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The Group of Automorphisms of the Coordinate Ring of Quantum Symplectic Space

J. Gómez-Torrecillas **L. EL Kaoutit***

Departamento de Algebra, Facultad de Ciencias
Universidad de Granada, E18071-Granada, Spain

Abstract. The group of automorphisms of the coordinate ring of quantum symplectic space $\mathcal{O}_q(\mathfrak{sp}\Bbbk^{2\times n})$ is isomorphic to the algebraic torus $(\Bbbk^\times)^{n+1}$, when q is not a root of unity.

Introduction.

In contrast with the classical commutative case, the rigidity of some quantized coordinate algebras in the 'generic case' allows an explicit description of their groups of automorphisms. Thus, J. Alev and M. Chamarie showed that the group of automorphisms of the quantum plane is a torus of rank two in the non root of unit case [1]. Under the same condition on the quantum parameters, J. Alev and F. Dumas [2] found later that the group of automorphisms of the first quantum Weyl algebra is an algebraic torus. This result was extended by L. Rigal to the n^{th} quantum Weyl algebra [12].

The aim of this paper is to describe explicitly the group of automorphisms of the coordinate ring of quantum space $\mathcal{O}_q(\mathfrak{sp}\Bbbk^{2\times n})$ (q is not a root of unity), which will be denoted by $\mathbf{Aut}_{\Bbbk}(\mathcal{O}_q(\mathfrak{sp}\Bbbk^{2\times n}))$. We use Rigal's method, namely, the fact that the set of height one prime ideals (explicitly determined in [5]) is invariant under the action of $\mathbf{Aut}_{\Bbbk}(\mathcal{O}_q(\mathfrak{sp}\Bbbk^{2\times n}))$.

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1. The quantum symplectic space

Let $\mathbb{k}$ be a field, and q a non-zero element in $\mathbb{k}$ which is not a root of unity. I. M. Musson found [8, §1.1] that the *coordinate ring* $\mathcal{O}_q(\mathfrak{sp}\mathbb{k}^{2\times n})$ *of the quantum symplectic space* (cf. [11, Definition 14] or [13, §4]) is the $\mathbb{k}$-algebra generated by $y_1, x_1, \ldots, y_n, x_n$ satisfying the following relations

$$\begin{array}{lll} y_j x_i = q^{-1} x_i y_j, & y_j y_i = q y_i y_j & (1 \leq i < j \leq n) \\ x_j x_i = q^{-1} x_i x_j, & x_j y_i = q y_i x_j & (1 \leq i < j \leq n) \\ x_i y_i - q^2 y_i x_i = (q^2-1)\sum_{l=1}^{i-1} q^{i-l} y_l x_l & & (1 \leq i \leq n) \end{array} \tag{1}$$

By [9, Proposition 1.10] or [10, Example 6], $\mathcal{O}_q(\mathfrak{sp}\mathbb{k}^{2\times n})$ can be written as an iterated Ore extension

$$R_0 \subseteq R_1 \subseteq \cdots \subseteq R_n = \mathcal{O}_q(\mathfrak{sp}\mathbb{k}^{2\times n})$$

where $R_0 = \mathbb{k}$ and $R_k = R_{k-1}[y_k; \alpha_k][x_k; \beta_k, \delta_k]$ for $k \geq 1$, with

$$\begin{array}{ll} \alpha_k(x_l) = q^{-1} x_l = \beta_k(x_l) & (1 \leq l < k \leq n) \\ \alpha_k(y_l) = q y_l = \beta_k(y_l) & (1 \leq l < k \leq n) \\ \beta_k(y_k) = q^2 y_k & (1 \leq k \leq n) \\ \delta_k(R_{k-1}) = 0,\ \delta_k(y_k) = (q^2-1)\sum_{l=1}^{k-1} q^{k-l} y_l x_l & (1 \leq k \leq n) \end{array}$$

Consider the elements $\Omega_i = \sum_{l=1}^{i} q^{i-l} y_l x_l$ $(i \geq 1)$. From [9, Lemma 1.3] we get

$$\begin{array}{lll} \Omega_i y_k = q^2 y_k \Omega_i, & \Omega_i x_k = q^{-2} x_k \Omega_i & (k \leq i) \\ \Omega_i x_k = x_k \Omega_i, & \Omega_i y_k = y_k \Omega_i & (i < k) \\ \Omega_i \Omega_k = \Omega_k \Omega_i & & (\text{for all } i, k) \end{array} \tag{2}$$

$$\begin{array}{l} \Omega_i = \sum_{j<l\leq i} q^{i-l} y_l x_l + q^{i-j}\Omega_j, \quad (j \leq i) \\ x_i y_i - q^2 y_i x_i = (q^2-1) q \Omega_{i-1} \\ x_i y_i - y_i x_i = (q^2-1)\Omega_i. \end{array} \tag{3}$$

