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# Geometric Probabilities for Convex Bodies of Large Revolution in the Euclidean Space $E_3$ (II)

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**Abstract.** In this paper we solve problems of Buffon type for an arbitrary convex body of revolution and four different types of lattices.

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Buffon's problem for an arbitrary convex body **K** and a lattice of parallelograms in the Euclidean space $E_2$ has been investigated in [1]. In [5] this problem is considered for two different types of lattices in the space $E_2$ namely, for those lattices whose fundamental cell is a triangle or a regular hexagon. Buffon's Needle Problem for a lattice of right-angled parallelepipeds in the $n$-dimensional Euclidean space was solved in [9]. In her dissertation, E. Bosetto has answered the corresponding questions for other types of lattices in the 3-dimensional space and for test bodies like the needle or the sphere. In [7] Buffon's problem is solved for a lattice of right-angled parallelepipeds in the 3-dimensional space (which will be denoted here by $\mathcal{R}_1$) and an arbitrary convex body of revolution. In the present paper we prove results of this type for arbitrary convex bodies of revolution and four types of lattices in $E_3$, considered also by E. Bosetto.

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Let $\mathbf{K}$ be an arbitrary convex body of revolution with centroid $S$ and oriented axis of rotation $\mathbf{d}$. Clearly, the axis $\mathbf{d}$ is determined by the angle $\theta$ between $\mathbf{d}$ and the $z$-axis and by the angle $\varphi$ between the projection of $\mathbf{d}$ on the $xy$-plane and the $x$-axis and we express this by writing $\mathbf{d} = \mathbf{d}(\theta, \varphi)$. If for a given $\mathbf{d} = \mathbf{d}(\theta, \varphi)$, the body $\mathbf{K}$ is tangent to the $xy$-plane such that the centroid $S$ lies in the upper half-space, we denote by $p(\theta, \varphi)$ the distance from $S$ to the $xy$-plane. Then the length of the projection of $\mathbf{K}$ on the $z$-axis is given by $L(\theta, \varphi) = p(\theta, \varphi) + p(\pi - \theta, \varphi)$. Note that $p(\theta, \varphi)$ does actually depend only on the angle $\theta$ and moreover, since $\mathbf{K}$ is a body of revolution about the axis $\mathbf{d}$ the value $p(\theta, \varphi)$ is invariant to any rotation about this axis, say by an $\psi$. Now let $\mathcal{F}$ be a fundamental cell of the lattice $\mathcal{R}$ and assume that the two 3-dimensional random variables defined by the coordinates of $S$ and by the triple $(\theta, \varphi, \psi)$ are uniformly distributed in the cell $\mathcal{F}$ and in $[0, \pi] \times [0, 2\pi] \times [0, 2\pi]$ respectively. We are interested in the probability $p_{\mathbf{K},\mathcal{R}}$ that the body $\mathbf{K}$ intersects the lattice $\mathcal{R}$. Furthermore, we will assume, as it is done in all papers cited here, that the body $\mathbf{K}$ is small with respect to the lattice $\mathcal{R}$. In order to recall briefly this concept, consider for fixed $(\theta, \varphi) \in [0, \pi] \times [0, 2\pi]$ the set of all points $P \in \mathcal{F}$ for which the body $\mathbf{K}$ with centroid $P$ and rotation axis $\mathbf{d} = \mathbf{d}(\theta, \varphi)$ does not intersect the boundary $\partial\mathcal{F}$ and let $\mathcal{F}(\theta, \varphi)$ be the closure of this open subset of $\mathcal{F}$. We say that the body $\mathbf{K}$ is small with respect to $\mathcal{R}$, if the polyhedrons sides of $\mathcal{F}(\theta, \varphi)$ and $\mathcal{F}$ are then clearly pairwise parallel.

Denote by $\mathcal{M}_{\mathcal{F}}$ the set of all test bodies $\mathbf{K}$ whose centroid $S$ lies in $\mathcal{F}$ and by $\mathcal{N}_{\mathcal{F}}$ the set of bodies $\mathbf{K}$ that are completely contained in $\mathcal{F}$. Of course, we can identify these sets with subsets of $\mathbb{R}^6$ and if $\mu$ denotes the Lebesgue measure then the probability is given by

$$p_{\mathbf{K},\mathcal{R}} = 1 - \frac{\mu(\mathcal{N}_{\mathcal{F}})}{\mu(\mathcal{M}_{\mathcal{F}})} \quad . \tag{1}$$

Using the cinematic measure (see [6])

$$d\mathbf{K} = dx \wedge dy \wedge dz \wedge d\Omega \wedge d\psi \ , \tag{2}$$

where $x, y, z$ are the coordinates of $S$, $d\Omega = \sin\theta d\theta \wedge d\varphi$ and $\psi$ is an angle of rotation about $\mathbf{d}$ we can compute

$$\mu(\mathcal{M}_{\mathcal{F}}) = \int_0^{2\pi} d\psi \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \iiint\limits_{\{(x,y,z)\in\mathcal{F}\}} dxdydz = 8\pi^2 \ \mathrm{Vol}\ (\mathcal{F}) \ , \tag{3}$$

$$\begin{aligned} \mu(\mathcal{N}_{\mathcal{F}}) &= \int_0^{2\pi} \Bigg( \int_0^{2\pi} \Bigg( \int_0^{\pi} \sin\theta \Bigg( \iiint\limits_{\{(x,y,z)\in\mathcal{F}(\theta,\varphi)\}} dxdydz \Bigg) d\theta \Bigg) d\varphi \Bigg) d\psi \\ &= 2\pi \int_0^{2\pi} \Bigg( \int_0^{\pi} \mathrm{Vol}\ \mathcal{F}(\theta, \varphi) \cdot \sin\theta d\theta \Bigg) d\varphi \ , \end{aligned} \tag{4}$$

which leads to

$$p_{\mathbf{K},\mathcal{R}} = 1 - \frac{1}{4\pi \ \mathrm{Vol}\ (\mathcal{F})} \int_0^{2\pi} \Bigg( \int_0^{\pi} \mathrm{Vol}\ \mathcal{F}(\theta, \varphi) \cdot \sin\theta d\theta \Bigg) d\varphi \ . \tag{1'}$$

