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Gelfand-Kirillov Dimension in some Crossed Products

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Abstract. Let k be a field, R a k-algebra and $A = R[\theta_1, \theta_2, \ldots, \theta_n]$ a Poincaré-Birkhoff-Witt extension of R. If each θ_i acts locally finitely on R, then we show that $GKdim(A) = GKdim(R) + n$. From this we deduce some results concerning incomparability, prime length and Tauvel's height formula in the crossed product $R \star g$ where g is a finite-dimensional Lie algebra acting as derivations on R. Similar results are obtained for $R \star G$, where G is a free abelian group of finite rank. As a corollary of the results for G, Tauvel's height formula is established in $R \otimes_k P(\lambda)$ where $P(\lambda)$ is the quantum torus.

0. Introduction

All rings (except the Lie algebras) in this paper are associative with identity. When we say that a ring is noetherian we mean left and right noetherian.

For the basic material concerning the Gelfand-Kirillov dimension (denoted by $GKdim$) we refer to [8]. We refer the reader to [5, 7, 12] for some definitions and undefined terminologies.

Let k be a field. We say that Tauvel's height formula holds in a noetherian k-algebra A with finite Gelfand-Kirillov dimension provided

$$ht(P) + GKdim(A/P) = GKdim(A)$$

for any prime ideal P of A where $ht(P)$ denotes the height of P.

Throughout, we fix a field k, an algebra R over k and a positive integer n.

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Let g be a k-Lie algebra of finite dimension n. We suppose that g acts by derivations on R and we denote by $A = R \star g$ the crossed product of R by the enveloping algebra $U(g)$ of g (see [1, 12, 13]).

For any g-prime ideal P of R, g-$ht(P)$ denotes the g-height of P; i.e.

$$\sup\{s : \text{there is a chain of } g\text{-prime ideals } P_0 \subset P_1 \subset \cdots \subset P_s = P \text{ of } R\}.$$

We denote by $spec_g(R)$ the set of g-prime ideals of R. If k has characteristic zero and if R is noetherian or g-locally finite with g solvable, any g-prime ideal of R is prime [2, Proposition 4.4]; so g-$ht(P) \leq ht(P)$.

If $X \in g$, we denote by $\bar{X}$ the canonical image of X in $R \star g$ and we set $\delta(X) = \delta_X$. We recall that there exist a linear map δ from g to the k-Lie algebra of k-derivations of R and a bilinear map $t : g \times g \to R$ such that $[\bar{X}, \bar{Y}] - \overline{[X,Y]} = t(X,Y)$. If t is the 0-map, $R \star g$ is usually denoted by $R\#U(g)$; in this case R is a g-module.

We say that condition (C_1) (resp. (C_2)) is satisfied in $spec_g(R)$ if Tauvel's height formula holds for all g-prime ideals of R (resp. if the g-height and the height are finite and coincide on the g-prime ideals of R).

Let R be finitely generated over k and $A = R[\theta_1, \theta_2, \ldots, \theta_n]$ a PBW extension of R. In [11], J. Matczuk showed that $GKdim(A) = GKdim(R) + n$. This result generalizes [12, 8.2.10] and from this, he deduced some results concerning incomparability and prime length. On the other hand a result of M. Lorenz [10] states that if R is δ-locally finite, then $GKdim(R[\theta]) = GKdim(R) + 1$ where δ is the derivation of R determined by θ.

In the first section of this paper, we show that all the results in [11] are also true if we replace the "finitely generated" hypothesis on R by the assumption that each θ_i acts locally finitely on R. This result generalizes [10] and it enables us to obtain the results in [11] with their g-invariant version in the ring $R \star h$ if R is g-locally finite or finitely generated, where h is an ideal of g. If further k is of characteristic 0, g is completely solvable, condition (C_1) is satisfied in $spec_g(R)$ and R is g-hypernormal, we have improved the preceding results. To close this section, we show that if R is noetherian of finite Gelfand-Kirillov dimension and if conditions (C_1) and (C_2) hold in $spec_g(R)$, then conditions (C_1) and (C_2) also hold in $spec_g(R \star g_i)$ for each i; where the g_i are terms of a composition series of g (see the definition after Corollary 1.11).

In the second section, G is a free abelian group of finite rank n. We prove similar statements as in Section 1.

1. When the action comes from a Lie algebra

In this section the characteristic of k is arbitrary. A Poincaré-Birkhoff-Witt extension (or PBW extension) of R is a ring extension $A \supseteq R$ generated by elements $\{\theta_1, \ldots, \theta_n\}$ such that

- $[\theta_i, r] \in R$ for each $i = 1, 2, \ldots, n$.
- $[\theta_i, \theta_j] \in R + R\theta_1 + R\theta_2 + \cdots + R\theta_n$.
- A is free as (left and right) R-module with basis $\theta_1^{i_1}\theta_2^{i_2}...\theta_n^{i_n}$; where the i_j are positive integers [11].

