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On Homogeneous Hypersurfaces in Complex Grassmannians*

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Abstract. In the present article we consider a class of real hypersurfaces of the Grassmann manifold of k-planes in $\mathbb{C}^n$, $G_k(\mathbb{C}^n)$, for $k > 2$. Namely the family of tubes around $G_k(\mathbb{C}^m)$ with $m < n$ and around the quaternionic Grassmann manifold of $k/2$-quaternionic planes in $\mathbb{H}^{n/2}$, $G_{k/2}(\mathbb{H}^{n/2})$, when k and n are even. We determine which of those tubes are homogeneous and for them we find the spectral decomposition of the shape operator. As a consequence we show that they are Hopf hypersurfaces.

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1. Introduction

The study of real hypersurfaces in complex projective spaces has a long and interesting history. A nice survey can be found in [11]. A particular subclass is that of the homogeneous hypersurfaces which at the present time seems to be very well understood. In this respect the reader may get well acquainted with this topic by reading the article [8] and references therein.

The study of real hypersurfaces (homogeneous or not) in $\mathbb{CP}^n$ suggests immediately the problem of understanding real hypersurfaces in other compact hermitian symmetric spaces.

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A very nice step in this direction is the paper [2] where the authors, using deep knowledge of the Grassmannians obtained in [1], study real hypersurfaces in the complex two-plane Grassmannian obtaining an important result about hypersurfaces in $G_2(\mathbb{C}^{m+2})$ (cf [2], Th.1, p.2). Those authors take particular advantage from the fact that $G_2(\mathbb{C}^{m+2})$ is the only compact, Kaehler, quaternionic Kaehler manifold with positive scalar curvature. They use then the Kaehler structure J and the quaternionic Kaehler structure $\mathcal{J}$ to determine all the real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ for which $J(\perp M)$ and $\mathcal{J}(\perp M)$ are invariant by the shape operator of M in $G_2(\mathbb{C}^{m+2})$. If one takes complete hypersurfaces it turns out that the only solutions are tubes around a totally geodesic $G_2(\mathbb{C}^{m+1})$ and tubes around a totally geodesic $\mathbb{HP}^n$ in $G_2(\mathbb{C}^{2n+2})$ which are real homogeneous hypersurfaces.

The present paper grew out of our attempt to study the more general situation of other complex Grassmannians as ground manifolds. As soon as one takes other Grassmannians, the quaternionic Kaehler condition is lost and the road, which is not easy in [2] gets rougher.

The goal of the present article is to initiate the study of real hypersurfaces in $G_k(\mathbb{C}^n)$ for $k > 2$ and we think that our results may be found interesting.

We study a certain class of real hypersurfaces of the Grassmann manifold of k-planes in $\mathbb{C}^n$, $G_k(\mathbb{C}^n)$ for $k > 2$. This class consists of tubes around $G_k(\mathbb{C}^m)$ with $m < n$ and those around the quaternionic Grassmann manifold of $k/2$-quaternionic planes in $\mathbb{H}^{n/2}$, $G_{k/2}(\mathbb{H}^{n/2})$, when k and n are even.

Our objective is to determine which tubes of the considered families are homogeneous and for them to find the spectral decomposition of the shape operator. According to J. Berndt (private communication) the classification of homogeneous hypersurfaces in complex Grassmanians seems to be obtainable from the results of A. Kollross [10] concerning hyperpolar actions on irreducible simply connected symmetric spaces of compact type. Our methods however are based on the root structure associated to the Grassmann manifold $G_k(\mathbb{C}^m)$.

In Section 2 we study the natural action on the tubes around $G_k(\mathbb{C}^m)$ for $m < n$ and around $G_{k/2}(\mathbb{H}^{n/2})$ when k and n are even and identify those for which the isotropy group of the center acts transitively on the zero centered spheres in the normal space. These are precisely the homogeneous tubes centered at $G_k(\mathbb{C}^{n-1})$ and $G_q(\mathbb{H}^{q+1})$ for $n = 2q + 2$ and $k = 2q$ (Theorem 1).

Section 3 deals with the computation of the spectrum of the Jacobi operator $R_Z = R(.,Z)Z$ in the direction of Z (for suitable Z) where R is the curvature of the Riemannian connection on $G_k(\mathbb{C}^n)$. This is a nontrivial task and we indicate some of the required calculations. The results are summarized in Proposition 1. Using these results and via some delicate computations, we finally obtain Theorem 2 which together Theorem 1 gives a complete classification of the homogeneous tubes of the considered families.

Section 4 contains our main result, Theorem 3, which gives the spectral decomposition of the shape operator of the family of tubes around $G_k(\mathbb{C}^{n-1})$ showing that they are Hopf hypersurfaces of $G_k(\mathbb{C}^n)$.

2. Some homogeneous tubes in $G_k(\mathbb{C}^n)$

The objective of this section is to study two families of homogeneous tubes in the Grassmannian manifold $G_k(\mathbb{C}^n)$ and describe their homogeneous structure.

The complex and quaternionic Grassmann manifolds are defined by

$$G_k(\mathbb{C}^s) = \frac{SU(s)}{S(U(k)\times U(s-k))} \quad \text{and} \quad G_q(\mathbb{H}^p) = \frac{Sp(p)}{Sp(q)\times Sp(p-q)}$$

with $k < s$ and $q < p$. Their real dimensions are $2k(s-k)$ and $4q(p-q)$ respectively.

For each of the above spaces we denote by o the class in the quotient of the identity element of the group.

Let us consider the natural embeddings

$$\begin{array}{rcll} j_1 & : & G_k(\mathbb{C}^m) \hookrightarrow G_k(\mathbb{C}^n) & \text{for } m < n \\ j_2 & : & G_q(\mathbb{H}^p) \hookrightarrow G_{2q}(\mathbb{C}^{2p}) & \text{for } q < p \end{array}$$

given by

$$\begin{aligned} j_1(\pi(A)) &= \pi(\tilde{A}) \quad \text{with } \tilde{A} = \begin{bmatrix} A & 0 \\ 0 & I_{n-m} \end{bmatrix}, \quad A \in SU(m) \\ j_2(\pi(A)) &= \pi(J_{p,q} A J_{p,q}^{-1}), \quad A \in Sp(p) \end{aligned}$$

where π denotes the corresponding projections map onto the quotient spaces,

$$J_{p,q} = \begin{bmatrix} I_q & 0 & 0 & 0 \\ 0 & 0 & I_q & 0 \\ 0 & I_{p-q} & 0 & 0 \\ 0 & 0 & 0 & I_{p-q} \end{bmatrix} \in U(2p) \tag{1}$$

and $Sp(q) \times Sp(p-q)$ is included into $Sp(p)$ via

$$\left(\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}, \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \right) \overset{j}{\hookrightarrow} \begin{bmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{bmatrix}. \tag{2}$$

On the ambient space $G_k(\mathbb{C}^n)$ we consider the $SU(n)$-invariant metric induced by the opposite of the Killing form of $\mathfrak{su}(n)$. This metric determines invariant metrics on $G_k(\mathbb{C}^m)$ and $G_q(\mathbb{H}^p)$ making the above defined j_1 and j_2, isometric imbeddings.

