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Another Counterexample to a Conjecture of Zassenhaus

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Abstract. A metabelian group G of order 1440 is constructed which provides a counterexample to a conjecture of Zassenhaus on automorphisms of integral group rings. The group is constructed in the spirit of [8]. An augmented automorphism of $\mathbb{Z}G$ which has no Zassenhaus factorization is given explicitly (this was already done in [7] for a group of order 6720), but this time only a few distinguished group ring elements are used for its construction, carefully exploiting certain congruence relations satisfied by powers of these elements.

1. Introduction

Let G be a finite group, and denote its integral group ring by $\mathbb{Z}G$. A *group basis* of $\mathbb{Z}G$ is a subgroup H of the group of units of $\mathbb{Z}G$ of augmentation 1 such that $\mathbb{Z}G = \mathbb{Z}H$ and $|G| = |H|$. H. Zassenhaus conjectured that each two group bases of $\mathbb{Z}G$ are conjugate by a unit of $\mathbb{Q}G$ (i.e., "rationally conjugate"). However, Roggenkamp and Scott constructed in [8] a counterexample to this conjecture.

Note that, by the Skolem-Noether theorem, the Zassenhaus conjecture asserts that G can be mapped onto any group basis of $\mathbb{Z}G$ by a *central* ring automorphism of $\mathbb{Z}G$ (an automorphism fixing the center element-wise). We shall say that an augmentation preserving automorphism α of $\mathbb{Z}G$ has a *Zassenhaus factorization* if it is the composition of a group automorphism of G (extended to a ring automorphism) and a central automorphism (cf. [11, p. 327]).

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Roggenkamp and Scott produced a metabelian group G of order 2880 such that there is an augmentation preserving automorphism α of $\mathbb{Z}G$ which has no Zassenhaus factorization. Then, G and its image $G\alpha$ are group bases of $\mathbb{Z}G$ which are not rationally conjugate. Their construction of the automorphism α is explicit in the semilocal situation. To show that their example is also a global counterexample, they developed some kind of a general theory—using Picard groups and Milnor's Mayer Vietoris sequence. This work is excellently outlined in [11].

Subsequently, Klingler [7] constructed a global automorphism explicitly. (However, he changed slightly the definition of the group G—replacing a normal subgroup of order 3 by a normal subgroup of order 7—in order to write $\mathbb{Z}G$ as a "multiple pullback", taken over *semisimple* finite skew group rings. Then, he essentially used the fact that certain elements of these rings, which, roughly speaking, correspond to elementary matrices, can be lifted to units of the factors.)

Since then, Blanchard [1, 2] constructed counterexamples in the semilocal case, using the idea from [11, Section 2] (see also [4, 2.1]).

In this paper, another counterexample is given:

Theorem. *There is a metabelian group G of order* 1440 $(= 2^5 \cdot 3^2 \cdot 5)$ *and an automorphism α of $\mathbb{Z}G$ which has no Zassenhaus factorization.*

The group G has been designed to satisfy a certain group-theoretical obstruction, in the very same way as Roggenkamp and Scott constructed their group. The elementary construction of the automorphism involves only a few elements of $\mathbb{Z}G$, carefully exploiting certain congruence relations satisfied by powers of these elements.

Finally, it should be remarked that a weaker question simply asks whether any two group bases of an integral group ring $\mathbb{Z}G$ are isomorphic (this is the so-called "isomorphism problem for integral group rings"). There are some exciting positive results, but also a counterexample has been found (see [3, 4]).

2. The group-theoretical obstruction

The group G is the semidirect product $G = (M \times N \times Q) \rtimes W$, where

- $W = \langle w : w^8 \rangle \rtimes (\langle b : b^2 \rangle \times \langle c : c^2 \rangle)$, with $w^b = w^{-1}$ and $w^c = w^5$;
- $M = \langle m : m^5 \rangle$, $N = \langle n : n^3 \rangle$ and $Q = \langle q : q^3 \rangle$;
- $\mathrm{C}_W(m) = \langle wc, b \rangle$, $\mathrm{C}_W(n) = \langle w^2, b, c \rangle$ and $\mathrm{C}_W(q) = \langle w, b \rangle$ (these are subgroups of index 2 in W).