The Poincaré-Birkhoff-Witt theorem implies that any crossed product $R \star g$ is a PBW extension of R. We will denote by δ_i the derivation of R determined by θ_i; i.e. $\delta_i(r) = [\theta_i, r]$ for all $r \in R$ and we will set $\Delta = \{\delta_1, \ldots, \delta_n\}$. We say that R is Δ-locally finite if every element of R is contained in a finite-dimensional Δ-stable subspace of R.

Let $A = R[\theta_1, \theta_2, \ldots, \theta_n]$ be a PBW extension of R. Our aim in the next theorem is to give a relation between the Gelfand-Kirillov dimension of R and A.

Theorem 1.1. *Suppose that $A = R[\theta_1, \theta_2, \ldots, \theta_n]$ is a PBW extension of R. If R is Δ-locally finite, then $GKdim(A) = GKdim(R) + n$.*

Proof. Denote by X the k-linear subspace of A spanned by $1, \theta_1, \theta_2, \ldots, \theta_n$. For any $l \geq 1$, let $A_l \subseteq X^l$ denote the subspace spanned by all monomials of the form $\theta_1^{i_1}\theta_2^{i_2}...\theta_n^{i_n}$ where $i_1 + i_2 + \cdots + i_n \leq l$, $0 \leq i_j$ for $j = 1, \ldots, n$.

Let W be a finite-dimensional k-vector subspace of A. Using locally finite-dimensionality of Δ, we can choose a finite-dimensional Δ-invariant subspace V of R such that $W \subseteq \sum_{i_1+i_2+\cdots+i_n \leq t} V\theta_1^{i_1}\theta_2^{i_2}...\theta_n^{i_n}$ for some $t \in \mathbb{N}$ where the $i_j's \in \mathbb{N}$ and $[\theta_i, \theta_j] \in V + V\theta_1 + \cdots + V\theta_n = VX$ for all $0 \leq i, j \leq n$. Thus, eventually replacing V by $k.1 + V$, we may assume that $1 \in V$. Now, it is clear that each power of V is Δ-stable and $V^l \subseteq V^{l+1}$ for any $l \in \mathbb{N}$. We can show as in the proof of [11, Theorem A] that between the subspaces V, X and A_l the following relations

$$X^tV \subseteq VX^t + V^2X^{t-1} + \cdots + V^tX + V^{t+1}$$

and

$$X^t \subseteq V^{t-1}A_t$$

hold for any $t \geq 1$. From these relations, we deduce that $W'^t \subseteq V^tX^t$ for any $t \geq 1$; where $W' = X + V \subseteq VX$. Now the same argument as in [11] shows that $GKdim(k[W']) \leq GKdim(R) + n$. It is clear that $W \subseteq k[W']$; so $GKdim(k[W]) \leq GKdim(R) + n$. This means that $GKdim(A) \leq GKdim(R) + n$.

Let V' be a finite-dimensional k-vector subspace of R. Since R is Δ-locally finite, we can find a finite-dimensional Δ-stable subspace V of R such that $V' \subseteq V$ and $[\theta_i, \theta_j] \in V + V\theta_1 + \cdots + V\theta_n = VX$ for all $0 \leq i, j \leq n$. We may suppose that $1 \in V$. Set $W = V + X \subseteq VX$. It is easy to see that $V'^tA_t \subseteq V^tA_t \subseteq W^{2t}$ for any $t \geq 1$. This implies that

$$GKdim(k[V']) + n \leq GKdim(k[V]) + n \leq GKdim(k[W]) \leq GKdim(A).$$

It follows that $GKdim(R) + n \leq GKdim(A)$.

Remark 1.2. All the results in [11] remain true if we replace the "finitely generated" hypothesis on R by the assumption that R is Δ-locally finite.

Let $b \in R$. Then $U(g).b$ is a g-stable k-vector subspace of R. We say that b is g-finite if $U(g).b$ has a finite dimension. It is clear that an element of $R \star h$ is g-finite if and only if it is g/h-finite. If b and b' are two g-finite elements of R, then bb' is a g-finite element of R. We say that R is g-locally finite if all its elements are g-finite.

The enveloping algebra of an ideal h of g is g-locally finite for the left adjoint action of g. If R is g-locally finite, any g-invariant factor of R is g-locally finite. If g acts trivially on R, then R is g-locally finite.

Let $(X_1, X_2, \ldots, X_n)$ be a basis of g. Then R is g-locally finite if and only if R is Δ-locally finite, where $\Delta = \{\delta_{X_i}; 1 \leq i \leq n\}$.

Lemma 1.3. *Let h be an ideal of g of dimension l and let $Y \in h$.*

(1) *If the k-bilinear map t is identically zero, then $\bar{Y}$ is a g-finite element of $R\#U(h)$.*

(2) *If R is g-locally finite, then $\bar{Y}$ is a g-finite element of $R \star h$.*

Proof. Let $(Y_1, Y_2, \ldots, Y_l)$ be a basis of h and X any element of g.

(1) We have $\delta_X(\bar{Y}) = \overline{[X,Y]} \in \overline{kY_1 + \cdots + kY_l} = k\overline{Y_1} + \cdots + k\overline{Y_l}$. It is clear that $k\overline{Y_1} + \cdots + k\overline{Y_l}$ is a finite-dimensional g-stable subspace of $R\#U(h)$.