The above reasoning is valid for all lattices $\mathcal{R}$ provided $\mathbf{K}$ is small with respect to the lattice. Our purpose here is "only" to show that for four different types of lattices that we denote as in [3] by $\mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_5$, the volume of $\mathcal{F}(\theta,\varphi)$ can be expressed in terms of the well known support- and width-function ($p$ and $L$) associated to the body $\mathbf{K}$ and to compute some of the integrals involved.

## 1. The lattice $\mathcal{R}_2$

The fundamental cell $\mathcal{F}_2$ of the lattice $\mathcal{R}_2$ is the parallelepiped spanned by the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, where $\mathbf{c}=(0,0,c)$ is perpendicular on $\mathbf{a}=(a\sin\alpha,\ a\cos\alpha,\ 0)$ and $\mathbf{b}=(0,b,0)$. We can assume without loss that the angle $\alpha$ between $\mathbf{a}$ and $\mathbf{b}$ belongs to $\left]0,\frac{\pi}{2}\right]$. One checks that $\mathbf{K}$ is small with respect to $\mathcal{R}_2$ if and only if its diameter is less than $\min(a\sin\alpha,\ b\sin\alpha,\ c)$.

Recall that given $\mathbf{d}=\mathbf{d}(\theta,\varphi)$, $L(\theta,\varphi)$ denotes the length of the orthogonal projection of $\mathbf{K}$ onto the $z$-axis. In order to simplify the expression for Vol $\mathcal{F}_2(\theta,\varphi)$ we use the functions $\theta_1,\ \varphi_1$ and $\theta_2,\ \varphi_2$ defined as follows:

$$\theta_1(\theta,\varphi):=\arccos(\sin\theta\cos\varphi),\ \varphi_1(\theta,\varphi):=\arctan\left(\frac{\cot\ \theta}{\sin\varphi}\right),$$
$$\theta_2(\theta,\varphi):=\arccos\left(\sin\theta\sin\left(\varphi+\alpha-\frac{\pi}{2}\right)\right),\ \varphi_2(\theta,\varphi):=\arctan\left(\tan\ \theta\sin(\varphi+\alpha)\right).$$

Thus, for $\mathbf{d}=\mathbf{d}(\theta,\varphi)$, the length of the orthogonal projection of $\mathbf{K}$ onto the $x$-axis is given by $L(\theta_1(\theta,\varphi),\ \varphi_1(\theta,\varphi))$ and also, the distance between the two planes that are parallel to the plane spanned by the vectors $\mathbf{a}$ and $\mathbf{c}$ and tangent to $\mathbf{K}$ equals $L(\theta_2(\theta,\varphi),\ \varphi_2(\theta,\varphi))$. This implies

$$\begin{aligned}
\text{Vol }\mathcal{F}_2(\theta,\varphi)=&\Big(a\sin\alpha-L\Big(\theta_1(\theta,\varphi),\varphi_1(\theta,\varphi)\Big)\Big)\Big(\Big(b-\frac{1}{\sin\alpha}L\Big(\theta_2(\theta,\varphi),\varphi_2(\theta,\varphi)\Big)\Big)\\
&\cdot\Big(c-L(\theta,\varphi)\Big)\\
=&\ abc\sin\alpha-ab\sin\alpha\ L(\theta,\varphi)-bc\ L\Big(\theta_1(\theta,\varphi),\ \varphi_1(\theta,\varphi)\Big)\\
&-ca\ L\Big(\theta_2(\theta,\varphi),\ \varphi_2(\theta,\varphi)\Big)+a\ L\Big(\theta_2(\theta,\varphi),\ \varphi_2(\theta,\varphi)\Big)\ L(\theta,\varphi)\\
&+b\ L(\theta,\varphi)\ L\Big(\theta_1(\theta,\varphi),\ \varphi_1(\theta,\varphi)\Big)+\frac{c}{\sin\alpha}\ L\Big(\theta_1(\theta,\varphi),\ \varphi_1(\theta,\varphi)\Big)L\Big(\theta_2(\theta,\varphi),\ \varphi_2(\theta,\varphi)\Big)\\
&-\frac{1}{\sin\alpha}\ L(\theta,\varphi)\ L\Big(\theta_1(\theta,\varphi),\ \varphi_1(\theta,\varphi)\Big)L\Big(\theta_2(\theta,\varphi),\ \varphi_2(\theta,\varphi)\Big).
\end{aligned}$$