It is well known that the tubes centered at $G_k(\mathbb{C}^m)$ and $G_q(\mathbb{H}^p)$ are globally defined for sufficiently small radii (see for instance [7]).

Note that $SU(n)$ acts transitively on $G_k(\mathbb{C}^n)$ by isometries and for $m < n$ the action of $SU(m)$ on $G_k(\mathbb{C}^m)$ is the restriction of the former one. Moreover $SU(m)$ acts on each one of the tubes around $G_k(\mathbb{C}^m)$ by

$$g.(\exp_p rX) = \exp_{g.p} r g_{*p} X \tag{3}$$

where $p \in G_k(\mathbb{C}^m)$ and $X \in (T_p G_k(\mathbb{C}^m))^\perp$ with $\|X\| = 1$.

This observation is also true when we replace $SU(m)$ by $Sp(p)$ and $G_k(\mathbb{C}^m)$ by $G_q(\mathbb{H}^p)$, for $n = 2p$ and $k = 2q$.

Let us now study the natural action of $S(U(k)\times U(m-k))$ (the isotropy group of $SU(m)$ at the point $o\in G_k(\mathbb{C}^m)$) on the normal space $(T_oG_k(\mathbb{C}^m))^\perp$.

Since $S(U(k)\times U(m-k))\subset S(U(k)\times U(n-k))$ via

$$g=\begin{bmatrix} A & 0\\ 0 & B\end{bmatrix}\hookrightarrow g=\begin{bmatrix} A & 0 & 0\\ 0 & B & 0\\ 0 & 0 & I_{n-m}\end{bmatrix}$$

the elements of $T_oG_k(\mathbb{C}^m)$ are included in $T_oG_k(\mathbb{C}^n)$ in the form

$$Y=\begin{bmatrix} 0 & Y_1 & 0\\ -Y_1^* & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\quad Y_1\in M_{k\times(m-k)}(\mathbb{C}) \tag{4}$$

and the elements of $(T_oG_k(\mathbb{C}^m))^\perp$ are of the form

$$Z=\begin{bmatrix} 0 & 0 & Z_1\\ 0 & 0 & 0\\ -Z_1^* & 0 & 0\end{bmatrix},\quad Z_1\in M_{k\times(n-m)}(\mathbb{C}). \tag{5}$$

Keeping this notation, we see that if $g\in S(U(k)\times U(m-k))$ and $Z\in(T_oG_k(\mathbb{C}^m))^\perp$, then

$$g_{*o}Z=gZg^{-1}=\begin{bmatrix} 0 & 0 & AZ_1\\ 0 & 0 & 0\\ -(AZ_1)^* & 0 & 0\end{bmatrix}. \tag{6}$$

Hence the action of $S(U(k)\times U(m-k))$ on $(T_oG_k(\mathbb{C}^m))^\perp$ is nothing but the action of the group $U(k)$ on $M_{k\times(n-m)}(\mathbb{C})$ by matrix multiplication. Then we obtain the following Lemma.

Lemma 1. *The natural action of $S(U(k)\times U(m-k))$ on the unit sphere of $(T_oG_k(\mathbb{C}^m))^\perp$ is transitive if and only if $m=n-1$.* □

In order to study the natural action of $Sp(q)\times Sp(p-q)$ (the isotropy group of $Sp(p)$ at the point $o\in G_q(\mathbb{H}^p)$) on the normal space $(T_oG_q(\mathbb{H}^p))^\perp\subset T_oG_{2q}(\mathbb{C}^{2p})$ we split each matrix A in $Sp(p)$ into blocks as follows:

$$\begin{aligned} J_{p,q}AJ_{p,q}^{-1} &= J_{p,q}\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14}\\ A_{21} & A_{22} & A_{23} & A_{24}\\ A_{31} & A_{32} & A_{33} & A_{34}\\ A_{41} & A_{42} & A_{43} & A_{44}\end{bmatrix}J_{p,q}^{-1}=\\ &= \begin{bmatrix} A_{11} & A_{13} & A_{12} & A_{14}\\ A_{31} & A_{33} & A_{32} & A_{34}\\ A_{21} & A_{23} & A_{22} & A_{24}\\ A_{41} & A_{43} & A_{42} & A_{44}\end{bmatrix}.\end{aligned}$$

(A_{11},A_{33} are matrices of order $q\times q$ and A_{22},A_{44} are of order $(p-q)\times(p-q)$).

This allows us to include the subgroup $Sp(q)\times Sp(p-q)$ of $Sp(p)$ into $S(U(2q)\times U(2(p-q)))$ via $(g_1, g_2) \hookrightarrow \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}$ and at the same time to view the elements of $T_oG_q(\mathbb{H}^p)$ and $(T_oG_q(\mathbb{H}^p)^\perp$ included into $T_oG_{2q}(\mathbb{C}^{2p})$ respectively as

$$\begin{bmatrix} 0 & 0 & X & Y \\ 0 & 0 & -\overline{Y} & \overline{X} \\ -X^* & Y^t & 0 & 0 \\ -Y^* & -X^t & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & X & Y \\ 0 & 0 & \overline{Y} & -\overline{X} \\ -X^* & -Y^t & 0 & 0 \\ -Y^* & X^t & 0 & 0 \end{bmatrix} \tag{7}$$

where $X, Y \in M_{q\times(p-q)}(\mathbb{C})$.

For $g = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \in Sp(q) \times Sp(p-q)$ and $Z = \begin{bmatrix} 0 & Z_1 \\ -Z_1^* & 0 \end{bmatrix} \in T_oG_{2q}(\mathbb{C}^{2p})$ it is clear that

$$g_{*o}Z = gZg^{-1} = \begin{bmatrix} 0 & g_1Z_1g_2^* \\ -g_2Z_1^*g_1^* & 0 \end{bmatrix}. \tag{8}$$

Since the matrices g_j are of the form $g_j = \begin{bmatrix} A_j & B_j \\ -\overline{B}_j & \overline{A}_j \end{bmatrix}$, it is easy to see that $gZg^{-1} \in (T_oG_q(\mathbb{H}^p))^\perp$, $\forall Z \in (T_oG_q(\mathbb{H}^p))^\perp$ and therefore $Sp(q) \times Sp(p-q)$ acts on the unit sphere of $(T_oG_q(\mathbb{H}^p))^\perp$.

In ([3], p. 301, 10.94) a complete classification of Lie groups acting transitively on spheres is given. According to this information $Sp(q) \times Sp(p-q)$ $(2 \le q < p)$ acts transitively on the sphere S^{l-1} only when $p - q = 1$, and in this case $l = 4q$.

Moreover, for $p - q = 1$ the action given in (8) is transitive on the unit sphere of $(T_oG_q(\mathbb{H}^p))^\perp$. We have then established the following result.