(2) The image Imt of t is a finite-dimensional subspace of R. Since R is g-locally finite, Imt is contained in a finite-dimensional g-stable subspace V of R. Now, $\delta_X(\bar{Y}) = \overline{[X,Y]} + t(X,Y) \in \overline{kY_1 + \cdots + kY_l} + V = k\overline{Y_1} + \cdots + k\overline{Y_l} + V$ and $k\overline{Y_1} + \cdots + k\overline{Y_l} + V$ is a finite-dimensional g-stable subspace of $R \star h$.

Corollary 1.4. *Let h be an ideal of g. If R is g-locally finite, then so is $R \star h$.*

The following is a generalization of the well known result [12, 8.2.7] which asserts that $GKdim(R \otimes_k U(g)) = GKdim(R) + n$.

Corollary 1.5. *Let h be an ideal of g of dimension l. Suppose that R is g-locally finite. Then $GKdim(R \star h) = GKdim(R) + l$.*

Proof. R is h-locally finite, since it is g-locally finite. The result follows from Theorem 1.1.

As an application of Corollary 1.5 we shall show some results concerning incomparability, prime length and Tauvel's height formula. We denote by *dim* the classical Krull dimension and by *g-dim* its g-invariant version; i.e. the maximal length of a chain of g-prime ideals of R. If k is of characteristic zero and if R is noetherian or g-locally finite with g solvable, then $g\text{-}dim(R) \leq dim(R)$.

Corollary 1.6. *Let R be noetherian of finite Gelfand-Kirillov dimension, h an ideal of g of dimension l, $A = R \star g$ and $B = R \star h$. Suppose that R is finitely generated or g-locally finite.*

(1) *Let k be of characteristic zero and P be a g-prime ideal of B such that $P \cap R = 0$. Then $g\text{-}ht(P) \leq ht(P) \leq l$. If R is g-simple, then $g\text{-}dim(B) \leq l$.*

(2) *Let P be a prime ideal of A such that $P \cap R = 0$. Then $ht(P) \leq n$. If R is g-simple, then $dim(A) \leq n$.*

Proof. (1) The lower bound was previously noted. Since $R = R/(P \cap R)$ is a subalgebra of B/P, we have $GKdim(R) \leq GKdim(B/P)$. Corollary 1.5 in the g-locally finite case ([12, 8.2.10] in the finitely generated case) implies that $GKdim(B) - GKdim(B/P) \leq l$. By [8, Corollary 3.16], $ht(P) \leq l$.

If R is g-simple, by the preceding paragraph, $ht(Q) \leq l$ for any g-prime ideal Q of B. Thus $g\text{-}dim(B) \leq l$.

(2) Note that the g-prime ideals of $R\star g$ are precisely its prime ideals and that in (1) we have assumed $char(k) = 0$ to be sure that all the g-prime ideals of $B = R \star h$ are prime.

The next result bounds $g\text{-}dim(B)$ in terms of $dim_k(h)$ and of g-dim(R). Although, the bound is surely not sharp.

Proposition 1.7. *Let R be noetherian of finite Gelfand-Kirillov dimension, h an ideal of g of dimension l, $A = R \star g$ and $B = R \star h$. Suppose that R is finitely generated or g-locally finite.*

1. *Let k be of characteristic zero. Suppose that $P_0 \subset P_1 \subset \cdots \subset P_{l+1}$ is a strictly increasing chain of g-prime ideals of B, then $P_0 \cap R \subset P_{l+1} \cap R$ and $g\text{-}dim(B) \ < (l+1)(g\text{-}dim(R)+1)$.*
2. *Suppose that $P_0 \subset P_1 \subset \cdots \subset P_{n+1}$ is a strictly increasing chain of prime ideals of A, then $P_0 \cap R \subset P_{n+1} \cap R$ and $dim(A) \ < (n+1)(g\text{-}dim(R)+1)$.*

Proof. (1) Suppose that $P_0 \cap R = P_{l+1} \cap R = I$. Then I is a g-prime ideal of R and $B/IB \simeq (R/I) \star h$. Set $\bar{R} = R/I$ and $\bar{B} = B/IB$. In $\bar{B}$, we have a strictly increasing chain of g-prime ideals $\overline{P_0} \subset \overline{P_1} \subset \cdots \subset \overline{P_{l+1}}$ of length $l+1$ such that $\overline{P_0} \cap \bar{R} = \overline{P_{l+1}} \cap \bar{R} = \bar{I} = 0$; where $\overline{P_i}$'s denote the natural images of P_i's in $\bar{B}$. It follows that $g\text{-}ht(\overline{P_{l+1}}) \geq l+1$. By Corollary 1.6(1), $g\text{-}ht(\overline{P_{l+1}}) \leq l$ and we get a contradiction.

