^2}$ by 0.63 in (12) we see that in order to show (12) for $b \ge 0.8$ it is sufficient to show that

$$\begin{aligned}&4a^6 - 20a^5 + 43a^4 - 46a^3 + 20a^2 + 4a - 4 + \\ &+ 5b^2 - a^4b^2 + 4a^3b^2 - 2a^2b^4 + 5a^2b^2 + 4ab^4 - 16ab^2 + \\ &3.78a^3b - 10.08a^2b + 2.52ab^3 + 3.78ab - 2.52b^3 + 2.52b \le 0,\end{aligned} \tag{13}$$

for $(a,b) \in D$.

Denote by $f(a,b)$ the function on the left side of (13). We see that the partial derivative

$$f'''_{bbb}(a,b) = -48a^2b + 96ab + 15.12a - 15.12 > 0.$$

Consequently, $f''_{bb}(a,b) \geq f''_{bb}(a,0.8) > 0$. This means that $f'_b(a,b)$ is increasing with respect to b. As a consequence, the function $f(a,b)$ does not have any maximum inside D.

Consider the points from the boundary of D. We see that

$$f(a,a) = a^6 - 12a^5 + 54.3a^4 - 74.6a^3 + 28.78a^2 + 6.52a - 4.$$

It is easy to see that this value is not greater than zero for $0.8077\ldots \leq a \leq 1$. Obviously, $f(1,b) = 2b^4 - 3b^2 + 1 \leq 0$ for $0.8 \leq b \leq 1$. Moreover,

$$f(a,0.8) = 4a^6 - 20a^5 + 42.36a^4 - 40.416a^3 + 14.3168a^2 - 0.28736a - 0.07424.$$

A computation shows that $f(a,0.8) \leq 0$ provided $\frac{0.87\sqrt{2}-0.8}{0.37\sqrt{2}} \leq a \leq 1$. It can be shown that also $f(a, 0.87\sqrt{2} - 0.37\sqrt{2}a) \leq 0$ for $0.8077\ldots \leq a \leq \frac{0.87\sqrt{2}-0.8}{0.37\sqrt{2}}$.

Subcase 3b, when $0.77 \leq b \leq 0.8$. Observe that now $2+a^2-2a-b^2 < 0.422$. Consequently, replacing $\sqrt{2+a^2-2a-b^2}$ by 0.65 on the left side of (12) we see that in order to confirm (12) it is sufficient to show that

$$\begin{aligned}&4a^6 - 20a^5 + 43a^4 - 46a^3 + 20a^2 + 4a - 4 +\\ &+ 5b^2 - a^4b^2 + 4a^3b^2 - 2a^2b^4 + 5a^2b^2 + 4ab^4 - 16ab^2 +\\ &3.9a^3b - 10.4a^2b + 2.6ab^3 + 3.9ab - 2.6b^3 + 2.6b \leq 0,\end{aligned} \tag{14}$$

for $0.77 \leq b \leq 0.8$, for $b \geq 0.87\sqrt{2} - 0.37\sqrt{2}a$ and for $a \leq 1$. Denote by $g(a,b)$ the function on the left side of (14). We have

$$g'''_{bbb}(a,b) = -48a^2b + 96ab + 15.6a - 15.6 > 0.$$

Consequently, $g''_{bb}(a,b) \geq g''_{bb}(a,0.77) > 0$ and $g'_b(a,b) \leq g'_b(a,0.8) < 0$. Hence, $g(a,b) \leq g(a,0.77) = 4a^6-20a^5+42.4071a^4-40.6254a^3+14.25343918a^2+0.10970744a-0.2204858 \leq 0$, provided $\frac{0.87\sqrt{2}-0.77}{0.37\sqrt{2}} \leq a \leq 1$, as is easy to check, and $g(a,b) \leq g(a, 0.87\sqrt{2}-0.37\sqrt{2}a) \leq 0$, provided $\frac{0.87\sqrt{2}-0.8}{0.37\sqrt{2}} \leq a \leq \frac{0.87\sqrt{2}-0.77}{0.37\sqrt{2}}$, as is easy to show.

Subcase 3c, when $b < 0.77$. Observe that $2+a^2-2a-b^2 \leq 0.5$. Taking 0.71 in place of $\sqrt{2+a^2-2a-b^2}$ on the left side of (12) we see that to show (12) it is sufficient to show that

$$\begin{aligned}&4a^6 - 20a^5 + 43a^4 - 46a^3 + 20a^2 + 4a - 4 +\\ &+ 5b^2 - a^4b^2 + 4a^3b^2 - 2a^2b^4 + 5a^2b^2 + 4ab^4 - 16ab^2 +\\ &4.26a^3b - 11.36a^2b + 2.84ab^3 + 4.26ab - 2.84b^3 + 2.84b \leq 0,\end{aligned} \tag{15}$$

for $0.87\sqrt{2} - 0.37\sqrt{2}a \leq b \leq 0.77$ and for $a \leq 1$.

An analysis similar to that in Subcase 3b shows that (15) holds. □

In [1] and [2] it is proved that any sequence of rectangles whose side lengths are not greater than 1 permits a parallel covering of I, provided the total area of this sequence is not smaller than 3.

The author conjectures that the unit square can be covered by any sequence of rectangles of side lengths not greater than 1 with total area greater than or equal to 2.

References

[1] Bezdek, A.; Bezdek, K.: *Eine hinreichende Bedingung für die Überdeckung des Einheitswürfels durch homothetische Exemplare im n-dimensionalen euklidischen Raum.* Beiträge Algebra Geom. **17** (1984), 5–21. Zbl 0556.52008

[2] Groemer, H.: *Covering and packing properties of bounded sequences of convex sets.* Mathematika **29** (1982), 18–31. Zbl 0496.52011

[3] Moon, J. W.; Moser, L.: *Some packing and covering theorems.* Colloq. Math. **17** (1967), 103–110. Zbl 0152.39502

[4] Moser, W.; Pach, J.: *Research Problems in Discrete Geometry.* Privately published collection of problems, 1994.

Received November 27, 2000