Lemma 2. *The natural action of $Sp(q) \times Sp(p-q)$ on the unit sphere of $(T_oG_q(\mathbb{H}^p))^\perp$ is transitive if and only if $p = q + 1$.* □

For any submanifold M of $G_k(\mathbb{C}^n)$ we denote by $(M)_r$ *the tube of radius $r > 0$ around M.*

Set $p_0 = \exp_o rZ$ where $Z \in (T_oG_k(\mathbb{C}^{n-1}))^\perp$ is such that Z_1 in the expression (5) is a suitable multiple of the first vector of the canonical basis of $\mathbb{C}^k$. From (3) and (6) we may conclude that the isotropy group of the action of $SU(n-1)$ on the tube $(G_k(\mathbb{C}^{n-1}))_r$ at p_0 is given by

$$S(\{1\} \times U(k-1) \times U(n-1-k)) \subset SU(n-1).$$

Set now $q_0 = \exp_o rZ$ where $Z \in (T_oG_q(\mathbb{H}^{q+1}))^\perp$ corresponds in the representation (7), to $X = 0$ and Y a suitable multiple of the first vector of the canonical basis of $\mathbb{C}^q$. Let us denote by H the isotropy group of the natural action of $Sp(q+1)$ on the tube $(G_q(\mathbb{H}^{q+1}))_r$ at q_0.

From (3) and (8) we conclude

$$g \in H \Longleftrightarrow g = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \in Sp(q) \times Sp(1) \text{ and } gZg^{-1} = Z.$$

Further, if $g_1 = \begin{bmatrix} A & B \\ -\overline{B} & \overline{A} \end{bmatrix}$ and $g_2 = \begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix}$, performing the computation indicated in (8) we obtain

$$\begin{cases} \overline{a}B_1 + \overline{b}A_1 = 0 \\ -bB_1 + aA_1 = e_1 \end{cases}$$

where e_1 is the first vector of the canonical basis of $\mathbb{C}^q$ and A_1, B_1 denote the first columns of A, B respectively. This implies that $A_1 = \overline{a}e_1$ and $B_1 = -\overline{b}e_1$ and so it is easy to conclude that

$$H = Sp(1) \times Sp(q-1)$$

which is embedded into $Sp(q+1)$ via the map

$$(P, Q) \hookrightarrow \begin{bmatrix} j(uPu^{-1}, Q) & 0 \\ 0 & P \end{bmatrix},$$

where $u = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and j is the inclusion given in (2).

Lemmas 1, 2 and the preceeding discusion may be summarized in the following theorem.

Theorem 1. *For small enough radii $r > 0$ the tubes $(M)_r$ in $G_k(\mathbb{C}^n)$ around $M = G_k(\mathbb{C}^{n-1})$ and around $M = G_q(\mathbb{H}^{q+1})$ for $n = 2q+2$ and $k = 2q$, are homogeneous real hypersurfaces of $G_k(\mathbb{C}^n)$. Moreover these tubes are of the form*

$$(G_k(\mathbb{C}^{n-1}))_r = SU(n-1)/S(\{1\} \times U(k-1) \times U(n-1-k))$$
$$(G_q(\mathbb{H}^{q+1}))_r = Sp(q+1)/Sp(1) \times Sp(q-1). \qquad \square$$

3. The spectrum of R_Z

If R is the curvature tensor of the Riemannian manifold N, the Jacobi operator in the Z-direction for $Z \in TN$, is defined by $R_Z := R(.,Z)Z$.

The goal of the present section is to compute the spectral decomposition of the Jacobi operator R_Z for suitable $Z \in T_oN$ in order to give a necessary condition for the tubes under consideration to be homogeneous.

It is well known and easy to see that for a Riemannian symmetric space $N = G/K$ the Jacobi operator R_Z, for $Z \in T_oN$, is given by

$$R_Z = -(adZ)^2$$

where ad is the adjoint representation of the Lie algebra of G.

Let $N = G_k(\mathbb{C}^n)$. We need to introduce some Lie algebraic ingredients.
In the complex simple Lie algebra $\mathfrak{sl}(n,\mathbb{C})$ we take the Cartan subalgebra $\mathfrak{h}$ consisting of diagonal matrices with zero trace and let Δ be the root system of $\mathfrak{sl}(n,\mathbb{C})$ relative to $\mathfrak{h}$. We may write

$$\mathfrak{sl}(n,\mathbb{C}) = \mathfrak{h} \oplus \sum_{\gamma \in \Delta^+} (\mathfrak{g}_\gamma + \mathfrak{g}_{-\gamma})$$

where Δ^+ indicates the set of positive roots with respect to the usual order. Let $\pi = \{\alpha_1, \ldots, \alpha_{n-1}\} \subset \Delta^+$ be the standard system of simple roots. We also take in $\mathfrak{sl}(n, \mathbb{C})$ a Weyl basis $\{X_\gamma : \gamma \in \Delta\} \cup \{H_\beta : \beta \in \pi\}$ (see [9] p. 176). The following set of vectors provides a basis of the compact real form $\mathfrak{su}(n)$

$$\begin{array}{cc} U_\gamma = \frac{1}{\sqrt{2}}(X_\gamma - X_{-\gamma}) & \gamma \in \Delta^+, \\ U_{-\gamma} = \frac{i}{\sqrt{2}}(X_\gamma + X_{-\gamma}) & \gamma \in \Delta^+, \\ iH_\beta & \beta \in \pi. \end{array} \tag{9}$$

We shall also denote by $\mathfrak{h}_u$ the real vector space generated by $\{iH_\beta : \beta \in \pi\}$ and set $\mathfrak{m}_\gamma = \mathbb{R}U_\gamma \oplus \mathbb{R}U_{-\gamma}$. Then we have

$$\mathfrak{su}(n) = \mathfrak{h}_u \oplus \sum_{\gamma \in \Delta^+} \mathfrak{m}_\gamma$$

For $m < n$ we include $\mathfrak{sl}(m, \mathbb{C})$ into $\mathfrak{sl}(n, \mathbb{C})$ in a natural way and hence we can write

$$\mathfrak{sl}(m, \mathbb{C}) = \mathfrak{h}_1 \oplus \sum_{\gamma \in \Delta_1} (\mathfrak{g}_\gamma + \mathfrak{g}_{-\gamma})$$

where $\mathfrak{h}_1 = \left\{ \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} : A \in \mathfrak{sl}(m, \mathbb{C}) \right\}$ and $\Delta_1 = \{\sum l_j \alpha_j \in \Delta^+ : l_j = 0, m \leq j \leq n-1\}$.