By the relations (1), $\mathcal{O}_q(\mathfrak{sp}\mathbb{k}^{2\times n})$ is a P.B.W $\mathbb{k}$-algebra in the sense of [3, 4] with respect to the lexicographical order $\leq_{lex}$ on $\mathbb{N}^{2n}$ with $(1,0,\cdots,0) <_{lex} (0,1,\cdots,0) <_{lex} \cdots <_{lex} (0,\cdots,0,1)$. This implies that every $f \in \mathcal{O}_q(\mathfrak{sp}\mathbb{k}^{2\times n})$ is uniquely written as

$$f = \sum_{\alpha \in \mathbb{N}^{2n}} c_\alpha X^\alpha$$

with respect to the $\mathbb{k}$-basis

$$\mathcal{B} = \left\{ X^\alpha = y_1^{\nu_1} x_1^{\mu_1} \cdots y_n^{\nu_n} x_n^{\mu_n} \,|\, \alpha = (\nu_1, \mu_1, \cdots, \nu_n, \mu_n) \in \mathbb{N}^{2n} \right\}.$$

For $f \neq 0$, define

$$exp(f) = \max_{\leq_{lex}}\{\alpha \mid \ c_\alpha \neq 0\}.$$

By [4, Proposition 1.3(i)],

$$exp(fg) = exp(f) + exp(g) \text{ for any } f,\, g \in \mathcal{O}_q(\mathfrak{sp}\Bbbk^{2\times n}) \setminus \{0\}.$$

In [5] we found an explicit description of the prime spectrum (the set of prime ideals) of $\mathcal{O}_q(\mathfrak{sp}\Bbbk^{2\times n})$. From [5, Proposition 3.2, Corollary 3.5], we can deduce that the set of all height one prime ideals is

$$\mathbf{P} = \{\langle y_1\rangle, \langle x_1\rangle, \langle \Omega_2\rangle, \cdots, \langle \Omega_n\rangle\}.$$

2. The group of automorphisms

To determine this group, we first show that the $\Bbbk$-vector space generated by an element $v \in \{y_1, x_1, \Omega_2, \cdots, \Omega_n\}$ is invariant by $\mathbf{Aut}_\Bbbk(\mathcal{O}_q(\mathfrak{sp}\Bbbk^{2\times n}))$. Then we compute the image of the rest of generators, by a fixed element of $\mathbf{Aut}_\Bbbk(\mathcal{O}_q(\mathfrak{sp}\Bbbk^{2\times n}))$. Let V_n be the $\Bbbk$-vector space generated by $\{\Omega_1, \cdots, \Omega_n\}$, by (3) the set $\{y_ix_i\}_{1\leq i\leq n}$ is a $\Bbbk$-basis of this vector space.

Lemma 2.1. *Let $\sigma \in \mathbf{Aut}_\Bbbk(\mathcal{O}_q(\mathfrak{sp}\Bbbk^{2\times n}))$. Then for any $i \in \{2, \cdots, n\}$ there exists $j \in \{2, \cdots, n\}$, such that*

$$\sigma(\Omega_i) = \lambda_{ij}\Omega_j, \ \textit{and}\ \sigma(x_1) = \mu_1 x_1,\, \sigma(y_1) = \nu_1 y_1,$$

where $\lambda_{ij}, \nu_1, \mu_1 \in \Bbbk^*$.