For a group X, let $\mathrm{Aut_c}(X)$ denote the group of *class-preserving automorphisms* of X, and write $\mathrm{Out_c}(X) = \mathrm{Aut_c}(X)/\mathrm{Inn}(X)$. It is well-known that the group W has a non-inner, class-preserving automorphism δ of order 2 (see [12]), which maps c to w^4c, and b, w stay fixed. (We remark that $\mathrm{Out}(W) \cong C_2 \times C_2$.) Here, an extension of this automorphism to an automorphism σ of G will be considered, defined by

$$\sigma : \begin{cases} c \mapsto w^4 c \\ b, w, m, n, q \text{ stay fixed} \end{cases} .$$

Let $\epsilon = \frac{1}{2}(1 + w^4)$, a central idempotent of the group algebra $\mathbb{Q}G$.

As in [11], write

$$G_3 = G/M, \quad G_5 = G/N \quad \text{and} \quad H = G/MN.$$

Note that we may identify G_3 with NQW, G_5 with MQW, and H with QW.

Lemma 2.1. *The automorphism σ induces an inner automorphism of $\mathbb{Z}[1/2]H$, given by conjugation with the unit*

$$\mu = \epsilon + (1 - \epsilon)(w + w^{-1}) \in \mathbb{Z}[1/2]\langle w \rangle.$$

Proof. It follows from $(1 - \epsilon)(w + w^{-1})^2 = 2(1 - \epsilon)$ that μ is a unit, and it is easily checked that $x\mu = \mu(x\sigma)$ for the given generators x of H. □

Together with the next lemma, this shows that, at the group-theoretic level, $[\sigma]$ is a "candidate obstruction" in the sense of [11, p. 330]. Write $x \not\sim y$ to indicate that group elements x and y are not conjugate.

Lemma 2.2. *We have* $\mathrm{Out}_{\mathrm{c}}(G_3) = 1$ *and* $\mathrm{Out}_{\mathrm{c}}(G_5) = 1$.

Proof. Let $\phi \in \mathrm{Aut}_{\mathrm{c}}(NQW)$; we have to show $\phi \in \mathrm{Inn}(NQW)$. We may assume that $w\phi = w$, $b\phi = b$ and either $c\phi = c$ or $c\phi = w^4c$. From $qw \not\sim q^{-1}w$ it follows that $q\phi = q$. If $c\phi = c$, then $nb \not\sim n^{-1}b$ implies that $\phi = \mathrm{id}$. So assume that $c\phi = w^4c$. From $[n, c] = 1$ it follows that nc and $(nc)\phi$ are conjugate in W. As $c^w = w^4c = c\phi$ and $\mathrm{C}_W(c) = \mathrm{C}_W(n)$, it follows that $n\phi = n^w = n^{-1}$, yielding the contradiction $nb \not\sim n^{-1}b = (nb)\phi$.

Similar for $\phi \in \mathrm{Aut}_{\mathrm{c}}(MQW)$. Again, we may assume that $w\phi = w$, $b\phi = b$ and either $c\phi = c$ or $c\phi = w^4c$. From $qw \not\sim q^{-1}w$ it follows that $q\phi = q$. If $c\phi = c$, then $mwc \not\sim m^{-1}wc$ implies that $\phi = \mathrm{id}$. So assume that $c\phi = w^4c$. From $[m, wc] = 1$ it follows that mwc and $(mwc)\phi$ are conjugate in W. As wc and w^5c $(= (wc)\phi)$ are not conjugate in $\mathrm{C}_W(m)$, $m\phi = m^{-1}$. From $[mqb, b] = 1$ it follows that mqb and $(mqb)\phi$ $(= m^{-1}qb)$ are conjugate in W. Hence $m^x = m^{-1}$ for some $x \in \mathrm{C}_W(q) \cap \mathrm{C}_W(b) = \langle w^4, b\rangle \subseteq \mathrm{C}_W(m)$, a contradiction. □

We shall need the following simple observation from [8].

Remark 2.3. Let x and y be elements of a group with $y^x = y^{-1}$. Then $(y - y^{-1})x = -x(y - y^{-1})$. □

For a group X, write $\hat{X}$ for the sum of its elements. The (two-sided) ideal generated by group ring elements $s, t, \dots$ will be denoted by $(s, t, \dots)$. The quotient

$$\Lambda = \mathbb{Z}G/(\hat{M}, \hat{N})$$

is the projection on a factor of $\mathbb{Q}G$ (to which all blocks having neither M nor N in their kernel belong).