Let $P_0 \subset P_1 \subset \cdots \subset P_s$ be a strictly increasing chain of g-prime ideals of B. By the preceding paragraph,

$$P_0 \cap R \subset P_{l+1} \cap R \subset P_{2(l+1)} \cap R \subset P_{3(l+1)} \cap R \subset \cdots$$

is a strictly increasing chain of g-prime ideals of R. Since this chain can contain at most $(1+g\text{-}dim(R))$ g-prime ideals, we conclude that $s \ < (l+1)(g\text{-}dim(R)+1)$.

(2) The proof is similar to that of (1). Use Corollary 1.6(2) in the place of Corollary 1.6(1).

Remarks 1.8. (1) Under the assumption that R is finitely generated, the results in Corollary 1.6(2) and Proposition 1.7(2) are consequences of [11]. In the g-locally finite case, they are new.

(2) If the characteristic of k is 0, Passman [14, Corollary 4.4] established (1.7(2)) in a more general setting without the assumptions that R has finite Gelfand-Kirillov dimension and is "finitely generated" or "g-locally finite".

(3) Let g be abelian. If the characteristic of k is 0 (resp. $p > 0$), Chin [1, Theorem 2.11] (resp. Chin and Quinn [2, Theorem 1.10]) established the first assertion of (1.7(2)) without the assumptions that R is noetherian of finite Gelfand-Kirillov dimension and is finitely generated or g-locally finite.

(4) If the characteristic of k is 0 and g is solvable, Chin [1, Corollary 2.19] established the first assertion of (1.7(2)) for a chain of prime ideals

$$P_0 \subset P_1 \subset \cdots \subset P_l$$

such that $l \geq 2^n$ without the assumptions that R is noetherian with finite Gelfand-Kirillov dimension and is finitely generated or g-locally finite.

The ring R is g-hypernormal if for any pair of distinct comparable g-invariant ideals $I \subset J$ in R, the factor J/I contains a nonzero normal element of $(R/I) \star g$. If R is g-simple, R is g-hypernormal. Let k be algebraically closed of characteristic 0. If g is solvable and if R is commutative g-locally finite then R is g-hypernormal.

We say that condition (C_1) (resp. (C_2)) is satisfied in $spec_g(R)$ if Tauvel's height formula is valid for all g-prime ideals of R (resp. if the g-height and the height are finite and coincide on the g-prime ideals of R).

By [5, Corollaire 2.8], condition (C_2) is satisfied in $spec_g(R)$ if $char(k) = 0$ and R is noetherian g-hypernormal.

If R is g-simple, conditions (C_1) and (C_2) are trivially satisfied in $spec_g(R)$.

Proposition 1.7 can be improved under additional hypotheses.

Corollary 1.9. *Let k be of characteristic 0, R noetherian of finite Gelfand-Kirillov dimension, h an ideal of g of dimension l, $A = R \star g$ and $B = R \star h$. Suppose that R is finitely generated or g-locally finite. Assume also that condition (C_1) is satisfied in $spec_g(R)$.*

(1) *If P is a g-prime ideal of B, then $g\text{-}ht(P) \leq ht(P) \leq ht(P \cap R) + l$.*

(2) *Then $g\text{-}dim(B) \leq dim(R) + l$. In particular, $dim(A) \leq dim(R) + n$.*

Proof. (1) Set $Q = P \cap R$. So R/Q is a subalgebra of B/P and $GKdim(R/Q) \leq GKdim(B/P)$. By condition (C_1), $GKdim(R) - ht(Q) \leq GKdim(B/P)$. Corollary 1.5 in the g-locally finite case and [12, 8.2.10] in the finitely generated case imply that $GKdim(B) - GKdim(B/P) \leq ht(Q) + l$, hence the upper bound follows from [8, Corollary 3.16].

Corollary 1.10. *Let k be of characteristic 0, R noetherian g-hypernormal of finite Gelfand-Kirillov dimension, h an ideal of g of dimension l, $A = R \star g$ and $B = R \star h$. Suppose that R is finitely generated or g-locally finite. Assume also that condition (C_1) and (C_2) are satisfied in $spec_g(R)$.*

(1) *If P is a g-prime ideal of B, then $g\text{-}ht(P) \leq ht(P) \leq g\text{-}ht(P \cap R) + l$.*

(2) *Then $g\text{-}dim(B) \leq g\text{-}dim(R) + l$. In particular, $dim(A) \leq g\text{-}dim(R) + n$.*

Proof. (1) By condition (C_2), we have $g\text{-}ht(P \cap R) = ht(P \cap R)$. The results follow from Corollary 1.9(1).

Corollary 1.11. *Let k be algebraically closed of characteristic 0, g solvable, h an ideal of g of dimension l, R commutative noetherian g-locally finite with finite Gelfand-Kirillov dimension, $A = R \star g$ and $B = R \star h$. Suppose that condition (C_1) is satisfied in $spec_g(R)$.*

(1) *If P is a g-prime ideal of B, then $g\text{-}ht(P) \leq g\text{-}ht(P \cap R) + l$.*

(2) *Then $g\text{-}dim(B) \leq g\text{-}dim(R) + l$. In particular, $dim(A) \leq g\text{-}dim(R) + n$.*

Proof. By [5, Corollaire 2.18] and [6, Remark 0.1], B is g-hypernormal.