From this we obtain

$$\int_0^{2\pi}\int_0^{\pi}\text{Vol }\mathcal{F}_2(\theta,\varphi)\sin\theta d\theta d\varphi=4\pi abc\sin\alpha-ab\sin\alpha\int_0^{2\pi}\int_0^{\pi}L(\theta,\varphi)\sin\theta d\theta d\varphi$$
$$-bc\int_0^{2\pi}\int_0^{\pi}L\Big(\theta_1(\theta,\varphi),\ \varphi_1(\theta,\varphi)\Big)\sin\theta d\theta d\varphi-ca\int_0^{2\pi}\int_0^{\pi}L\Big(\theta_2(\theta,\varphi),\ \varphi_2(\theta,\varphi)\Big)\sin\theta d\theta d\varphi$$

$$+a\int_0^{2\pi}\int_0^{\pi} L\Big(\theta_2(\theta,\varphi),\ \varphi_2(\theta,\varphi)\Big)L(\theta,\varphi)\sin\theta d\theta d\varphi$$

$$+\frac{c}{\sin\alpha}\int_0^{2\pi}\int_0^{\pi} L\Big(\theta_1(\theta,\varphi),\ \varphi_1(\theta,\varphi)\Big)L\Big(\theta_2(\theta,\varphi),\ \varphi_2(\theta,\varphi)\Big)\sin\theta d\theta d\varphi$$

$$+b\int_0^{2\pi}\int_0^{\pi} L(\theta,\varphi)L\Big(\theta_1(\theta,\varphi),\ \varphi_1(\theta,\varphi)\Big)\sin\theta d\theta d\varphi$$

$$-\frac{1}{\sin\alpha}\int_0^{2\pi}\int_0^{\pi} L(\theta,\varphi)L\Big(\theta_1(\theta,\varphi),\ \varphi_1(\theta,\varphi)\Big)L\Big(\theta_2(\theta,\varphi),\ \varphi_2(\theta,\varphi)\Big)\sin\theta d\theta d\varphi\ ,$$

and by $(1')$

$$(5_2)\quad p_{\mathbf{K},\mathcal{R}_2} = \frac{1}{4\pi a\sin\alpha}\int_0^{2\pi}\int_0^{\pi} L\Big(\theta_1(\theta,\varphi),\ \varphi_1(\theta,\varphi)\Big)\sin\theta d\theta d\varphi$$

$$+\frac{1}{4\pi b\sin\alpha}\int_0^{2\pi}\int_0^{\pi} L\Big(\theta_2(\theta,\varphi),\ \varphi_2(\theta,\varphi)\Big)\sin\theta d\theta d\varphi + \frac{1}{4\pi c}\int_0^{2\pi}\int_0^{\pi} L(\theta,\varphi)\sin\theta d\theta d\varphi$$

$$-\frac{1}{4\pi bc\sin\alpha}\int_0^{2\pi}\int_0^{\pi} L\Big(\theta_2(\theta,\varphi),\ \varphi_2(\theta,\varphi)\Big)L(\theta,\varphi)\sin\theta d\theta d\varphi$$

$$-\frac{1}{4\pi ab\sin^2\alpha}\int_0^{2\pi}\int_0^{\pi} L\Big(\theta_1(\theta,\varphi),\ \varphi_1(\theta,\varphi)\Big)L\Big(\theta_2(\theta,\varphi),\ \varphi_2(\theta,\varphi)\Big)\sin\theta d\theta d\varphi$$

$$-\frac{1}{4\pi ca\sin\alpha}\int_0^{2\pi}\int_0^{\pi} L(\theta,\varphi)L\Big(\theta_1(\theta,\varphi),\ \varphi_1(\theta,\varphi)\Big)\sin\theta d\theta d\varphi$$

$$+\frac{1}{4\pi abc\sin^2\alpha}\int_0^{2\pi}\int_0^{\pi} L(\theta,\varphi)L\Big(\theta_1(\theta,\varphi),\ \varphi_1(\theta,\varphi)\Big)L\Big(\theta_2(\theta,\varphi),\ \varphi_2(\theta,\varphi)\Big)\sin\theta d\theta d\varphi\ .$$

Thus, we have proved:

**Theorem 1.** *The probability $p_{\mathbf{K},\mathcal{R}_2}$ is given by the equality* $(5_2)$.

**Remarks.** 1) For $\alpha = \frac{1}{2}$ one obtains (for the lattice $\mathcal{R}_1$) the equality (1) in [7], since in this case the expression involved is symmetric in $a, b$ and $c$.

2) If $\mathbf{K}$ has constant width then the above result becomes

$$\Big(\frac{1}{a\sin\alpha}+\frac{1}{b\sin\alpha}+\frac{1}{c}\Big)k - \Big(\frac{1}{ab\sin^2\alpha}+\frac{1}{bc\sin\alpha}+\frac{1}{ca\sin\alpha}\Big)k^2 + \frac{1}{abc\sin^2\alpha}k^3\ .$$

In the case of sphere this expression is exactly the right-hand side of the formula (1.21) in [3].

3) If $\mathbf{K}$ is a needle of length $l < \min(a\sin\alpha, b\sin\alpha, c)$, we have $L(\theta,\varphi) = l|\cos\theta|$, which implies $L(\theta_2(\theta,\varphi),\ \varphi_2(\theta,\varphi)) = l|\sin\theta\cos(\varphi+\alpha)|$ and $L(\theta_1(\theta,\varphi),\ \varphi_1(\theta,\varphi)) = l|\sin\theta\cos\varphi|$ and the computations give the same result as in formula (1.13) in [3], i.e..

$$p_{\mathbf{K},\mathcal{R}_2} = \frac{ab\sin\alpha + ac + bc}{2abc\sin\alpha}\, l - 2\, \frac{a+b+[1+(\frac{\pi}{2}-\alpha)\cot\alpha]c}{3\pi abc\sin\alpha}\, l^2 + \frac{1+(\frac{\pi}{2}-\alpha)\cot\alpha}{4\pi abc\sin\alpha}\, l^3\,.$$