Analogously we have

$$\mathfrak{su}(m) = \mathfrak{h}_{1u} \oplus \sum_{\gamma \in \Delta_1} \mathfrak{m}_\gamma$$

Let $\mathfrak{k}$ and $\mathfrak{k}_1 \subset \mathfrak{k}$ be the Lie algebras of $S(U(k)\times U(n-k))$ and $S(U(k)\times U(m-k))$ respectively. Then by [[5], 4.1] we may write

$$\begin{array}{rlr} \mathfrak{su}(n) = \mathfrak{k} \oplus \sum_{\gamma \in \Gamma} \mathfrak{m}_\gamma & \text{where} & \Gamma = \{\sum l_j \alpha_j \in \Delta^+ : l_k = 1\} \\ \mathfrak{su}(m) = \mathfrak{k}_1 \oplus \sum_{\gamma \in \Gamma_1} \mathfrak{m}_\gamma & \text{where} & \Gamma_1 = \{\gamma \in \Gamma : l_j = 0, m \leq j \leq n-1\}. \end{array}$$

As usual we identify

$$T_o G_k(\mathbb{C}^n) \equiv \sum_{\gamma \in \Gamma} \mathfrak{m}_\gamma \quad \text{and} \quad T_o G_k(\mathbb{C}^m) \equiv \sum_{\gamma \in \Gamma_1} \mathfrak{m}_\gamma \tag{10}$$

and hence

$$(T_o G_k(\mathbb{C}^m))^\perp = \sum_{\gamma \in \Gamma - \Gamma_1} \mathfrak{m}_\gamma. \tag{11}$$

The following notation will be useful for the roots of $\mathfrak{sl}(n, \mathbb{C})$

$$\varepsilon_{i,j} = \sum_{l=i}^{j} \alpha_l \tag{12}$$

however we shall denote the maximal root by μ or $\varepsilon_{1,n-1}$ indistinctly.

When $n = 2p$ and $k = 2q$, for

$$1 \leq i \leq q \text{ and } 2q \leq j \leq p + q - 1$$

we define the following tangent vectors of $T_oG_{2q}(\mathbb{C}^{2p})$

$$\begin{array}{ll} X^+_{i,j} = \frac{1}{\sqrt{2}}(U_{\varepsilon_{i,j}} + U_{\varepsilon_{q+i,j+p-q}}) & V^+_{i,j} = \frac{1}{\sqrt{2}}(U_{\varepsilon_{i,j}} - U_{\varepsilon_{q+i,j+p-q}}) \\ X^-_{i,j} = \frac{1}{\sqrt{2}}(U_{-\varepsilon_{i,j}} - U_{-\varepsilon_{q+i,j+p-q}}) & V^-_{i,j} = \frac{1}{\sqrt{2}}(U_{-\varepsilon_{i,j}} + U_{-\varepsilon_{q+i,j+p-q}}) \\ Y^+_{i,j} = \frac{1}{\sqrt{2}}(U_{\varepsilon_{i,j+p-q}} - U_{\varepsilon_{q+i,j}}) & W^+_{i,j} = \frac{1}{\sqrt{2}}(U_{\varepsilon_{i,j+p-q}} + U_{\varepsilon_{q+i,j}}) \\ Y^-_{i,j} = \frac{1}{\sqrt{2}}(U_{-\varepsilon_{i,j+p-q}} + U_{-\varepsilon_{q+i,j}}) & W^-_{i,j} = \frac{1}{\sqrt{2}}(U_{-\varepsilon_{i,j+p-q}} - U_{-\varepsilon_{q+i,j}}) \end{array} \tag{13}$$

It is easy to see that

$$\begin{array}{c} T_oG_q(\mathbb{H}^p) \equiv span_{\mathbb{R}}\{X^\pm_{i,j}, Y^\pm_{i,j}\} \\ (T_oG_q(\mathbb{H}^p))^\perp \equiv span_{\mathbb{R}}\{V^\pm_{i,j}, W^\pm_{i,j}\}. \end{array} \tag{14}$$

Our next task is to find the spectral decomposition of the Jacobi operator R_Z for $Z = U_\mu = U_{\varepsilon_{1,n-1}}$ and $Z = W^+_{1,p+q-1}$ respectively. To that end we introduce some extra notation. Let ε and ρ be elements of Δ^+ such that $\varepsilon - \rho \in \Delta$. We define

$$sg(\varepsilon - \rho) = \begin{cases} 1 & \text{if } \varepsilon - \rho \in \Delta^+ \\ -1 & \text{if } \rho - \varepsilon \in \Delta^+ \end{cases}$$

and

$$|\varepsilon - \rho| = \begin{cases} \varepsilon - \rho & \text{if } \varepsilon - \rho \in \Delta^+ \\ \rho - \varepsilon & \text{if } \rho - \varepsilon \in \Delta^+. \end{cases}$$

Then for $\varepsilon \neq \rho$ we get the following formulae (see [5] p. 223)

$$\begin{array}{l} [U_\varepsilon, U_{-\varepsilon}] = iH_\varepsilon \\ [U_\varepsilon, U_\rho] = \frac{1}{\sqrt{2}}\{N_{\varepsilon,\rho}U_{\varepsilon+\rho} + sg(\rho - \varepsilon)N_{\varepsilon,-\rho}U_{|\varepsilon-\rho|}\} \\ [U_\varepsilon, U_{-\rho}] = \frac{1}{\sqrt{2}}\{N_{\varepsilon,\rho}U_{-(\varepsilon+\rho)} + N_{\varepsilon,-\rho}U_{-|\varepsilon-\rho|}\}. \end{array} \tag{15}$$

Here we understand that if $\varepsilon \pm \rho$ are not roots then the terms $N_{\varepsilon,-\rho}U_{|\varepsilon-\rho|}$, $N_{\varepsilon,\rho}U_{\varepsilon+\rho}$, etc., vanish. Therefore, if $\gamma \in \Delta^+$ we have

$$[U_\mu, U_\gamma] = \begin{cases} -\frac{1}{\sqrt{2}}N_{\mu,-\gamma}U_{\mu-\gamma} & \text{if } \mu - \gamma \in \Delta \\ 0 & \text{if } \mu - \gamma \notin \Delta. \end{cases}$$

Using again the formulae (15) we obtain

$$[U_\mu, [U_\mu, U_\gamma]] = \begin{cases} \frac{1}{2}N_{\mu,-\gamma}N_{\mu,\gamma-\mu}U_\gamma & \text{if } \mu - \gamma \in \Delta \\ 0 & \text{if } \mu - \gamma \notin \Delta. \end{cases}$$

Due to $\mu - \gamma + (\gamma - \mu) = 0$ we have $N_{\mu,-\gamma} = -N_{\mu,\gamma-\mu}$. Further, since for the algebra $\mathfrak{sl}(n, \mathbb{C})$ (of type $\mathfrak{a}_{n-1}$) the constants $N_{\alpha,\beta}$ are given by

$$N_{\alpha,\beta} = \begin{cases} \pm\frac{1}{\sqrt{2n}} & \text{if } \alpha + \beta \in \Delta \\ 0 & \text{if } \alpha + \beta \notin \Delta \end{cases}$$

we may write

$$[U_\mu, [U_\mu, U_\gamma]] = \begin{cases} -\frac{1}{4n}U_\gamma & \text{if } \mu - \gamma \in \Delta \\ 0 & \text{if } \mu - \gamma \notin \Delta \end{cases}$$

and so, for $\gamma \in \Delta^+$ we obtain

$$R_{U_\mu}(U_\gamma) = \begin{cases} \frac{1}{4n}U_\gamma & \text{if } \mu - \gamma \in \Delta \\ 0 & \text{if } \mu - \gamma \notin \Delta. \end{cases}$$