Now we shall study Tauvel's height formula in $R \star g$. Our aim is to obtain the results of [4] in the case where the ring R is g-locally finite instead of finitely generated.

Let g be completely solvable. We fix a composition series of g; i.e. a chain

$$0 = g_0 \subset g_1 \subset \cdots \subset g_n = g$$

of ideals of g such that g_{i+1}/g_i has dimension one. We shall set $R_i = R \star g_i$; $0 \leq i \leq n$ and $B = R_m$; so $R_0 = R$ and $R \star g_n = R \star g$. Choose X_i in $g_i - g_{i-1}$ such that $X_i + g_{i-1}$ is a basis of g_i/g_{i-1}. So $(X_1, X_2, \ldots, X_n)$ is a basis of g. We have $R_i \simeq R_{i-1}[\theta_i, \delta_i]$; where $\delta_i(r) = \delta_{X_i}(r)$ for any $r \in R_{i-1}$.

Suppose that condition (C_2) is satisfied in $spec_g(R)$. Let P be a g-prime ideal of $B = R * g_m$, $0 \leq m \leq n$ and $Q = P \cap R$. Set $l = g\text{-}ht(Q)$. So there exists a strictly saturated increasing chain of g-prime ideals of R ending at Q

$$Q_0 \subset Q_1 \subset \cdots \subset Q_l = Q.$$

Set $P_i = Q_i B$; $0 \leq i \leq l$ and $P_{l+i} = (P \cap R_i)B$; $0 \leq i \leq m$. So $P_l = Q_l B = QB$ and $P_{l+m} = P$. Consider the chain of g-prime ideals of B ending at P

$$P_0 \subset P_1 \subset \cdots \subset P_l \subseteq P_{l+1} \subseteq \cdots \subseteq P_{l+m} = P \qquad (\alpha)$$

Proposition 1.12. *Let k be of characteristic 0, g completely solvable, R noetherian g-locally finite with finite Gelfand-Kirillov dimension, $B = R * g_m; 0 \leq m \leq n$ and P a g-prime ideal of B. Suppose that conditions (C_1) and (C_2) are satisfied in $spec_g(R)$. Then the length of the chain (α) is $GKdim(B) - GKdim(B/P)$.*

Proof. The proof is similar to that of [4, Proposition 3.1] using "g-locally finite" in the place of "finitely generated". If $m = 0$, the result is true by the hypotheses. Assume that the result is true in R_i, $0 \leq i < m$. Set $B' = R * g_{m-1}$, $P' = P \cap B'$; so $P \cap R_i = P' \cap R_i$ for $0 \leq i \leq m-1$. Set $P'_i = Q_i B'$; $0 \leq i \leq l$ and $P'_{l+i} = (P' \cap R_i)B'$; $0 \leq i \leq m-1$; so $P' = P'_{l+m-1}$. By the induction hypothesis, the length of the chain

$$P'_0 \subset P'_1 \subset \cdots \subset P'_l = QB' \subseteq P'_{l+1} \subseteq \cdots \subseteq P'_{l+m-1} = P' \qquad (\beta)$$

is $GKdim(B') - GKdim(B'/P')$. By Corollary 1.5, its length is $GKdim(B) - GKdim(B/P'B)$. Clearly, $P_i = P'_i B$ for $0 \leq i \leq l$; $P_{l+i} = P'_{l+i}B$ and $P_{l+i} \cap B' = P'_{l+i}$ for $0 \leq i \leq m-1$. We deduce that $P_i = P_{i+1}$ if and only if $P'_i = P'_{i+1}$, $0 \leq i \leq l$ and $P'_{l+i+1} = P'_{l+i}$ if and only if $P_{l+i+1} = P_{l+i}$. It follows that the chain of g-prime ideals of B

$$P_0 \subset P_1 \subset \cdots \subset P_l \subseteq P_{l+1} \subseteq \cdots \subseteq P_{l+m-1} = P'B$$

has the same length as the chain (β). Thus, its length is $GKdim(B) - GKdim(B/P'B)$. If $P = P'B$, then by Corollary 1.5, the result is true.

If $P'B \subset P$, the chain (α) has for length $GKdim(B) - GKdim(B/P'B) + 1$. Now we shall show that $GKdim(B/P) = GKdim(B/P'B) - 1$. As B'/P' is a subalgebra of B/P, we have $GKdim(B'/P') \leq GKdim(B/P)$; hence $GKdim(B/P'B) - 1 \leq GKdim(B/P)$, by Corollary 1.5. On the other hand, $P/P'B$ is a nonzero g-invariant ideal of the prime noetherian ring

$B/P'B$. By [12, 2.3.5 (ii)], $P/P'B$ contains a regular element. By [8, Proposition 3.15], $GKdim(B/P) \leq GKdim(B/P'B) - 1$ and the proposition is proved.

We are now ready to prove Tauvel's height formula in $R \star g$.