## 2. The lattice $\mathcal{R}_3$

The fundamental cell $\mathcal{F}_3$ of the lattice $\mathcal{R}_3$ is the parallelepiped spanned by the vectors $\mathbf{a} = (a\sin\alpha, a\cos\alpha, 0)$, $\mathbf{b} = (0, b, 0)$ and $\mathbf{c}$ (with $\|\mathbf{c}\| = c$). Let $\alpha, \beta$ and $\gamma$ the angles between $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{b}$ and $\mathbf{c}$ and $\mathbf{c}$ and $\mathbf{a}$ respectively. We can assume without loss that all three angles belong to the interval $\left]0, \frac{\pi}{2}\right]$. We denote also by $E_1$, $E_2$ and $E_3$ the planes spanned by $\mathbf{b}$ and $\mathbf{c}$, $\mathbf{c}$ and $\mathbf{a}$ and $\mathbf{a}$ and $\mathbf{b}$ respectively. Of course, $E_3$ is the $xy$-plane. Further, if $\xi_{ij}$ with $0 < \xi_{ij} \le \dfrac{\pi}{2}$ is the angle between $E_i$ and $E_j$ then $d_1 = a\sin\xi_{13}\sin\alpha = a\sin\xi_{12}\sin\gamma$ , $d_2 = b\sin\xi_{12}\sin\beta = b\sin\xi_{23}\sin\alpha$ and $d_3 = c\sin\xi_{23}\sin\gamma = c\sin\xi_{13}\sin\beta$ are the heights of the parallelepiped. Note that $(\alpha,\beta,\gamma)$ is uniquely determined by $\xi_{12}$, $\xi_{23}$, $\xi_{13}$ and viceversa. Thus, we can write $\mathcal{R}_3$ as a union of lattices of parallel equidistant planes denoted by $\mathcal{E}^1$, $\mathcal{E}^2$ and $\mathcal{E}^3$ such that the distance between the planes of $\mathcal{E}^i$ equals $d_i$. The normal vector to $E_3$ is $\mathbf{n}_3 = (0,0,1)$. As we did before, we denote by $\theta$ and $\varphi$ the angles between $\mathbf{d}$ and $\mathbf{n}_3$ and between $(1,0,0)$ and the projection of $\mathbf{d}$ on $E_3$.

Let $\mathbf{c}'$ be the orthogonal projection of $\mathbf{c}$ on the $xz$-plane and $\mathbf{c}_1 = \dfrac{1}{\|\mathbf{c}'\|}\mathbf{c}' = (\cos\xi_{13}, 0, \sin\xi_{13})$. The vector $\mathbf{n}_1 = (\sin\xi_{13}, 0, -\cos\xi_{13})$ is orthogonal to $E_1$ and $(\mathbf{b}, \mathbf{c}_1, \mathbf{n}_1)$ is a (positively oriented) triple of orthonormal vectors. Let $\theta_1$ and $\varphi_1$ be the angles formed by $\mathbf{d}$ and $\mathbf{n}_1$ and the projection of $\mathbf{d}$ on $E_1$ and $\mathbf{b}$. We have

$$\begin{aligned}\theta_1 &= \theta_1(\theta,\varphi) = \arccos(\sin\xi_{13}\sin\theta\cos\varphi - \cos\xi_{13}\cos\theta)\,,\\ \varphi_1 &= \varphi_1(\theta,\varphi) = \arctan\Big(\cos\xi_{13}\cot\varphi + \tfrac{\sin\xi_{13}\cot\theta}{\sin\varphi}\Big)\,.\end{aligned}$$

$x\sin\xi_{23}\cos\alpha - y\sin\xi_{23}\sin\alpha + z\cos\xi_{23} = 0$ is an equation for the plane $E_2$. The corresponding normal vector is $\mathbf{n}_2 = (\sin\xi_{23}\cos\alpha, -\sin\xi_{23}\sin\alpha, \cos\xi_{23})$. The vectors $\mathbf{c}_2 = (-\cos\xi_{23}\cos\alpha, \cos\xi_{23}\sin\alpha, \sin\xi_{23})$, $\mathbf{a}$ and $\mathbf{n}_2$ form a positively oriented triple of orthogonal vectors. If we consider the angles $\theta_2$ and $\varphi_2$ between $\mathbf{d}$ and $\mathbf{n}_2$ and between the projection of $\mathbf{d}$ on $E_2$ and $\mathbf{c}_2$ we have

$$\begin{aligned}\theta_2 &= \theta_2(\theta,\varphi) = \arccos(-\sin\xi_{23}\sin\theta\cos(\varphi+\alpha) - \cos\xi_{23}\cos\theta)\,,\\ \varphi_2 &= \varphi_2(\theta,\varphi) = \arctan\left(\tfrac{\sin\theta\sin(\alpha+\varphi)}{\sin\xi_{23}\cos\theta - \sin\theta\cos\xi_{23}\cos(\alpha+\varphi)}\right)\,.\end{aligned}$$

The parallelepiped $\mathcal{F}_3$ has the volume

$$\begin{aligned}\text{Vol }\mathcal{F}_3 &= ab\sin\alpha\cdot d_3 = abc\sin\alpha\sin\gamma\sin\xi_{23}\\ &= \tfrac{d_1}{\sin\xi_{13}}\cdot\tfrac{d_2}{\sin\alpha\sin\xi_{23}}\cdot d_3 = \tfrac{d_1 d_2 d_3}{\sin\xi_{13}\sin\xi_{23}\sin\alpha}\,.\end{aligned}$$

Now when $\mathbf{K}$ is small with respect to $\mathcal{R}_3$, that is, when the diameter $\sup\limits_{(\theta,\varphi)} L(\theta,\varphi)$ of $\mathbf{K}$ is smaller than $\min(d_1,d_2,d_3)$, then $\mathcal{F}_3(\theta,\varphi)$ is at its turn a parallelepiped whose faces and sides are parallel to the corresponding faces and sides of $\mathcal{F}_3$ for all values $(\theta,\varphi)\in[0,\pi]\times[0,2\pi]$. The heights of $\mathcal{F}_3(\theta,\varphi)$ are given by

$$d_1(\theta,\varphi)=d_1-L(\theta_1,\varphi_1),\ d_2(\theta,\varphi)=d_2-L(\theta_2,\varphi_2),\ d_3(\theta,\varphi)=d_3-L(\theta,\varphi)\ .$$