Though the next lemma is not really needed in the construction of the automorphism α of $\mathbb{Z}G$, a short proof is included. (It could also be proved character-theoretically, analyzing the inertia groups of the characters of Λ, just as in [8]). It already shows that the group G provides a semilocal counterexample (cf. [11, p. 333–34]).

Lemma 2.4. *The automorphism σ induces a central automorphism of Λ.*

Proof. We have to show that σ induces an inner automorphism of $\mathbb{Q}\Lambda$. On Λ, the group elements m and n correspond to, roughly speaking, primitive fifth and third roots of unity, respectively. Hence the images u and v of $m - m^{-1}$ and $n - n^{-1}$ in $\mathbb{Q}\Lambda$ are units. It follows from Remark 2.3 that both units normalize Λ, and that on Λ,

$$\operatorname{conj}(u) : \begin{cases} c \mapsto -c \\ w \mapsto -w \\ b, m, n, q \text{ stay fixed} \end{cases},$$

$$\operatorname{conj}(v) : \begin{cases} w \mapsto -w \\ b, c, m, n, q \text{ stay fixed} \end{cases},$$

$$\operatorname{conj}(uv) : \begin{cases} c \mapsto -c \\ b, w, m, n, q \text{ stay fixed} \end{cases}.$$

Note that $\mathbb{Q}\Lambda = \epsilon\mathbb{Q}\Lambda \oplus (1-\epsilon)\mathbb{Q}\Lambda$. As $w^4 = -1$ on $(1-\epsilon)\Lambda$, it follows that σ on $(1-\epsilon)\mathbb{Q}\Lambda$ is given by conjugation with $(1-\epsilon)uv$. Since σ induces the identity on $\epsilon\mathbb{Q}\Lambda$, this proves the lemma. $\square$

3. The automorphism α

The quotient $\Lambda = \mathbb{Z}G/(\hat{M}, \hat{N})$ is the projection of $\mathbb{Z}G$ on a factor of $\mathbb{Q}G$. The projection of $\mathbb{Z}G$ on the complementary factor is the image Γ of $\mathbb{Z}G$ under the natural map $\mathbb{Z}G \to \mathbb{Z}G_3 \oplus \mathbb{Z}G_5$. Hence there are pull-back diagrams

$$\begin{array}{ccc} \Gamma & \longrightarrow & \mathbb{Z}G_5 \\ \downarrow & & \downarrow \\ \mathbb{Z}G_3 & \longrightarrow & \mathbb{Z}H \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ \Lambda & \longrightarrow & \Omega \end{array}.$$

Note that N and M can be viewed as normal subgroups of G_3 and G_5, respectively. As in [11], let

$$\Lambda_3 = \mathbb{Z}G_3/(\hat{N}) \quad \text{and} \quad \Lambda_5 = \mathbb{Z}G_5/(\hat{M}).$$

Roggenkamp and Scott proved in [8, Section 2] (see also [3, 4]) that the ring Ω has the form

$$\Lambda_3/5\Lambda_3 \oplus \Lambda_5/3\Lambda_5.$$

Let

$$\begin{aligned} \nu &= 1 + (1 - w^4)(1 + w + w^{-1}) \\ &= \epsilon + (1-\epsilon)(3 + 2(w + w^{-1})). \end{aligned}$$

From $(1-\epsilon)(w + w^{-1})^2 = 2(1-\epsilon)$ it follows that ν is a unit of $\mathbb{Z}\langle w\rangle$, with inverse $\nu^{-1} = \epsilon + (1-\epsilon)(3 - 2(w + w^{-1}))$.

From $\nu^3 = \epsilon + (1-\epsilon)(99 + 70(w + w^{-1}))$ it follows that

$$\begin{aligned} \nu^3 &\equiv \epsilon - (1-\epsilon) = w^4 && \bmod 5\mathbb{Z}\langle w\rangle, \\ \nu^3 &\equiv \mu = \epsilon + (1-\epsilon)(w + w^{-1}) && \bmod 3\mathbb{Z}[1/2]\langle w\rangle. \end{aligned}$$

Given a normal subgroup Y of a finite group X, there is a well-known pull-back diagram of rings

$$\begin{tikzcd} \mathbb{Z}X \arrow[r] \arrow[d] & \mathbb{Z}X/(\hat{Y}) \arrow[d] \\ \mathbb{Z}X/Y \arrow[r] & (\mathbb{Z}/|Y|\mathbb{Z})\,(X/Y) \end{tikzcd} .$$

Note that the pull-back diagram describing Γ can be extended to the right-hand side by a diagram of this type.