In an analogous way, for $\gamma \in \Delta^+$ and $\gamma \neq \mu$ we have

$$R_{U_\mu}(U_{-\gamma}) = \begin{cases} \frac{1}{4n}U_{-\gamma} & \text{if } \mu - \gamma \in \Delta \\ 0 & \text{if } \mu - \gamma \notin \Delta. \end{cases}$$

Since

$$[U_\mu, H_\mu] = -\frac{1}{\sqrt{2}}\{\mu(H_\mu)X_\mu - (-\mu(H_\mu))X_{-\mu}\} = i\mu(H_\mu)U_{-\mu} \tag{16}$$

we get

$$[U_\mu, [U_\mu, U_{-\mu}]] = [U_\mu, iH_\mu] = -\mu(H_\mu)U_{-\mu}.$$

Let E_{ij} denote a square matrix with entry 1 where the i-th row and the j-th column meet, all other entries being 0. Then $H_\mu = \frac{1}{2n}(E_{11} - E_{nn})$ and consequently

$$\mu(H_\mu) = B(H_\mu, H_\mu) = 2ntr(H_\mu^2) = \frac{1}{n}.$$

Thus we deduce that $R_{U_\mu}(U_{-\mu}) = \frac{1}{n}U_{-\mu}$. For $\gamma \in \Delta^+$, the above calculations may be resumed into

$$R_{U_\mu}(X) = \begin{cases} \frac{1}{4n}X & \text{if } X = U_{\pm\gamma} \text{ and } \mu - \gamma \in \Delta \\ 0 & \text{if } X = U_\mu \text{ or } U_{\pm\gamma} \text{ and } 0 \neq \mu - \gamma \notin \Delta \\ \frac{1}{n}X & \text{if } X = U_{-\mu}. \end{cases} \tag{17}$$

Let now $n = 2p$ and $k = 2q$. For $Z = W^+_{1,p+q-1}$ by proceeding as in the previous situation, with some more effort, we obtain the following formulae.

$$R_Z(X) = \begin{cases} \frac{1}{8n}X & \left\{ \begin{array}{c} \text{if } X \text{ is one of the following } X^\pm_{s,t}, Y^\pm_{s,t}, V^\pm_{s,t}, W^\pm_{s,t} \text{ with} \\ (s,t) = (1,j),\ 2q \leq j \leq p+q-2 \text{ or} \\ (s,t) = (i, p+q-1), 2 \leq i \leq q \end{array} \right. \\ \frac{1}{2n}X & \text{if } X = V^\pm_{1,p+q-1} \text{ or } X = Y^-_{1,p+q-1} \text{ or } X = W^-_{1,p+q-1} \\ 0 & \text{for the remaining vectors of (13)} \end{cases} \tag{18}$$

The above calculations may be summarized in the following proposition.

Proposition 1. *For each one of the tangent vectors $Z \in T_oG_k(\mathbb{C}^n)$ given below, with the same notation of* (9) *and* (13)*, we list the eigenvalues c of R_Z, the corresponding eigenspace V_c and their dimensions.*

i) $Z = U_\mu = U_{\varepsilon_{1,n-1}}$

c	V_c	$\dim V_c$
$1/4n$	$\sum \mathfrak{m}_\gamma$, for $\gamma \in \Delta^+$, $\mu - \gamma \in \Delta$	$2(n-2)$
$1/n$	$\mathbb{R}U_{-\mu}$	1
0	$\mathbb{R}U_\mu \oplus \sum \mathfrak{m}_\gamma$, for $\gamma \in \Delta^+$, $0 \neq \mu - \gamma \notin \Delta$	$2(k-1)(n-1-k)+1$

ii) $n = 2p$, $k = 2q$, $Z = W^+_{1,p+q-1}$

c	V_c	$\dim V_c$
$1/8n$	$\sum_{(s,t)\in I_1} \mathfrak{m}_{s,t}$	$8(p-2)$
$1/2n$	$span\{V^\pm_{1,p+q-1}, Y^-_{1,p+q-1}, W^-_{1,p+q-1}\}$	4
0	$\sum_{(s,t)\in I-I_1} \mathfrak{m}_{s,t} \oplus \mathfrak{n}$	$8(q-1)(p-q-1)+4$

where $\mathfrak{m}_{s,t} = span\{X^\pm_{s,t}, Y^\pm_{s,t}, V^\pm_{s,t}, W^\pm_{s,t}\}$,
$\mathfrak{n} = span\{X^\pm_{1,p+q-1}, Y^+_{1,p+q-1}, W^+_{1,p+q-1}\}$,
$I = \{(i,j) : 1 \leq i \leq q$ *and* $2q \leq j \leq p+q-2\}$ *and*
$I_1 = \{(1,j) : 2q \leq j \leq q+p-2\} \cup \{(i, q+p-1) : 2 \leq i \leq q\}$. □

Remark 1. Note that from (10), (14) and the eigenspaces decomposition in the preceeding proposition it follows that $T_oG_k(\mathbb{C}^m)$ and $(T_oG_k(\mathbb{C}^m))^\perp$ are invariant by R_{U_μ} and the spaces $T_oG_q(\mathbb{H}^p)$ and $(T_oG_q(\mathbb{H}^p))^\perp$ are invariant by $R_{W^+_{1,p+q-1}}$.

Our next immediate objective is to decide which of the considered tubes are homogeneous. To that end we need to perform the following calculations.

Set $Z = aZ_1 + bZ_2 + cZ_3$ where $Z_i \in T_oG_k(\mathbb{C}^n)$ and $a, b, c \in \mathbb{R}$. By bilinearity of the Lie bracket and Jacobi´s identity we have for $\gamma \in \Delta$

$$\begin{aligned} [Z,[Z,U_\gamma]] &= a^2\,[Z_1,[Z_1,U_\gamma]] + b^2\,[Z_2,[Z_2,U_\gamma]] + c^2\,[Z_3,[Z_3,U_\gamma]] + \\ &\quad +2ab\,[Z_1,[Z_2,U_\gamma]] + 2ac\,[Z_1,[Z_3,U_\gamma]] + 2bc\,[Z_2,[Z_3,U_\gamma]] + \\ &\quad +ab\,[U_\gamma,[Z_1,Z_2]] + ac\,[U_\gamma,[Z_1,Z_3]] + bc\,[U_\gamma,[Z_2,Z_3]] \end{aligned} \tag{19}$$

Let now be $k < n-2$.

Let us take now $Z_1 = U_{\varepsilon_{1,n-2}}$, $Z_2 = U_{\varepsilon_{1,n-1}}$, $Z_3 = U_{\varepsilon_{2,n-2}}$ and pick the coefficients $a, b, c \in \mathbb{R}$ so that $a^2 + b^2 + c^2 = 1$ (hence $\|Z\| = 1$).