Then $\operatorname{Vol}\mathcal{F}_3(\theta,\varphi)=\dfrac{d_1(\theta,\varphi)d_2(\theta,\varphi)d_3(\theta,\varphi)}{\sin\xi_{13}\sin\xi_{23}\sin\alpha}$ and from $(1')$ we get

$$p_{\mathbf{K},\mathcal{R}_3}=1-\tfrac{1}{4\pi\ \operatorname{Vol}\ \mathcal{F}_3}\int_0^{2\pi}\int_0^{\pi}\operatorname{Vol}\mathcal{F}_3(\theta,\varphi)\sin\theta d\theta d\varphi$$
$$=1-\tfrac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\Big[1-\tfrac{L(\theta_1,\varphi_1)}{d_1}-\tfrac{L(\theta_2,\varphi_2)}{d_2}-\tfrac{L(\theta,\varphi)}{d_3}+\tfrac{L(\theta_1,\varphi_1)L(\theta_2,\varphi_2)}{d_1d_2}+$$
$$\tfrac{L(\theta_2,\varphi_2)L(\theta,\varphi)}{d_2d_3}+\tfrac{L(\theta,\varphi)L(\theta_1,\varphi_1)}{d_3d_1}-\tfrac{L(\theta,\varphi)L(\theta_1,\varphi_1)L(\theta_2,\varphi_2)}{d_1d_2d_3}\Big]\sin\theta d\theta d\varphi.$$

We have proved

**Theorem 2.** *If $\boldsymbol{K}$ is small with respect to $\mathcal{R}_3$, the probability $p_{\mathbf{K},\mathcal{R}_3}$ is given by*

$$(5_3)\quad p_{\mathbf{K},\mathcal{R}_3}=\frac{1}{4\pi}\Big[\frac{1}{d_1}\int_0^{2\pi}\int_0^{\pi}L(\theta_1,\varphi_1)\sin\theta d\theta d\varphi+\frac{1}{d_2}\int_0^{2\pi}\int_0^{\pi}L(\theta_2,\varphi_2)\sin\theta d\theta d\varphi$$
$$+\frac{1}{d_3}\int_0^{2\pi}\int_0^{\pi}L(\theta,\varphi),\sin\theta d\theta d\varphi-\frac{1}{d_1d_2}\int_0^{2\pi}\int_0^{\pi}L(\theta_1,\varphi_1)L(\theta_2,\varphi_2)\sin\theta d\theta d\varphi$$
$$-\frac{1}{d_2d_3}\int_0^{2\pi}\int_0^{\pi}L(\theta_2,\varphi_2)L(\theta,\varphi)\sin\theta d\theta d\varphi-\frac{1}{d_3d_1}\int_0^{2\pi}\int_0^{\pi}L(\theta,\varphi)L(\theta_1,\varphi_1)\sin\theta d\theta d\varphi$$
$$+\frac{1}{d_1d_2d_3}\int_0^{2\pi}\int_0^{\pi}L(\theta,\varphi)L(\theta_1,\varphi_1)L(\theta_2,\varphi_2)\sin\theta d\theta d\varphi\Big]\ .$$

**Remarks.** 1) The result is a generalization of Theorem 1 which is obtained for $\xi_{13}=\xi_{23}=\frac{\pi}{2}\ ,\ \beta=\gamma=\frac{\pi}{2}$ .

2) If $\mathbf{K}$ has constant width $k<\min\ (d_1,d_2,d_3)$ we obtain the special case

$$p_{\mathbf{K},\mathcal{R}_3}=\Big(\frac{1}{d_1}+\frac{1}{d_2}+\frac{1}{d_3}\Big)k-\Big(\frac{1}{d_1d_2}+\frac{1}{d_2d_3}+\frac{1}{d_3d_1}\Big)k^2+\frac{k^3}{d_1d_2d_3}\ .$$

3) For a needle of length $l<\min\ (d_1,d_2,d_3)$ one can find more detailed computations in [2].