The automorphism σ of G induces automorphisms of G_5 and H, which, for simplicity, shall be denoted by the same symbol. Recall that $\sigma = \mathrm{conj}(\mu)$ on H, so $\sigma = \mathrm{conj}(\nu^3)$ on $\mathbb{F}_3H$, and there is $\gamma \in \mathrm{Aut}(\Gamma)$, inducing σ on $\mathbb{Z}G_5$ and a central automorphism β on $\mathbb{Z}G_3$, as shown below.

$$\begin{tikzcd} \boxed{\gamma} & & \boxed{\beta} \arrow[ll, dashed] & & \boxed{\mathrm{conj}(\nu^3)} \arrow[ll, dashed] & \\ & \Gamma \arrow[rr] \arrow[dd] & & \mathbb{Z}G_3 \arrow[rr] \arrow[dd] & & \Lambda_3 \arrow[dd] \\ \boxed{\sigma} \arrow[uu, dashed] & {} \arrow[l, mapsto] \arrow[r] & \boxed{\mathrm{conj}(\mu)} \arrow[uu, dashed] & {} \arrow[l, mapsto] \arrow[r] & \boxed{\mathrm{conj}(\mu)} \arrow[uu, dashed] & \\ & \mathbb{Z}G_5 \arrow[rr] & & \mathbb{Z}H \arrow[rr] & & \mathbb{F}_3H \end{tikzcd}$$

As $\nu^3 \equiv w^4 \mod 5$ and $w^4 \in \mathrm{Z}(G)$,

$$\gamma \text{ induces the identity on } \Lambda_3/5\Lambda_3, \text{ and induces } \sigma \text{ on } \Lambda_5/3\Lambda_5. \tag{1}$$

On Λ, the group element mn corresponds to a "primitive 15-th root of unity", so the image t of $mn - (mn)^{-1}$ in Λ is a unit in Λ (cf. [13, Proposition 2.8]). Binomial expansion gives $(mn-(mn)^{-1})^{15} = 1 - \left(\sum_{i=1}^{14}(-1)^i\binom{15}{i}(mn)^{2i}\right) - 1$, and for $1 \le i \le 14$ it follows from $\binom{15}{1} \equiv 1 \mod 2$ and $\binom{15}{i} + \binom{15}{i+1} = \binom{16}{i+1} \equiv 0 \mod 2$ that $\binom{15}{i} \equiv 1 \mod 2$. Hence $(mn-(mn)^{-1})^{15} \equiv 1 + \hat{M}\hat{N} \equiv 1 \mod 2$ and consequently $t^{15} \in 1 + 2\Lambda$.

Recall from Lemma 2.4 the definition of the automorphisms $\mathrm{conj}(u)$ and $\mathrm{conj}(v)$ of Λ.

Note that Λ can be written as a pull-back

$$\begin{tikzcd} \Lambda \arrow[r] \arrow[d] & (1-\epsilon)\Lambda \arrow[d] \\ \epsilon\Lambda \arrow[r] & \overline{\Lambda} \end{tikzcd} ,$$

where $2 = 0$ in $\overline{\Lambda}$ since $2\epsilon \in \Lambda$. The automorphism $\mathrm{conj}(v)\mathrm{conj}(t^{15})$ of Λ induces an automorphism ϕ of $(1-\epsilon)\Lambda$, which in turn induces the identity on $\overline{\Lambda}$. It follows that there is an automorphism λ of Λ which induces ϕ on $(1-\epsilon)\Lambda$, and the identity on $\epsilon\Lambda$.