By (15) and computing the corresponding coefficients $N_{\alpha,\beta}$ we have that

$$\begin{aligned} [Z_1,Z_2] &= -\tfrac{1}{2\sqrt{n}}U_{\varepsilon_{n-1,n-1}} \\ [Z_1,Z_3] &= -\tfrac{1}{2\sqrt{n}}U_{\varepsilon_{1,1}} \\ [Z_2,Z_3] &= 0 \end{aligned} \tag{20}$$

It is also easy to obtain that

$$\begin{array}{ll}
\left[Z_1, U_{\varepsilon_{1,k}}\right] = \frac{1}{2\sqrt{n}} U_{\varepsilon_{k+1,n-2}} & \left[Z_2, U_{\varepsilon_{1,k}}\right] = \frac{1}{2\sqrt{n}} U_{\varepsilon_{k+1,n-1}} \\
\left[Z_3, U_{\varepsilon_{1,k}}\right] = 0 & \left[Z_1, \left[Z_1, U_{\varepsilon_{1,k}}\right]\right] = -\frac{1}{4n} U_{\varepsilon_{1,k}} \\
\left[Z_1, \left[Z_2, U_{\varepsilon_{1,k}}\right]\right] = 0 & \left[Z_2, \left[Z_2, U_{\varepsilon_{1,k}}\right]\right] = -\frac{1}{4n} U_{\varepsilon_{1,k}} \\
\left[U_{\varepsilon_{2,k}}, [Z_1, Z_2]\right] = 0 & \left[U_{\varepsilon_{2,k}}, [Z_1, Z_3]\right] = -\frac{1}{4n} U_{\varepsilon_{2,k}}
\end{array}$$

and

$$\begin{array}{ll}
\left[Z_1, U_{\varepsilon_{2,k}}\right] = 0 & \left[Z_2, U_{\varepsilon_{2,k}}\right] = 0 \\
\left[Z_3, U_{\varepsilon_{2,k}}\right] = \frac{1}{2\sqrt{n}} U_{\varepsilon_{k+1,n-2}} & \left[Z_1, \left[Z_3, U_{\varepsilon_{2,k}}\right]\right] = -\frac{1}{4n} U_{\varepsilon_{1,k}} \\
\left[Z_2, \left[Z_3, U_{\varepsilon_{2,k}}\right]\right] = 0 & \left[Z_3, \left[Z_3, U_{\varepsilon_{2,k}}\right]\right] = -\frac{1}{4n} U_{\varepsilon_{2,k}} \\
\left[U_{\varepsilon_{2,k}}, [Z_1, Z_2]\right] = 0 & \left[U_{\varepsilon_{2,k}}, [Z_1, Z_3]\right] = \frac{1}{4n} U_{\varepsilon_{1,k}}.
\end{array}$$

Using these relations and (20) in the expresion (19) we obtain

$$\left[Z, \left[Z, U_{\varepsilon_{1,k}}\right]\right] = -\frac{1}{4n}\{(a^2+b^2)U_{\varepsilon_{1,k}} + acU_{\varepsilon_{2,k}}\}$$

and

$$\left[Z, \left[Z, U_{\varepsilon_{2,k}}\right]\right] = -\frac{1}{4n}\{acU_{\varepsilon_{1,k}} + c^2U_{\varepsilon_{2,k}}\}.$$

Choosing $a = \frac{1}{2}\sqrt{\frac{3}{2}}, b = \frac{1}{2\sqrt{2}}$ and $c = \frac{1}{\sqrt{2}}$ we set

$$Z_0 = \frac{1}{2}\sqrt{\frac{3}{2}} U_{\varepsilon_{1,n-2}} + \frac{1}{2\sqrt{2}} U_{\varepsilon_{1,n-1}} + \frac{1}{\sqrt{2}} U_{\varepsilon_{2,n-2}} \tag{21}$$

and then

$$\left[Z_0, \left[Z_0, U_{\varepsilon_{1,k}} + U_{\varepsilon_{2,k}}\right]\right] = -\frac{2+\sqrt{3}}{16n}(U_{\varepsilon_{1,k}} + U_{\varepsilon_{2,k}}). \tag{22}$$

Let now $n = 2p$, $k = 2q$ and $q \leq p-2$.

Using the formula (19) and performing a straightforward but very tedious calculation, for $Z_1 = U_{\varepsilon_{1,2p-2}} + U_{\varepsilon_{q+1,p+q-2}}$, $Z_2 = U_{\varepsilon_{1,2p-1}} + U_{\varepsilon_{q+1,p+q-1}}$, $Z_3 = U_{\varepsilon_{2,2p-2}} + U_{\varepsilon_{q+2,p+q-2}}$, $W = aZ_1 + bZ_2 + bZ_3$ with $a^2 + 2b^2 = \frac{1}{2}$ and $Y = U_{\varepsilon_{2,p+q-2}} + U_{\varepsilon_{q+2,2p-2}} - U_{\varepsilon_{1,p+q-1}} - U_{\varepsilon_{q+1,2p-1}}$, we obtain that

$$[W,[W,Y]] = -\frac{a^2}{4n}Y. \tag{23}$$

The preceeding discussion gives us a necessary condition under which the tubes around $G_k(\mathbb{C}^m)$ and $G_q(\mathbb{H}^p)$ are homogeneous with the natural action of a subgroup of the isometry group of $G_k(\mathbb{C}^n)$.

Remark 2. It is known that if a subgroup of the isometry group of $G_k(\mathbb{C}^n)$ acts transitively on the tube $(M)_r$ then $(R_{Z_1})_{p_1}$ and $(R_{Z_2})_{p_2}$ have the same eigenvalues for $p_i \in (M)_r$ and $Z_i \in (T_{p_i}(M)_r)^\perp, \|Z_i\| = 1, i = 1, 2$.

Theorem 2. *If the tube $(M)_r$ in $G_k(\mathbb{C}^n)$ around M is homogeneous then*
i) $m = n - 1$ *if* $M = G_k(\mathbb{C}^m)$;
ii) $p = q + 1$ *if* $M = G_q(\mathbb{H}^p)$, $k = 2q$ *and* $n = 2p$.

Proof. i) Let $M = G_k(\mathbb{C}^m)$. By (11), if $n - m \geq 2$ then U_μ and Z_0 given by (21) are normal vectors of M at o. We consider now γ and η, the radial geodesics in $G_k(\mathbb{C}^n)$ starting from o with velocities U_μ and Z_0 respectively.

Since $G_k(\mathbb{C}^n)$ is a Riemannian symmetric space, the Jacobi operators $R_{\dot\gamma}$ and $R_{\dot\eta}$ have constant eigenvalues along γ and η respectively. Thus, by Proposition 1 the spectrum of $R_{\dot\gamma(r)}$ is $\{0, \frac{1}{4n}, \frac{1}{n}\}$ and by formula (22), $\frac{2+\sqrt{3}}{16n}$ belongs to the spectrum of $R_{\dot\eta(r)}$. This shows that both spectrums are different and hence, by the Remark 2 the tube $(M)_r$ is not homogeneous.
ii) Let now $M = G_q(\mathbb{H}^p)$. By (14), if $p - q \geq 2$, then $W^+_{1,p+q-1}$ and W (see above (23)) are normal vectors of M at o.

