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Osculating Plane Preserving Diffeomorphisms

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For someone familiar with the notion of self-parallel group for an immersion into euclidean space [3], [6] it is only natural to wonder what happens if, in the case of space curves, normal planes are replaced by, say, osculating planes. We give here a necessary and sufficient condition for the non-triviality of the osculating group of a simple space curve. This is, in the new situation, the group corresponding to the self-parallel group.

Problems of a similar nature have also been considered in [1] and [2].

1.

In what follows X will stand for R or S^1 and we will be dealing with smooth space curves, that is, C^∞ immersions $f : X \to R^3$. In the case $X = S^1$, we will write f for both $f : S^1 \to R^3$ and $f \circ \exp$, with $\exp(t) = (\cos 2\pi t, \sin 2\pi t)$. It will be clear from the context which one we are considering.

The curvature k_f of f is assumed not to vanish and (T_f, N_f, B_f) will denote the Frenet-Serret frame. Also we do not assume parametrization by arc-length and denote by v_f the velocity of f.

Definition 1. *The osculating group $O(f)$ of $f : X \to R^3$ is the subgroup of Diff (X) formed by the diffeomorphisms $\delta : X \to X$ such that, for $x \in X$, the osculating planes of f at x and $\delta(x)$ coincide.*

If f is a plane curve then $O(f)$ is Diff (X) precisely.

Proposition 1. *Let $f : X \to R^3$ be a smooth curve with non-vanishing curvature and torsion. Then $O(f)$ is*

a) *cyclic of finite order if* $X = S^1$,
b) *trivial or not finite if* $X = R$.

Proof. The proof is almost a duplicate of proofs given in [1], [2]. It is only included for completeness.

Denote by A_2^3 the open Grassmannian of affine planes in R^3 and define $\widetilde{O} : X \to A_2^3$, where $\widetilde{O}(x)$ is the osculating plane at x. Since we are assuming non-vanishing torsion $\widetilde{O}$ is an immersion. This fact implies that the action $\phi : O(f) \times X \to X$, with $\phi(\delta, x) = \delta(x)$, is properly discontinuous.

In fact let $\delta \in O(f)$ and suppose that $x \in X$ is such that $\delta(x) = x$. Since $\widetilde{O}$ is an immersion there is an open neighbourhood U of x such that $\widetilde{O} \mid U$ is injective. Then, for $y \in U \cap \delta^{-1}(U)$, $\delta(y) = y$ because $\widetilde{O}(\delta(y)) = \widetilde{O}(y)$. Therefore the fixed point set Δ of δ is open. Since Δ is also closed it follows that either δ has no fixed points or is the identity. Consequently the action of $O(f)$ on X is free.

Furthermore if $U \cap \delta(U) \neq \emptyset$ then δ is the identity and $O(f)$ acts in a properly discontinuous way. Hence the projection $p : X \to X/O(f)$ is a covering projection and $\pi_1(X/O(f), p(x))/p_*(\pi_1(X, x)) \approx O(f)$ [4].

If $X = S^1$ then $X/O(f)$ is diffeomorphic to S^1 and it follows that $O(f)$ is cyclic of finite order.

Assume now that $X = R$ and that $O(f)$ is finite. Then $X/O(f)$ is either R or S^1. Since $O(f) \approx \pi_1(X/B(f))$ it follows that it must be trivial. □

We will also use the *tangent group* $T(f)$ formed by the diffeomorphisms $\delta : X \to X$ such that, for $x \in X$, the tangent lines of f at x and $\delta(x)$ coincide. As above one can show that if the curvature k_f never vanishes the natural action of $T(f)$ on X is properly discontinuous.

2. Non-vanishing torsion

We start by recalling that a *simple point* for f is a point $y \in X$ such that $f^{-1}(f(y)) = \{y\}$.