The elements $mn-(mn)^{-1}$ and $n-n^{-1}$ map to the same element in $\mathbb{Z}G_3$, and it follows that the inner automorphism $\mathrm{conj}(t)$ and the automorphism $\mathrm{conj}(v)$ induce the same automorphism (of order 2) on $(1-\epsilon)\Lambda_3/5\Lambda_3$. Similarly, $mn-(mn)^{-1}$ and $m-m^{-1}$ map to the same element in $\mathbb{Z}G_5$, so that $\mathrm{conj}(t)$ and $\mathrm{conj}(u)$ induce the same automorphism (of order 2) on $(1-\epsilon)\Lambda_5/3\Lambda_5$. As $\mathrm{conj}(v)\mathrm{conj}(u)$ and σ induce the same automorphism of $(1-\epsilon)\Lambda$ (see the proof of Lemma 2.4), it follows that

$$\lambda \text{ induces the identity on } \Lambda_3/5\Lambda_3, \text{ and induces } \sigma \text{ on } \Lambda_5/3\Lambda_5. \tag{2}$$

Using the given description of $\mathbb{Z}G$ as a pull-back, it follows from (1) and (2) that there is an automorphism α of $\mathbb{Z}G$, inducing γ on Γ and λ on Λ.

As γ induces σ on $\mathbb{Z}G_5$ (which implies that α preserves the augmentation), and a central automorphism on $\mathbb{Z}G_3$, it follows from Lemma 2.2 and the discussion in [11, p. 327–28] that α has no Zassenhaus factorization, and the theorem is proved.

4. Conjugacy of Sylow subgroups

Let G be a finite group, H a group basis of $\mathbb{Z}G$, and fix a prime p. It has been shown in [6] that if G is solvable, then Sylow p-subgroups of G and H are conjugate in the units of $\mathbb{Q}G$.

One might also ask if the Sylow p-subgroups are conjugate in the units of $\mathbb{Z}_pG$, where $\mathbb{Z}_p$ are the p-adic integers.

For nilpotent groups, this problem is solved by Theorem 2 of [14], except for the determination of a global obstruction in a certain genus class group. Here we provide an example, based on the group W and the unit ν.

Example 4.1. Let $G = W \times C$, with C a cyclic group of order 3. Then there is a group basis H of $\mathbb{Z}G$ such that Sylow 2-subgroups of G and H are not conjugate in the units of $\mathbb{Z}_2G$.

Proof. Recall from the beginning of Section 2 that W has a non-inner class-preserving automorphism δ, given by conjugation with the unit μ of $\mathbb{Z}[1/2]\langle w\rangle$. As $\nu^3 \equiv \mu \mod 3$, it follows that there is an augmentation preserving automorphism α of $\mathbb{Z}G$ which induces δ on $\mathbb{Z}G/C \cong \mathbb{Z}W$ and is given by conjugation with the image of ν^3 on $\mathbb{Z}G/(\hat{C})$, as indicated below.

$$\begin{tikzcd}
\boxed{\alpha} & & & \boxed{\delta} \arrow[lll, dashed] \arrow[ddd, mapsto] \\
& \mathbb{Z}G \arrow[r] \arrow[d] & \mathbb{Z}W \arrow[d] & \\
& \mathbb{Z}G/(\hat{C}) \arrow[r] & \mathbb{F}_3W & \\
\boxed{\mathrm{conj}(\nu^3)} \arrow[uuu, dashed] \arrow[rrr, mapsto] & & & \boxed{\mathrm{conj}(\mu)}
\end{tikzcd}$$

In $\mathbb{Z}_2G$, there are central idempotents $\epsilon = \frac{1}{3}\hat{C}$ and $\eta = 1-\epsilon$, and $W\alpha$ is conjugate in the units of $\mathbb{Z}_2G$ to $\{\epsilon(x\delta)+\eta x \,:\, x \in W\}$. Assume that this group is conjugate to W within the units of $\mathbb{Z}_2G$. Then there is an automorphism β of W such that either $\delta\beta^{-1}$ induces an inner automorphism of the group ring $\epsilon\mathbb{Z}_2G \cong \mathbb{Z}_2W$ or β induces an inner automorphism of the group ring $\eta\mathbb{Z}_2G \cong \mathbb{Z}_2[\zeta]W$, where ζ is a primitive third root of unity. As δ is a non-inner group automorphism, this contradicts a result of D. B. Coleman (see [5, 2.6]). $\square$

However, it should be remarked that it is not known whether the projections of the Sylow p-subgroups of G and H on the principal block of $\mathbb{Z}_pG$ are conjugate within the units of the block. This is a special case of Scott's defect group (conjugacy) question (see [9, p. 267], [10]).

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