By an analogous argument and using the Proposition 1 and formula (23) with $0 < a^2 < \frac{1}{2}$, we may conclude that the tube $(M)_r$ is not homogeneous. □

Theorems 1 and 2 classify completely the homogeneous tubes centered at $G_k(\mathbb{C}^m)$ for $m < n$ and centered at $G_q(\mathbb{H}^p)$ when $k = 2q$ and $n = 2p$. In the last case, the tubes are homogeneous only when the ambient space is $G_{2q}(\mathbb{C}^{2q+2}) \equiv G_2(\mathbb{C}^{2q+2})$. This case has received much attention by J. Berndt and Y. Suh ([2]) who obtained very interesting results as we mention in Section 1. For this reason, from now on we shall only consider tubes in $G_k(\mathbb{C}^n)$ around $G_k(\mathbb{C}^{n-1})$ for $k > 2$.

4. $G_k(\mathbb{C}^{n-1})$-centered tubes in $G_k(\mathbb{C}^n)$

In this paragraph we shall make use of the spectral decomposition of the Jacobi operator R_Z obtained in Section 3, to determine the focal set of $G_k(\mathbb{C}^{n-1})$ in $G_k(\mathbb{C}^n)$ and to obtain some information about the geometry of the family of $G_k(\mathbb{C}^{n-1})$-centered tubes.

Let M be a submanifold of a complete Riemannian manifold N and let $Z \in T_pM^\perp$ be a unit normal vector at some point $p \in M$. Let γ_Z be the geodesic in N with $\gamma_Z(0) = p$ and $\dot\gamma_Z(0) = Z$. We recall that the point $\gamma_Z(t_0)$ with $t_0 > 0$ is said to be a *focal point* of M along γ_Z if the differential of the normal exponential map of M is singular at t_0Z. Equivalently $\gamma_Z(t_0)$ with $t_0 > 0$ is a focal point of M along γ_Z if there exists a Jacobi vector field $J(t)$ along γ_Z satisfying

$$\begin{array}{rl} \text{(i)} & J(0) \in T_pM \\ \text{(ii)} & J'(0) + A_Z(J(0)) \in T_pM^\perp \\ \text{(iii)} & J(t_0) = 0 \end{array} \tag{24}$$

where A_Z is the shape operator of M in the direction of Z (see for instance [6]).

If there are focal points along γ_Z, one defines

$$t_Z := \min\{t_0 > 0 : \gamma_Z(t_0) \text{ is a focal point of } M \text{ along } \gamma_Z\}$$

and call $\gamma_Z(t_Z)$ *the first focal point of* M *along* γ_Z. By the *focal set of* M we mean the set FM consisting of first focal points of M along all the geodesics γ_Z departing from M with Z normal unit vectors to M.

From now on $M = G_k(\mathbb{C}^{n-1}) \subset G_k(\mathbb{C}^n) = N$.

Note that from (11) we know that $U_\mu \in (T_oM)^\perp$ and since M is totally geodesic in N condition (ii) of (24) becomes $J'(0) \in (T_oM)^\perp$.

Let $d = 2k(n-k) = dimN$ and $d_1 = 2k(n-1-k) = dimM$. Then Proposition 1 provides an orthonormal basis of eigenvectors of R_{U_μ},$\{E_1, \ldots, E_d\}$ such that

$$\begin{aligned} \{E_1, \ldots, E_{d_1}\} &\subset T_oM \\ \{E_{d_1+1}, \ldots, E_d\} &\subset (T_oM)^\perp \end{aligned}$$

Reordering we may assume that $E_{d-1} = U_\mu$, $E_d = U_{-\mu}$, the $2(n-1-k)$ first vectors $E_1, \ldots, E_{2(n-1-k)}$ and the $2(k-1)$ vectors $E_{d_1+1}, \ldots, E_{d-2}$ are eigenvectors corresponding to the eigenvalue $\frac{1}{4n}$.

If $E_k(t)$ denotes the parallel transport of E_k along $\gamma_{U_\mu}(t)$ then it is not difficult to see that any Jacobi vector field along $\gamma_{U_\mu}(t)$ satisfying (24) at $p = o$ and for some $t_0 > 0$ is given by

$$J(t) = \sum_{i=1}^{2(n-1-k)} p_i \cos(\frac{t}{2\sqrt{n}})E_i(t) + q\sin(\frac{t}{\sqrt{n}})E_d(t) \tag{25}$$

or

$$J(t) = \sum_{i=d_1+1}^{d-2} p_i \sin(\frac{t}{2\sqrt{n}})E_i(t) + q\sin(\frac{t}{\sqrt{n}})E_d(t) \tag{26}$$

where q and p_i are arbitrary real constants.

The Jacobi vector fields given by (25) vanish on $t_0 = \sqrt{n}(2l+1)\pi, l \in \mathbb{Z}$ while those given by (26) vanish on $t_0 = \sqrt{n}2l\pi, l \in \mathbb{Z}$. Hence the first focal point of M along γ_{U_μ} is precisely $\gamma_{U_\mu}(\sqrt{n}\pi)$ i.e. corresponds to $t_{U_\mu} = \sqrt{n}\pi$.

If $J(t)$ is a Jacobi vector field given by (25), for any $g \in SU(n)$, $g_*J(t)$ is a Jacobi vector field along the geodesic $\gamma_{g_*U_\mu}(t) = g.\gamma_{U_\mu}(t)$. Further if $g \in SU(n-1) \subset SU(n)$ via $g \hookrightarrow \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix}$, $g_*J(t)$ satisfies (24) at $p = g.o \in M$ for $t_0 = \sqrt{n}(2l+1)\pi, l \in \mathbb{Z}$.

Since $SU(n-1)$ acts transitively on M by isometries of N and the natural action of $S(U(k) \times U(n-1-k))$ on the unit sphere of $(T_oM)^\perp$ is transitive (see Lemma 1), we may conclude:

i) *The focal set* FM *of* M *consists of those points in* $G_k(\mathbb{C}^n)$ *which are at distance* $\sqrt{n}\pi$ *from* M.

ii) *The natural action of* $SU(n-1)$ *on the focal set* FM *given by (3) is transitive.*

It is known that the geodesic $\gamma_{U_\mu}(t)$ is given by $\gamma_{U_\mu}(t) = e^{tU_\mu}.o$ where

$$e^{tU_\mu} = \begin{bmatrix} \cos(\frac{t}{2\sqrt{n}}) & 0 & \sin(\frac{t}{2\sqrt{n}}) \\ 0 & I_{n-2} & 0 \\ -\sin(\frac{t}{2\sqrt{n}}) & 0 & \cos(\frac{t}{2\sqrt{n}}) \end{bmatrix}.$$

Then the focal point on γ_{U_μ} turns out to be $\gamma_{U_\mu}(\sqrt{n}\pi) = g_0.o$, where

$$g_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & I_{n-2} & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Thus by a straightforward calculation we may conclude that:

$h \in SU(n-1) \subset SU(n)$ belongs to the isotropy subgroup of $SU(n-1)$ at $\gamma_{U_\mu}(\sqrt{n}\pi)$ if and only if $g_0^{-1} h g_0 \in H = S(\{1\} \times U(k-1) \times U(n-k))$.