Proposition 1.13. *Let k be of characteristic 0, g completely solvable, R noetherian g-locally finite with finite Gelfand-Kirillov dimension, $B = R * g_m$; $0 \leq m \leq n$ and P a g-prime ideal of B. Suppose that conditions (C_1) and (C_2) are satisfied in $spec_g(R)$. Then $GKdim(B) = GKdim(B/P) + ht(P)$.*

Proof. By (1.12), $GKdim(B) - GKdim(B/P) \leq htP$. By [8, Corollary 3.16], $GKdim(B) - GKdim(B/P) \geq htP$.

The following result is not already known.

Corollary 1.14. *Let k be of characteristic 0, g completely solvable, R noetherian g-locally finite g-simple with finite Gelfand-Kirillov dimension. Set $B = R * g_m$, $0 \leq m \leq n$ and let P be a g-prime ideal of B. Then we have $GKdim(B) = GKdim(B/P) + ht(P)$.*

Corollary 1.15. *Let k be of characteristic 0, g completely solvable, R noetherian g-locally finite with finite Gelfand-Kirillov dimension, $B = R*g_m$; $0 \leq m \leq n$ and P a g-prime ideal of B. Suppose that conditions (C_1) and (C_2) are satisfied in $spec_g(R)$. Then g-$ht(P) = ht(P)$.*

Proof. Let h' be an ideal of g of codimension 1 in h. Assume that the property is true in $B' = R \star h'$. Set $B = R \star h$ and $P' = P \cap B'$. So $B = B' \star (h/h')$, $B/P'B = (B'/P') \star (h/h')$, g-$ht(P') = ht(P')$ and (cf [4, Lemme 4.9]),

$$ht(P') = g - ht(P') \leq g - ht(P'B) \leq g - ht(P) \qquad (**)$$

By Corollary 1.5, $GKdim(B/P'B) = GKdim(B'/P') + 1$ and $GKdim(B) = GKdim(B') + 1$, since B' and B'/P' are h/h'-locally finite. We deduce from Proposition 1.13 that $ht(P'B) = ht(P')$.

If $P = P'B$, $(**)$ implies that $ht(P) \leq g$-$ht(P)$, as claimed.

If $P'B \subset P$ then $ht(P') = ht(P'B) < ht(P)$. As B'/P' is a subalgebra of B/P, we have $GKdim(B'/P') \leq GKdim(B/P)$. On the other hand, Proposition 1.13 implies that $ht(P) \leq ht(P') + 1$. It follows that $ht(P) = ht(P') + 1$. By [4, Lemme 4.9], g-$ht(P) \geq g$-$ht(P'B) + 1$. But g-$ht(P'B) = ht(P'B) = ht(P')$, so g-$ht(P) \geq ht(P') + 1 \geq ht(P)$ and the proof is complete.

2. When the action comes from an abelian group

In this section R is a k-algebra. Let G be a group and kG the group algebra of G. We suppose that G acts on R by k-algebra automorphisms. Denote by $R \star G$ the crossed product of R by G. For any $x \in G$, we denote by $\bar{x}$ the canonical image of x in $R \star G$. We recall that there exists a map $t : G \times G \to U(R)$; ($U(R)$ is the multiplicative group of unit elements of R) such that

$$\bar{x}\bar{y} = \overline{xy}t(x, y)$$

and $\bar{x}r = x(r)\bar{x}$. It follows that

$$x(\bar{y}) = \overline{x(y)}t(xy, x^{-1})t(x, x^{-1})^{-1}x(t(x, y))$$

where $x(y) = xyx^{-1}$.

If $t(x, y) = 1 \quad \forall\, x, y \in G$ then R is a G-module, $R \star G$ is usually denoted by $R\#G$ [12] and is the k-algebra generated by R and kG.

For any G-prime ideal P of R, G-$ht(P)$ denotes the G-height of P. We denote by G-dim the G-invariant version of dim.

Let P be a G-prime ideal of R. Then P is semi-prime invariant and G permutes transitively the minimal prime ideals over P. It follows that $ht(P) = ht(I)$ and $GKdim(R/P) = GKdim(R/I)$, where I is any minimal prime ideal over P.

From now on G is a free abelian group of finite rank n with basis $x_1, x_2, \ldots, x_n$.

We denote by G_i the subgroup of G whose basis is $x_1, x_2, \ldots, x_i$. We will set $R_i = R\star G_i$; $0 \leq i \leq n$. So we have $R_0 = R$, $R \star G_n = R \star G$ and $R_i = R_{i-1}[\bar{x}_i, \bar{x}_i{}^{-1}, \phi_i]$; where $\phi_i(r) = x_i(r) = \bar{x}_i r \bar{x}_i{}^{-1}$ for any $r \in R_{i-1}$. If Imt is contained in the center of R, the ϕ_i are commuting automorphisms of R.

In the remainder of this section we suppose (except 2.7) that the image of t is contained in a field W which is a finite-dimensional G-stable subalgebra of R. This condition is satisfied if the image Imt of t is contained in k; in particular if Imt is identically 1.

Lemma 2.1.