Proof. By the Theorem of the principal ideal [7, Theorem 4.1.11] (or [6]), $\mathbf{P}$ is invariant by $\mathbf{Aut}_\Bbbk(\mathcal{O}_q(\mathfrak{sp}\Bbbk^{2\times n}))$. Fix $\sigma \in \mathbf{Aut}_\Bbbk(\mathcal{O}_q(\mathfrak{sp}\Bbbk^{2\times n}))$, and let $x \in \{y_1, x_1, \Omega_2, \cdots, \Omega_n\}$, then there exist $h, h' \in \mathcal{O}_q(\mathfrak{sp}\Bbbk^{2\times n}) \setminus \{0\}$ and $y \in \{y_1, x_1, \Omega_2, \cdots, \Omega_n\}$, such that $\sigma(x) = hy$ and $\sigma^{-1}(y) = h'x$. Hence $h\sigma(h') = \sigma(h')h = 1$, so h, h' are invertible thus $h, h' \in \Bbbk^*$. Suppose now that there exist $i \in \{2, \cdots, n\}$ and $j \in \{2, \cdots, n\}$, such that $\sigma(\Omega_i) = \alpha y_1$, $\sigma(\Omega_j) = \beta x_1$ for some $\alpha, \beta \in \Bbbk^*$. From (2), we have $\Omega_i\Omega_j = \Omega_j\Omega_i$. Applying σ to this equality we get $\alpha\beta = 0$, because q is not a root of unity. So for each $i \in \{2, \cdots, n\}$ there exists $j \in \{2, \cdots, n\}$, such that $\sigma(\Omega_i) = \lambda_{ij}\Omega_j, \lambda_{ij} \in \Bbbk^*$. By the same way, we show that x_1 (resp, y_1) can not be the image of y_1 (resp, x_1) by σ. Therefore $\sigma(x_1) = \mu_1 x_1$, $\sigma(y_1) = \nu_1 y_1$, $\nu_1, \mu_1 \in \Bbbk^*$. □

Remark 2.2. *By the previous lemma, for each $\sigma \in \mathbf{Aut}_\Bbbk(\mathcal{O}_q(\mathfrak{sp}\Bbbk^{2\times n}))$ and $i \in \{2, \cdots, n\}$, there exist $k \in \mathbb{N}$, such that $\sigma^k(\Omega_i) = \lambda\Omega_i, \lambda \in \Bbbk^*$.*

Lemma 2.3. *Let $a, b \in \mathcal{O}_q(\mathfrak{sp}\Bbbk^{2\times n}) \setminus \Bbbk$, such that $ab = \sum_{1\leq k_i\leq n} k_iy_ix_i \in V_n$ with $k_i \in \Bbbk$, and $k_n \neq 0$. Then there exist $\lambda, \lambda' \in \Bbbk^*$, such that $a = \lambda y_n$, and $b = \lambda' x_n$ (or $a = \lambda' x_n$, and $b = \lambda y_n$).*

Proof. It is clear that $exp(ab) = (0, 0, \cdots, 0, 1, 1)$, so we have

$$\begin{cases} exp(a) = (0, \cdots, 0, 1, 0), \quad exp(b) = (0, \cdots, 0, 0, 1) \\ \text{or} \\ exp(a) = (0, \cdots, 0, 0, 1), \quad exp(b) = (0, \cdots, 0, 1, 0). \end{cases}$$

Then we have, for example, $a = \lambda y_n + a_0$, $b = \lambda' x_n + b_0$, where $\lambda, \lambda' \in \mathbb{k}^*$ and $a_0 \in R_{n-1}$, $b_0 \in R_{n/2} = R_{n-1}[y_n; \alpha_n]$. So $ab = \lambda\lambda' y_n x_n + \lambda' a_0 x_n + \lambda y_n b_0 + a_0 b_0$, and $u = \lambda' a_0 x_n + \lambda y_n b_0 + a_0 b_0 \in V_n$. But $exp(\lambda' a_0 x_n) = exp(u) = (\nu, 0, 1)$, with $\nu \in \mathbb{N}^{2(n-1)}$. Hence $a_0 = b_0 = 0$. By the same way we get the other case. $\square$

Proposition 2.4. *Let $\sigma \in \mathbf{Aut}_{\mathbb{k}}(\mathcal{O}_q(\mathfrak{sp}\mathbb{k}^{2\times n}))$. Then for each $i \in \{1, \cdots, n\}$ we have $\sigma(\Omega_i) = \lambda_i \Omega_i$, $\lambda_i \in \mathbb{k}^*$, and $\sigma(x_i) = \mu_i x_i$, $\sigma(y_i) = \nu_i y_i$, $\nu_i, \mu_i \in \mathbb{k}^*$.*