Hence the isotropy subgroup of the point $\gamma_{U_\mu}(\pi\sqrt{n})$ is $g_0 H g_0^{-1}$ and therefore

$$FM = SU(n-1)/g_0 H g_0^{-1} \simeq i(SU(n-1))/H$$

where $i : SU(n-1) \hookrightarrow SU(n)$ is the inclusion $g \hookrightarrow \begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix}$. It turns out that FM is the Grassmann manifold

$$FM \simeq SU(n-1)/S(U(k-1) \times U(n-k)) = G_{k-1}(\mathbb{C}^{n-1}).$$

Now we compute the shape operator of the tube $(M)_r$.

Denote by J_X the Jacobi vector field along the geodesic $\gamma(t) = \gamma_{U_\mu}(t)$ with initial conditions $J_X(0) = 0$ and $J'_X(0) = X$ if $X \in (T_oM)^\perp$ and with $J_X(0) = X$ and $J'_X(0) = 0$ if $X \in T_oM$.

Using the results of Proposition 1 we obtain

$$\begin{array}{ll} J_X(t) = \sqrt{n}\sin(\frac{t}{\sqrt{n}})X(t) & \text{if } X \in V_{\frac{1}{n}} \\ J_X(t) = 2\sqrt{n}\sin(\frac{t}{2\sqrt{n}})X(t) & \text{if } X \in V_{\frac{1}{4n}} \cap (T_oM)^\perp \\ J_X(t) = \cos(\frac{t}{2\sqrt{n}})X(t) & \text{if } X \in V_{\frac{1}{4n}} \cap T_oM \\ J_X(t) = X(t) & \text{if } X \in V_0 \cap T_oM \\ J_X(t) = t\dot{\gamma}(t) & \text{if } X = \dot{\gamma}(0) \end{array} \tag{27}$$

where $X(t)$ is the parallel transport of X along $\gamma(t)$.

For $0 < r < \pi\sqrt{n}$ let A^r be the shape operator of the tube $(M)_r$ with respect to the "outward" normal unit field.

It is well known that for the Jacobi vector fields listed in (27) with $X \neq U_\mu = \dot{\gamma}(0)$ the following formula holds

$$A^r_{\dot{\gamma}(r)} J_X(r) = -J'_X(r)$$

(see for example [4] p. 152).

From this formula, Proposition 1 and (27) we obtain that the eigenvalues of $A^r_{\dot\gamma(r)}$ are

$$-\frac{1}{\sqrt{n}}\cot(\frac{r}{\sqrt{n}}), \quad -\frac{1}{2\sqrt{n}}\cot(\frac{r}{2\sqrt{n}}), \quad \frac{1}{2\sqrt{n}}\tan(\frac{r}{2\sqrt{n}}), \quad 0 \tag{28}$$

and its respective multiplicities are

$$1, \quad 2(k-1), \quad 2(n-1-k), \quad 2(k-1)(n-1-k).$$

Since $SU(n-1)$ acts on $(M)_r$ transitively by isometries of N, A^r has the same eigenvalues at every point of $(M)_r$ and hence $(M)_r$ *is a real isoparametric hypersurface* whose principal curvatures are given in (28).

On the other hand, it is known that the natural complex Kaehler structure $\mathbb{J}$ of N is given by

$$\mathbb{J}_o = Ad(h)|_{T_oN} \quad \text{with} \quad h = \begin{bmatrix} -I_k & 0 \\ 0 & iI_{n-k} \end{bmatrix}.$$

Keeping the notation of the matrix E_{ij} given below (16), we may write $U_\mu = \frac{1}{2\sqrt{n}}(E_{1n} - E_{n1})$ and $U_{-\mu} = \frac{i}{2\sqrt{n}}(E_{1n} + E_{n1})$, then $\mathbb{J}_oU_\mu = U_{-\mu}$. By homogeneity of N, the parallel translated vectors of U_μ and $U_{-\mu}$ satisfy $\mathbb{J}_{\gamma(r)}E_{d-1}(r) = E_d(r)$ and consequently $A^r_{\dot\gamma(r)}\mathbb{J}_{\gamma(r)}E_{d-1}(r) = \frac{1}{n}E_d(r)$.

Hence, if Z is the "outward" unit normal field of $(M)_r$ then $\mathbb{J}Z$ is an eigenvector of the shape operator A^r with eigenvalue $\frac{1}{n}$. This says that $(M)_r$ *is a Hopf hypersurface of* N ([11] p. 244).

We summarize the discussion of the present section in the following result which gives a generalization of Theorem 18 in [1].

Theorem 3. *Let $M = G_k(\mathbb{C}^{n-1}) \subset G_k(\mathbb{C}^n) = N$. The focal set FM of M in N is the Grassmannian manifold $G_{k-1}(\mathbb{C}^{n-1})$ and consists precisely of all points in N at distance $\sqrt{n}\pi$ from M. Any tube $(M)_r$ of radius $0 < r < \sqrt{n}\pi$ around M is an embedded isoparametric real hypersurface of N with four (resp. three for $r = \frac{\sqrt{n}}{2}\pi$) distinct principal curvatures. The principal curvatures k_i of $(M)_r$ and its respective multiplicities $m(k_i)$, with respect to the outward unit normal field Z, are*

k_i	$m(k_i)$
$-\frac{1}{\sqrt{n}}\cot(\frac{r}{\sqrt{n}})$	1
$-\frac{1}{2\sqrt{n}}\cot(\frac{r}{2\sqrt{n}})$	$2(k-1)$
$\frac{1}{2\sqrt{n}}\tan(\frac{r}{2\sqrt{n}})$	$2(n-1-k)$
0	$2(k-1)(n-1-k)$

and the corresponding eigenspaces of A^r at $\gamma_{U_\mu}(r)$ are the parallel translated, along the radial geodesic γ_{U_μ}, of the following subspaces of T_oN

$$
\begin{aligned}
T(k_1) &= \mathbb{R}U_{-\mu} \\
T(k_2) &= \sum_{j=2}^{k} \mathfrak{m}_{\varepsilon_{j,n-1}} \\
T(k_3) &= \sum_{j=k}^{n-2} \mathfrak{m}_{\varepsilon_{1,j}} \\
T(k_4) &= \sum \mathfrak{m}_{\varepsilon_{i,j}} \textit{ with } 2 \leqslant i \leqslant k,\ k \leqslant j \leqslant n-2
\end{aligned}
$$

where $\mathfrak{m}_{\varepsilon_{i,j}} = \mathbb{R}U_{\varepsilon_{i,j}} \oplus \mathbb{R}U_{-\varepsilon_{i,j}}$ *and* $\varepsilon_{i,j}$ *is defined in (12). Furthermore* $(M)_r$ *is a Hopf hypersurface of* N. □

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