(1) *If $y \in G_i$, then $\bar{y}$ is a G-finite element of R_i.*

(2) *If R is G-locally finite, then R_i is G-locally finite.*

Proof. (1) Since G is abelian we have

$$x(\bar{y}) = \bar{y}t(xy, x^{-1})t(x, x^{-1})^{-1}x(t(x, y)).$$

(Note in passing that we have also $x(\bar{y}) = \bar{y}t(y, x^{-1})t(x, y)$.) So $x(\bar{y}) \in \bar{y}W$ and $\bar{y}W$ is a finite-dimensional G-stable k-vector subspace of R.

(2) Suppose that R_{i-1} is G-locally finite. Let $u \in R_{i-1}$. Then $g(u) \in V$ where V is a finite-dimensional G-stable subspace of R and $g(u\bar{x}_i) = g(u)g(\bar{x}_i) \in V\bar{x}_iW$. Now it is easy to see that $V\bar{x}_iW$ is a finite-dimensional G-stable subspace of R_i. Then R_i is G-locally finite.

Lemma 2.2. *Let R be G-locally finite. Then $GKdim(R_i) = GKdim(R_{i-1}) + 1$; $1 \leq i \leq n$.*

Proof. By (2.1), R_i is G-locally finite. So R_i is ϕ_{i+1}-locally finite. By [9, Proposition 1], $GKdim(R_{i+1}) = GKdim(R_i) + 1$.

Let $\lambda = (\lambda_{ij})$ be an $n \times n$ matrix of nonzero elements of k such that $\lambda_{ii} = 1$ and $\lambda_{ji} = \lambda_{ij}^{-1}$ for $1 \leq i, j \leq n$. The multiparameter coordinate ring of quantum affine n-space is the k-algebra $O_\lambda(k^n)$ generated by elements $x_1, \ldots, x_n$ subject only to the relation $x_i x_j = \lambda_{ij} x_j x_i$ for $1 \leq i, j \leq n$. Denote by $P(\lambda)$ the localisation of $O_\lambda(k^n)$ with respect to the multiplicative set generated by the x_i. The $P(\lambda)$ are exactly the k-algebras which are crossed product $k \star \mathbb{Z}^n$ of the field k by the group $\mathbb{Z}^n$.

We deduce from (2.2) that the Gelfand-Kirillov dimension of $R \otimes_k P(\lambda)$ is $GKdim(R)+n$, since $R \otimes_k P(\lambda) \simeq R \star G$ where G acts trivially on R and Imt is contained in k.

Let P be a G-prime ideal of R. Then G permutes transitively the minimal prime ideals over P. A consequence of this fact is that $ht(P) = ht(I)$ for any minimal prime I over P.

Proposition 2.3. *Let R be noetherian G-locally finite with finite Gelfand-Kirillov dimension. Set $A = R \star G$ and $B = R \star G_m$.*

1. *Let P be a G-prime ideal of B such that $P \cap R = 0$. Then $ht(P) \leq m$.*
2. *If R is G-simple, then G-$dim(B) \leq m$.*
3. *Suppose that $P_0 \subset P_1 \subset \cdots \subset P_{m+1}$ is a strictly increasing chain of G-prime ideals of B, then $P_0 \cap R \subset P_{m+1} \cap R$.*
4. *Then G-$dim(B) < (m+1)(G$-$dim(R)+1)$.*

Proof. (1) Since $R = R/(P \cap R)$ is a subalgebra of B/P, we have $GKdim(R) \leq GKdim(B/P)$. Lemma 2.2 implies that $GKdim(B) - GKdim(B/P) \leq m$. Note that $GKdim(B/P) = GKdim(B/I)$ for any minimal prime ideal I over P. By [8, Corollary 3.16], $ht(I) \leq GKdim(B) - GKdim(B/I)$. But $ht(I) = ht(P)$; so $ht(P) \leq m$.

(2) By [7, Proposition 3.6, Corollaire 3.8], B is G-hypernormal and G-$ht(P) = ht(P)$.

(3) and (4) Adapt the proof of Proposition 1.7 (1).

Remarks 2.4. (1) If G is abelian finitely generated, Chin [1, Theorem 3.9] established (2.3(3)) for $m = n$ without the assumptions that R is noetherian G-locally finite of finite Gelfand-Kirillov dimension.

(2) If G is nilpotent finitely generated of Hirsch number n, Chin [1, Theorem 3.10] established (2.3(3)) for $m = n$ and for a chain of prime ideals

$$P_0 \subset P_1 \subset \cdots \subset P_l$$

such that $l \geq 2^n$ without the assumptions that R is noetherian G-locally finite with finite Gelfand-Kirillov dimension.

We say that condition (C_1) is satisfied in $spec_G(R)$ if Tauvel's height formula is valid for all G-prime ideals of R.

By [7, Remarque 3.5(ii)], G-$ht(P) \leq ht(P)$ for any G-prime ideal P of R if R is commutative noetherian. Proposition 2.3(3) can be improved under additional hypothesis.