Proof. For $i = 1$, we know by the Lemma 2.1 that $\sigma(x_1) = \mu_1 x_1$, $\sigma(y_1) = \nu_1 y_1$, $\nu_1, \mu_1 \in \mathbb{k}^*$, and so $\sigma(\Omega_1) = \lambda_1 \Omega_1$, $\lambda_1 = \nu_1 \mu_1$. Suppose that there exist $i \neq j \in \{2, \cdots, n\}$, such that $\sigma(\Omega_i) = \lambda \Omega_j$, $\lambda \in \mathbb{k}^*$. If $\sigma(\Omega_n) = \lambda \Omega_n$ then there does not exist $k \in \mathbb{N}^*$ such that $\sigma^k(\Omega_i) = \lambda_0 \Omega_n$, with $\lambda_0 \in \mathbb{k}^*$. Let m be the maximal element in the set

$$\{j \mid \sigma(\Omega_l) = \lambda \Omega_j \text{ for some } \lambda \in \mathbb{k}^* \text{ and } l \neq j\}.$$

Let $\sigma(\Omega_i) = \lambda_0 \Omega_m$, $\lambda_0 \in \mathbb{k}^*$. Note that $1 < i < m \leq n$, $\sigma(\Omega_{i-1}) = \lambda'_0 \Omega_r$, $\sigma(\Omega_{i+1}) = \lambda'' \Omega_s$ for some $\lambda', \lambda'' \in \mathbb{k}^*$ and $r, s < m$. Applying σ to $\Omega_i = y_i x_i + q\Omega_{i-1}$, we get $\sigma(y_i)\sigma(x_i) = \sum_{1 \leq l \leq m} k_l y_l x_l$, with $k_m = \lambda_0$. By Lemma 2.3 (with $m = n$) applied to $\sigma(y_i)\sigma(x_i)$, we have for example $\sigma(y_i) = \mu' x_m$, $\mu' \in \mathbb{k}^*$. If we apply σ to $\Omega_{i+1} = y_{i+1} x_{i+1} + q\Omega_i$, then we get $\sigma(y_{i+1}) = \mu'_0 x_m$, $\mu'_0 \in \mathbb{k}^*$ (or $\sigma(x_{i+1}) = \mu'_0 x_m$), which contradicts the injectivity of σ. In conclusion we have $\sigma(\Omega_i) = \lambda_i \Omega_i, \lambda_i \in \mathbb{k}^*$. Now applying σ, to $\Omega_i = y_i x_i + q\Omega_{i-1}, i = 2, \ldots, n$, we get $\sigma(y_i)\sigma(x_i) \in V_n$. By Lemma 2.3 (with $i = n$) applied to $\sigma(y_i)\sigma(x_i)$, we have $\sigma(y_i) = \nu_i y_i$, $\sigma(x_i) = \mu_i x_i$, $\nu_i, \mu_i \in \mathbb{k}^*$. Observe that we can not have, $\sigma(x_i) = \nu_i y_i, \sigma(y_i) = \mu_i x_i$. If it is the case, then one have $\mu_i \nu_i y_i x_i - q^2 \mu_i \nu_i x_i y_i = (q^2 - 1) q \lambda_{i-1} \Omega_{i-1}$, so $\mu_i \nu_i = 0$. $\square$

Theorem 2.5. *Let $\mathcal{O}_q(\mathfrak{sp}\mathbb{k}^{2\times n})$, with q not a root of unity. Then*

$$\mathbf{Aut}_{\mathbb{k}}(\mathcal{O}_q(\mathfrak{sp}\mathbb{k}^{2\times n})) \cong (\mathbb{k}^\times)^{n+1}.$$

Proof. Pick $\sigma \in \mathbf{Aut}_{\mathbb{k}}(\mathcal{O}_q(\mathfrak{sp}\mathbb{k}^{2\times n}))$, by the Proposition 2.4 we have for each $i = 1, \ldots, n$, $\sigma(x_i) = \mu_i x_i$, $\sigma(y_i) = \nu_i y_i$, $\mu_i, \nu_i \in \mathbb{k}^*$ and $\sigma(\Omega_i) = \lambda_i \Omega_i$, $\lambda_i \in \mathbb{k}^*$, with $\lambda_1 = \mu_1 \nu_1$. Applying σ to $\Omega_2 = y_2 x_2 + (q^2 - 1) q y_1 x_1$, we get $\lambda_2 = \lambda_1 = \mu_1 \nu_1 = \mu_2 \nu_2$. We continue to applying σ to other equalities listed in (3), we get in the end, $\lambda_1 = \cdots = \lambda_n = \mu_1 \nu_1 = \cdots = \mu_n \nu_n$. The theorem now is clear. $\square$

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