Proposition 1. *Let* $f : X \to R^3$ *be a smooth curve with non-vanishing curvature. If* f *has a simple point then* $T(f)$ *is trivial.*

Proof. Let $\delta \in T(f)$. Then, for $x \in X$,

$$< f(x) - f(\delta(x)) \mid N_f(x) > = < f(x) - f(\delta(x)) \mid B_f(x) > = 0.$$

It then follows that $k_f(x)\ v_f(x) < f(x) - f(\delta(x)) \mid T_f(x) > = 0$. Consequently $f(x) - f(\delta(x)) = 0$, for $x \in X$.

If $y \in X$ is a simple point then $y = \delta(y)$ and, since the action of $T(f)$ is properly discontinuous, $\delta = id_X$. □

Proposition 2. *Let* $f : X \to R^3$ *be a smooth curve with non-vanishing curvature and torsion. If* f *has a simple point then* $O(f)$ *is trivial.*

Proof. Let $\delta \in O(f)$. From

$$f(x) - f(\delta(x)) = \alpha(x)\ T_f(x) + \beta(x)\ N_f(x)$$

one can conclude, by differentiation, that $\beta(x) = 0$, for $x \in X$. That is, $f(\delta(x))$ belongs to the tangent line of f at x.

Since $B_f(x) = \pm\ B_f(\delta(x))$, for $x \in X$, we can conclude that also $T_f(x) = \pm\ T_f(\delta(x))$, for $x \in X$. Therefore $\delta \in T(f)$. By Proposition 1, $\delta = id_X$ and $O(f)$ is trivial. □

3. Plane arcs

From now on we will assume that τ_f vanishes but that the curve is not plane.

Lemma 1. *Let $f : X \to R^3$ be a smooth, simple curve with non-vanishing curvature. If x_0 is a point such that $\tau_f(x_0) \neq 0$ and $\delta \in O(f)$ then $\delta(x_0) = x_0$.*

Proof. There is an open interval I containing x_0 where τ_f does not vanish. Then, for $x \in I$, $f(\delta(x)) = f(x) + \alpha(x)\ T_f(x)$.

Since $B_f(x) = \pm\ B_f(\delta(x))$, for $x \in X$, we also have $T_f(x) = \pm\ T_f(\delta(x))$. By differentiation, $\alpha(x) = 0$, for $x \in I$, and, due to the injectivity of f, $\delta \mid I = id_I$. □

Proposition 1. *Let $f : X \to R^3$ be a smooth, simple curve with non-vanishing curvature and such that τ_f vanishes but not everywhere. Then $O(f)$ is non-trivial if and only if f has a plane arc.*

Proof. Assume that $f : [a, b] \to R^3$ is a plane arc for f. Without loss of generality we assume $0 < a < b < 1$.

Let δ be a diffeomorphism of $[a, b]$ such that $\delta(a) = a$, $\delta(b) = b$, $\delta'(a) = \delta'(b) = 1$ and $\delta^{(k)}(a) = \delta^{(k)}(b) = 0$, for $k \geq 2$. Any such δ can be extended to a diffeomorphism of R by letting the extension be the identity outside $[a, b]$ if $X = R$ or, in the case $X = S^1$, by letting the extension $\overline{\delta}$ be the identity in $[0, 1] \setminus [a, b]$ and satisfy $\overline{\delta}(x + 1) = \overline{\delta}(x) + 1$. The resulting diffeomorphism or the diffeomorphism that it induces for S^1, in the case $X = S^1$, is then an element of $O(f)$.

Assume now that there are no plane arcs for f and that $\delta \in O(f)$. If x_0 is such that $\tau_f(x_0) = 0$ then x_0 belongs to the topological closure of $A = \{x \in X \mid \tau_f(x) \neq 0\}$. Therefore there exists a sequence $(x_n), x_n \in A$, which converges to x_0. The sequence $(\delta(x_n))$ converges to $\delta(x_0)$. Since by Lemma 1 $\delta(x_n) = x_n, n \in N$, it follows that $\delta(x_0) = x_0$. Using Lemma 1 again, δ must be id_X. □

Examples of curves with plane arcs can be constructed using convenient bump functions [5].

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