Corollary 2.5. *Let R be commutative noetherian G-locally finite with finite Gelfand-Kirillov dimension. Set $A = R \star G$ and $B = R \star G_m$. Suppose that condition (C_1) is satisfied in $spec_G(R)$.*

1. *Let P be a G-prime ideal of B, then G-$ht(P) \leq ht(P) \leq ht(P \cap R) + m$.*
2. *Then G-$dim(B) \leq dim(R) + m$.*

Proof. (1) Adapt the proof of Corollary 1.9.

Lemma 2.6. *Let k be algebraically closed.*

(1) *Assume that W is contained in the center of R. If I is a nonzero G-invariant ideal of R, then there exists in I a nonzero element u such that $x(u) = \lambda_x u$ for any $x \in G$; where $\lambda_x \in k$.*

(2) *If R is commutative, then R is G-hypernormal.*

Proof. (1) See the proof of [7, Lemme 4.12].

Corollary 2.7. *Let k be algebraically closed, R commutative noetherian G-locally finite with finite Gelfand-Kirillov dimension. Set $A = R \star G$ and $B = R \star G_m$. Suppose that condition (C_1) is satisfied in $spec_G(R)$.*

(1) *Let P be a G-prime ideal of B, then $G\text{-}ht(P) \leq G\text{-}ht(P \cap R) + m$.*

(2) *Then $G\text{-}dim(B) \leq G\text{-}dim(R) + m$. In particular, $dim(A) \leq G\text{-}dim(R) + n$.*

Proof. By Corollary 2.6(2), R is G-hypernormal. By [7, Corollaire 3.8], B is G-hypernormal. By [7, Proposition 3.6], $G\text{-}ht(P \cap R) = ht(P \cap R)$ and $G\text{-}ht(P) = ht(P)$. The results follow from Corollary 2.5.

The following proposition and its two corollaries are generalizations of [7, Proposition 4.14, Corollaries 4.15, 4.16]

Proposition 2.8. *Let k be algebraically closed, R commutative noetherian G-locally finite with finite Gelfand-Kirillov dimension. Set $A = R\star G$ and $B = R\star G_m$; $0 \leq m \leq n$. Suppose that condition (C_1) is satisfied in $spec_G(R)$. Then Tauvel's height formula holds in $spec_G(B)$.*

Proof. The result follows from [7, Corollaire 4.5].

From Proposition 2.8, we get the two following corollaries.

Corollary 2.9. *Let k be algebraically closed, R commutative, finitely generated, G-locally finite and G-prime with finite Gelfand-Kirillov dimension. Set $A = R \star G$ and $B = R \star G_m$; $0 \leq m \leq n$. Then Tauvel's height formula holds in $spec_G(B)$.*

Corollary 2.10. *Let k be algebraically closed, R commutative, noetherian, G-locally finite and G-simple with finite Gelfand-Kirillov dimension. Set $A = R \star G$ and $B = R \star G_m$; $0 \leq m \leq n$. Then Tauvel's height formula holds in $spec_G(B)$.*

As last application of [7, Proposition 4.5], we get the following result which was proved for $R = k$ in [3].

Proposition 2.11. *Let R be noetherian hypernormal with finite Gelfand-Kirillov dimension. If Tauvel's height formula holds in R then it also holds in $R \otimes_k P(\lambda)$.*

3. When the action comes from an abelian monoid

Let $S = R[x_1, \phi_1][x_2, \phi_2] \dots [x_n, \phi_n]$ be an iterated skew polynomial ring over k. For $1 \le i \le n$ set $S_i = R[x_1, \phi_1][x_2, \phi_2] \dots [x_i, \phi_i]$ with $S_0 = R$, $S_n = S$ and assume that the following conditions hold:

(1) for $1 \le i \le n$, ϕ_i is a k-algebra automorphism of R,

(2) for $1 \le i < j \le n$, ϕ_j is a k-algebra automorphism of S_i,

(3) for $1 \le i < j \le n$, there exists $\lambda_{i,j} \in k^*$ such that $\phi_j(x_i) = \lambda_{i,j} x_i$.

Set $\Phi_{i,j} = \{\phi_i, \phi_{i+1}, \dots, \phi_j\}$; $1 \le i < j \le n$. We say that R is $\Phi_{i,j}$-locally finite if each element of R is contained in a finite-dimensional ϕ_l-stable subspace of R; $l = i, i+1, \dots, j$.

Lemma 3.1. *If R is $\Phi_{1,n}$-locally finite, then S_i is $\Phi_{i+1,n}$-locally finite.*

Proposition 3.2. *If R is $\Phi_{1,n}$-locally finite, then $GKdim(S) = GKdim(R) + n$.*

Corollary 3.3. *The Gelfand-Kirillov dimension of $R \otimes_k O_\lambda(k^n)$ is $GKdim(R) + n$.*

Proof. $R \otimes_k O_\lambda(k^n) \simeq R[x_1, \phi_1][x_2, \phi_2] \dots [x_n, \phi_n]$ where each ϕ_i acts trivially on R.

The result in Corollary (3.3) is known for $R = k$ (see [3]).

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