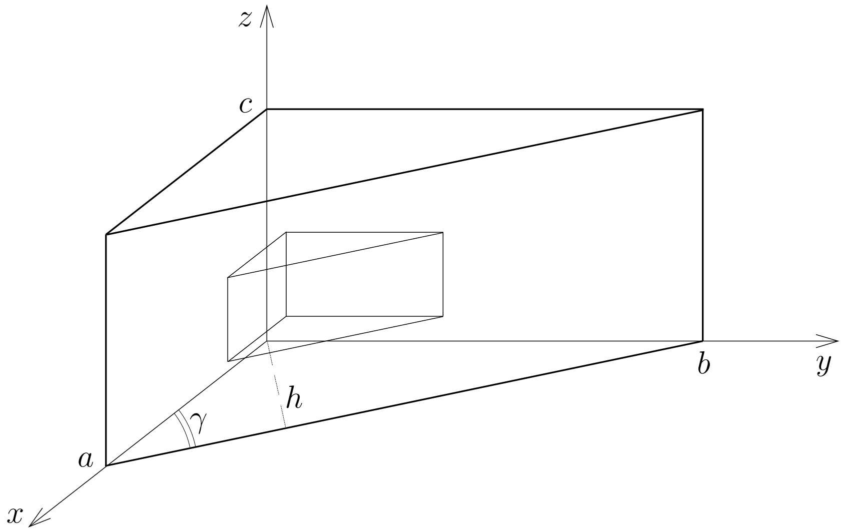
## 3. The lattice $\mathcal{R}_4$

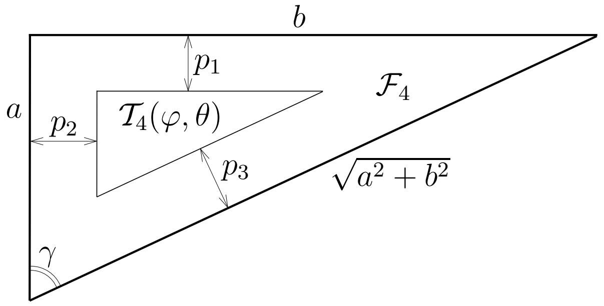
The fundamental cell $\mathcal{F}_4$ of the lattice $\mathcal{R}_4$ is a right-angled prism whose base $\mathcal{B}_4$ is a right-angled triangle with catheti $a$ and $b$. If $c$ is the height of the prism, then we can assume that the vertices of $\mathcal{F}_4$ are $(0,0,0)$ , $(a,0,0)$ , $(0,b,0)$ , $(0,0,c)$ , $(a,0,c)$ and $(0,b,c)$. We denote $\gamma := \arctan \dfrac{b}{a}$ and $h := \dfrac{ab}{\sqrt{a^2+b^2}}$. The body $\mathbf{K}$ is small with respect to $\mathcal{R}_4$ if

$$\text{Diam}\,(\mathbf{K}) < \min\Big(\frac{3ab}{2(a+b+\sqrt{a^2+b^2})}\Big)$$

(see [6]). In this case the set $\mathcal{F}_4(\theta,\varphi)$ is also a right-angled prism with height $c - L(\theta,\varphi)$, and whose base $\mathcal{B}_4(\theta,\varphi)$ is a right-angled triangle. We denote by $p_1, p_2$ and $p_3$ the lengths $p\Big(\theta_1(\theta,\varphi),\ \varphi_1(\theta,\varphi)\Big)$, $p\Big(\theta_2(\theta,\varphi),\ \varphi_2(\theta,\varphi)\Big)$ and $p\Big(\theta_3(\theta,\varphi),\ \varphi_3(\theta,\varphi)\Big)$. Let $\theta_1, \varphi_1, \theta_2, \varphi_2, \theta_3$ and $\varphi_3$ be the functions defined by

$$\begin{aligned}
&\theta_1(\theta,\varphi) := \arccos(\sin\theta\cos\varphi),\ \varphi_1(\theta,\varphi) := \arctan\Big(\tfrac{\cot\,\theta}{\sin\varphi}\Big)\ ,\\
&\theta_2(\theta,\varphi) := \arccos(\sin\theta\sin\varphi),\ \varphi_2(\theta,\varphi) := \arctan(\tan\,\theta\cos\varphi)\ ,\\
&\theta_3(\theta,\varphi) := \arccos(-\sin\theta\sin(\varphi+\gamma)),\ \varphi_3(\theta,\varphi) := \text{arccot}(-\tan\,\theta\cos(\varphi+\gamma))\ .
\end{aligned}$$




By a simple geometric argument (see e.g. [2]) is follows that

$$\frac{\text{Area } \mathcal{B}_4(\theta,\varphi)}{\text{Area } \mathcal{B}_4} = \Big(1 - \frac{p_1}{a} - \frac{p_2}{b} - \frac{p_3}{h}\Big)^2 \ .$$

Using also the fact that $L(\theta,\varphi)=L$ we obtain

$$\frac{\mathrm{Vol}\ \mathcal{F}_4(\theta,\varphi)}{\mathrm{Vol}\ \mathcal{F}_4}=\Big(1-\frac{p_1}{a}-\frac{p_2}{b}-\frac{p_3}{h}\Big)^2\ \Big(1-\frac{L}{c}\Big)\ .$$

We now prove

**Theorem 3.** *The probability $p_{\mathbf{K},\mathcal{R}_4}$ is given by*

$$(5_4)\quad p_{\mathbf{K},\mathcal{R}_4}=\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi}\Big(\frac{p_1}{a}+\frac{p_2}{b}+\frac{p_3}{h}+\frac{L}{2c}\Big)\sin\theta d\theta d\varphi$$

$$-\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi}\Big(\frac{p_1p_2}{ab}+\frac{p_2p_3}{bh}+\frac{p_3p_1}{ha}+\frac{p_1L}{ac}+\frac{p_2L}{bc}+\frac{p_3L}{hc}\Big)\sin\theta d\theta d\varphi$$

$$-\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\Big(\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{h^2}\Big)\sin\theta d\theta d\varphi$$

$$+\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi}\Big(\frac{p_1p_2L}{abc}+\frac{p_2p_3L}{bhc}+\frac{p_3p_1L}{hac}\Big)\sin\theta d\theta d\varphi$$

$$+\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\Big(\frac{p_1^2L}{a^2c}+\frac{p_2^2L}{b^2c}+\frac{p_3^2L}{h^2c}\Big)\sin\theta d\theta d\varphi\ .$$

*Proof.* We have

$$\Big(1-\frac{p_1}{a}-\frac{p_2}{b}-\frac{p_3}{h}\Big)^2\Big(1-\frac{L}{c}\Big)=1-2\Big(\frac{p_1}{a}+\frac{p_2}{b}+\frac{p_3}{h}+\frac{L}{2c}\Big)$$
$$+2\Big(\frac{p_1p_2}{ab}+\frac{p_2p_3}{bh}+\frac{p_3p_1}{ha}+\frac{p_1L}{ac}+\frac{p_2L}{bc}+\frac{p_3L}{hc}\Big)+\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{h^2}$$
$$-2\Big(\frac{p_1p_2L}{abc}+\frac{p_2p_3L}{bhc}+\frac{p_3p_1L}{hac}\Big)-\Big(\frac{p_1^2L}{a^2c}+\frac{p_2^2L}{b^2c}+\frac{p_3^2L}{h^2c}\Big)$$

and from $(1')$ we obtain $(5_4)$ .

**Remarks.** 1) In the case when $\mathbf{K}$ is a needle of length $l<\min\ (h,c)$ one can deduce from $(5_4)$, after some tedious calculations, the result of Theorem 1.3.3 in [3].

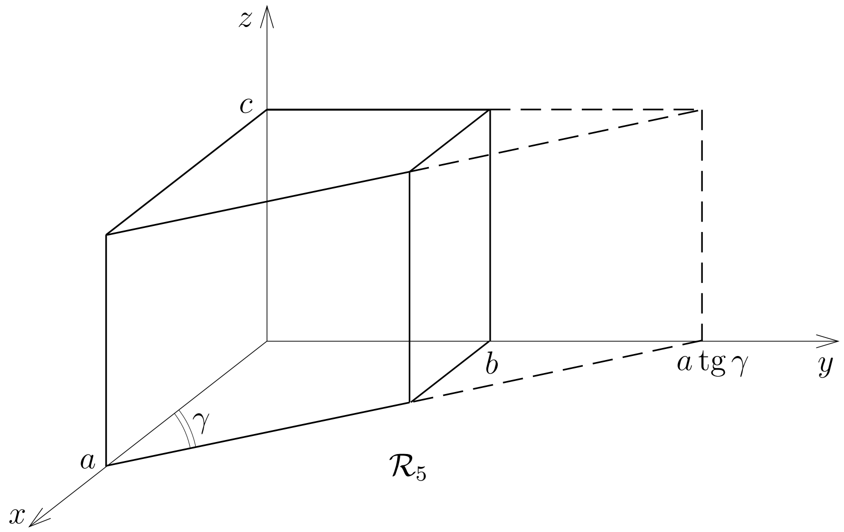
2) In the case when $\mathbf{K}$ is a sphere of radius $r<\min\Big(\frac{c}{2},\frac{ab}{a+b+\sqrt{a^2+b^2}}\Big)$, one obtains the probability

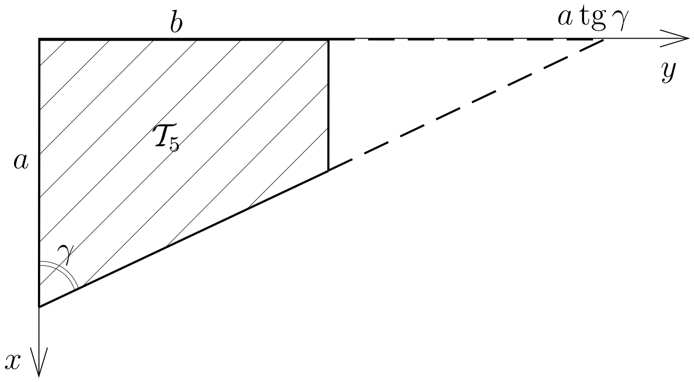
$$2\Big(\frac{1}{a}+\frac{1}{b}+\frac{1}{h}+\frac{1}{c}\Big)r-2\Big(\frac{1}{ab}+\frac{1}{bh}+\frac{1}{ha}\Big)r^2-4\Big(\frac{1}{ac}+\frac{1}{bc}+\frac{1}{hc}\Big)r^2$$
$$-\Big(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{h^2}\Big)r^2+4\Big(\frac{1}{abc}+\frac{1}{bhc}+\frac{1}{hac}\Big)r^3+2\Big(\frac{1}{a^2c}+\frac{1}{b^2c}+\frac{1}{h^2c}\Big)r^3\quad ,$$

which can be shown to be equivalent to the formula (1.23) in [3].

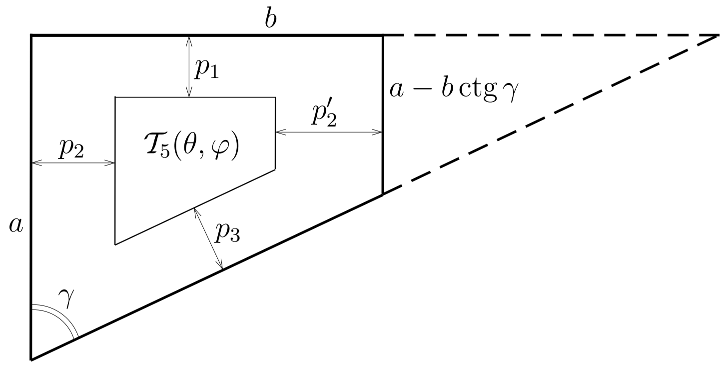
## 4. The lattice $\mathcal{R}_5$

The fundamental cell $\mathcal{F}_5$ of the lattice $\mathcal{R}_5$ is a right-angled prism whose base $\mathcal{T}_5$ is a right-angled trapezoid, as it is shown in the figure below.




The convex body $\mathbf{K}$ is small with respect to $\mathcal{R}_5$ if it satisfies the inequality $\mathrm{Diam}(\mathbf{K}) < \min(a - b\cot\gamma, b, c)$. In this case $\mathcal{F}_5(\theta,\varphi)$ is again a right-angled prism having the height $c - L(\theta,\varphi)$ (or in short form $c-L$) and the trapezoid $\mathcal{T}_5(\theta,\varphi)$ as a base. Using the notations from the previous section, we have again that the prism is completely determined by the distances $p_1, p_2, p_3$ and $p_2' = p(\pi - \theta_2, \varphi_2)$ :



If we denote $L := L(\theta,\varphi)$ and $L_2 := p_2 + p_2'$ we can write

$$\text{Area } \mathcal{T}_5(\theta,\varphi) = (b - p_2 - p_2')\Big(a - \frac{b}{2}\cot\gamma - p_1 - \frac{p_2 - p_2'}{2}\cot\gamma - \frac{p_3}{\sin\gamma}\Big) =$$

$$\text{Area } \mathcal{T}_5 - b\Big(p_1 + \frac{p_3}{\sin\gamma}\Big) + \frac{b}{2}(p_2 - p_2')\cot\gamma - \Big(a - \frac{b}{2}\cot\gamma\Big)L_2 + \frac{1}{2}\,(p_2^2 - p_2'^2)\cot\gamma$$
$$+L_2\Big(p_1 + \frac{p_3}{\sin\gamma}\Big)\ ,$$

$$\text{Vol } \mathcal{F}_5(\theta,\varphi) = (c - L)\text{ Area } \mathcal{T}_5(\theta,\varphi) = \text{Vol } \mathcal{F}_5 - bc\Big(p_1 + \frac{p_3}{\sin\gamma}\Big)$$
$$+\frac{bc}{2}(p_2' - p_2)\cot\gamma - \Big(a - \frac{b}{2}\cot\gamma\Big)cL_2 + \frac{c}{2}(p_2^2 - p_2'^2)\cot\gamma$$
$$-\Big(a - \frac{b}{2}\cot\gamma\Big)bL + b\Big(p_1 + \frac{p_3}{\sin\gamma}\Big)L - \frac{b}{2}(p_2' - p_2)L\cot\gamma + cL_2\Big(p_1 + \frac{p_3}{\sin\gamma}\Big)$$
$$+\Big(a - \frac{b}{2}\cot\gamma\Big)LL_2 - \frac{1}{2}(p_2^2 - p_2'^2)L\cot\gamma - LL_2\Big(p_1 + \frac{p_3}{\sin\gamma}\Big)\ .$$

Using now $(1')$ and the equalities

$$\int_0^{2\pi}\int_0^{\pi} p_2^i \sin\theta d\theta d\varphi = \int_0^{2\pi}\int_0^{\pi} p_2'^i \sin\theta d\theta d\varphi\ , \quad i = 1,2\ ,$$
$$\int_0^{2\pi}\int_0^{\pi} p_2^i L\sin\theta d\theta d\varphi = \int_0^{2\pi}\int_0^{\pi} p_2'^i L\sin\theta d\theta d\varphi\ , \quad i = 1,2$$

we obtain a proof of the following result.

**Theorem 4.** *The probability $p_{\mathbf{K},\mathcal{R}_5}$ that a uniformly distributed convex body of revolution $\boldsymbol{K}$, which is small with respect to $\mathcal{R}_5$, hits $\mathcal{R}_5$ is*

$$(5_5)\quad p_{\mathbf{K},\mathcal{R}_5} = \frac{1}{4\pi}\Bigg[\frac{1}{a - \frac{b}{2}\cot\gamma}\int_0^{2\pi}\int_0^{\pi}\Big(p_1 + \frac{p_3}{\sin\gamma}\Big)\sin\theta d\theta d\varphi + \frac{1}{b}\int_0^{2\pi}\int_0^{\pi} L_2\sin\theta d\theta d\varphi$$
$$+\frac{1}{c}\int_0^{2\pi}\int_0^{\pi} L\sin\theta d\theta d\varphi - \frac{1}{(a - \frac{b}{2}\cot\gamma)c}\int_0^{2\pi}\int_0^{\pi}\Big(p_1 + \frac{p_3}{\sin\gamma}\Big)L\sin\theta d\theta d\varphi$$
$$-\frac{1}{(a - \frac{b}{2}\cot\gamma)b}\int_0^{2\pi}\int_0^{\pi} L_2\Big(p_1 + \frac{p_3}{\sin\gamma}\Big)\sin\theta d\theta d\varphi - \frac{1}{bc}\int_0^{2\pi}\int_0^{\pi} LL_2\sin\theta d\theta d\varphi$$
$$+\frac{1}{(a - \frac{b}{2}\cot\gamma)bc}\int_0^{2\pi}\int_0^{\pi} LL_2\Big(p_1 + \frac{p_3}{\sin\gamma}\Big)\sin\theta d\theta d\varphi\Bigg]\ .$$

**Remarks.** 1) In the case $\mathbf{K}$ is a sphere of radius $r$, the conditions for $\mathbf{K}$ to be small with respect to $\mathcal{R}_5$ can be weakened; the upper bound $a - b\cot\gamma$ can be replaced by the larger number $\dfrac{2a - b\cot\gamma}{1 + \tan\frac{\gamma}{2}}$, and the condition in the theorem becomes

$$2r < \min\left(2\,\frac{a - b\cot\gamma}{1 + \tan\frac{\gamma}{2}}\,, b, c\right)\ .$$

From ($5_5$) we obtain

$$p_{\mathbf{K},\mathcal{R}_5} = \frac{1+\frac{1}{\sin\gamma}}{a-\frac{b}{2}\cot\gamma}\, r + \frac{2r}{b} + \frac{2r}{c} - 2\, \frac{1+\frac{1}{\sin\gamma}}{(a-\frac{b}{2}\cot\gamma)b}\, r^2$$
$$-2\, \frac{1+\frac{1}{\sin\gamma}}{(a-\frac{b}{2}\cot\gamma)c}\, r^2 - 4\, \frac{r^2}{bc} + 4\, \frac{1+\frac{1}{\sin\gamma}}{(a-\frac{b}{2}\cot\gamma)bc}\, r^3 \quad .$$

The same result follows from the formula (1.24) from [3] after some manipulations.

2) If $\mathbf{K}$ is a needle of length $l < \min\ (a - b\cot\gamma, b, c)$ then one can use ($5_5$) to deduce the formula (1.18) in [3], however some integrals are to be computed for this purpose.

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