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Two-distance Preserving Functions

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Abstract. This paper provides a proof the following result: Suppose $n \geq 2$, $0 < c < s$, $0 < c/s < \frac{\sqrt{5}-1}{2}$, and $f\colon E^n \to E^m$ is any function such that for all $p, q \in E^n$, the following two properties hold: (1) If $|p-q| = c$, then $|f(p)-f(q)| \leq c$. (2) If $|p-q| = s$, then $|f(p)-f(q)| \geq s$. Then f is congruence. This result originally proved Bezdek and Connelly [2]. However Bezdek and Connelly's result depends on a result of Rado et al. [6], where the similar result holds but for $\frac{1}{\sqrt{3}}$ replacing $\frac{\sqrt{5}-1}{2}$. Rado's proof is long and time consuming to verify. In this paper, we provide our own separate proof, using the mathematical software, Maple, which is independent of Rado's proof. Our result is shorter, more systematic and, we believe, easier to verify.

1. Introduction

Let E^n be a real Euclidean space of dimension n and let $f\colon E^n \to E^m$ be a function from E^n to E^m. A function f is *α-distance preserving* if:

For all p, q elements of E^n , $|p-q| = \alpha$ implies $|f(p) - f(q)| = \alpha$.

We said that a function f is a *congruence* if all distances are preserved, i.e.:

For all p, q element of E^n , $|p-q| = |f(p) - f(q)|$.

For $2 \leq n = m < \infty$, F. S. Beckman and D. A. Quarles [1] proved in 1953 that any unit distance preserving function $f\colon E^n \to E^m$ is a congruence. In 1985, Dekster [4] showed that there is a function $f\colon E^2 \to E^6$ that is a distance 1 preserving function but is not congruence. A problem is then what properties of a function $f\colon E^n \to E^m$ imply that f is a congruence. In 1986, F. Rado, D. Andreescu and E. Vâlcan proved the following interesting result:

[1] This paper is the author's senior thesis at Cornell University

Theorem 1. [6] *Suppose* $n \geq 2$, $0 < c < s$, $0 < c/s < \frac{1}{\sqrt{3}}$, *and* $f\colon E^n \to E^m$ *is such that for all* p, $q \in E^n$, *the following properties hold:*

(1) *If* $|p - q| = c$, *then* $|f(p) - f(q)| \leq c$.
(2) *If* $|p - q| = s$, *then* $|f(p) - f(q)| \geq s$.

Then f *is a congruence.*

Even though Rado's result is excellent and explicit, his proof is lengthy and difficult, with many cases. In fact, H. Lenz stated in his review [5] that "the proof is not easy. The reviewer tried in vain to simplify it." In 1999, K. Bezdek and R. Connelly [2] improved the constant $\frac{1}{\sqrt{3}}$ in Rado's Theorem to the golden ratio $\frac{\sqrt{5}-1}{2}$.
Let (as defined in Bezdek and Connelly's paper [2])

$$F_k(c, s) = \{f\colon E^n \to E^m|\ (1) \text{ and } (2) \text{ in Rado's Theorem hold for all } n \geq k\},$$
$$X_k = \{r \in E^2|\ \text{if } c/s = r, \text{ then } f \in F_k(c, s) \text{ is a congruence}\}.$$

It is quite easy to show that $X_n \subset X_{n+1}$ for $n = 2, 3, \ldots$ So, Rado's theorem [6] says that $(0, \frac{1}{\sqrt{3}}] \subset X_2$. And the theorem of Bezdek and Connelly [1] states that $(0, \frac{\sqrt{5}-1}{2}) \subset X_2$. Bezdek and Connelly's techniques were easy to verify but nevertheless their theorem was still based on Rado's result. In this paper, we will use the theory of tensegrity structures and techniques that Bezdek and Connelly used. Our aim is to prove that $(0, \frac{\sqrt{5}-1}{2}) \subset X_2$ without using the result of Rado et al.'s Theorem. The most difficult part of our proof is to show that the interval $[1/3, 1/2]$ is in X_2 . This will replace Rado's proof of his theorem. After successfully showing that $[1/2, 1/3]$, we will use the same techniques which Bezdek and Connelly [1] used in their paper to complete the proof. Compared to Rado et al.'s original proof, our proof is more systematic, easy to verify with routine calculations, and it involves fewer cases. We only need to check a finite number of steps in all cases, while Rado had some situations that involved an unbounded number iterations of certain functions. In fact, we used mathematical software, Maple, to help us find the right combination of tensegrity functions.

2. Tensegrity techniques

In the fall of 1948, Kenneth Snelson constructed an original sculpture. Snelson showed this amazing sculpture to his mentor R. Buckminster Fuller. Fuller adapted Snelson's invention as a centerpiece of his system of synergetics. He named them *tensegrities.* Snelson's sculpture consisted of two different types of elements, cables and struts. A cable has the property that it holds two vertices close together. Two vertices joined by cable can be as close as desired but they may never be further apart more than the length of the cable. Struts, on the other hand, keep two vertices apart. Two vertices that joined by a strut can be arbitrarily far apart but they can never be closer than the length of the strut.
Consider any fixed function $f\colon E^n \to E^m$, where $n \geq 2$. Define:

C_f as a set of cable lengths for f, where

$$C_f = \{c \in E^1|\ \text{for all } p, q \in E^n, \text{ where } |p - q| = c \text{ implies } |f(p) - f(q)| \leq c\},$$

S_f as a set of strut lengths for f where

$$S_f = \{s \in E^1 | \text{ for all } p, q \in E^n, \text{ where } |p-q| = s \text{ implies } |f(p) - f(q)| \geq s\}.$$

Suppose that we have a set of positive real numbers $C \subset C_f$ and another set of positive real numbers $S \subset S_f$. If we have some other real number c is in C_f or s is in S_f , then we call c an implied cable length for f, or we call s an implied strut length for f respectively.

Lemma 1. *If c_1 and c_2 are cable lengths for f, then*

$$cable1(c_1,c_2) = c_1 + c_2$$

is an implied cable.

Lemma 2. *If s is strut length and c is cable length and $s > c$, then*

$$strut1(c,s) = s - c$$

is an implied strut.

This first two lemmas are quite simple. In fact, they come from the same tensegrity configuration (Figure 1). The proofs of these lemmas are also in Rado's original paper [6].

Figure 1. Tensegrity of Lemma 1 and Lemma 2

Note that all three points in Figure 1 lie on a same line.

Lemma 3. *Let c_1, c_2, s be positive real number such that $c_1 \leq c_2$, $c_1^2 + s^2 \leq c_2^2$, where c_1 , c_2 are cable lengths and s is strut length, $n \geq 2$, where $f\colon E^n \to E^m$ is a function. Then*

$$cable2 = (c_1, c_2, s) = 2\sqrt{c_2^2 - \left(\frac{c_2^2 + s^2 - c_1^2}{2s}\right)^2}$$

is an implied cable length.

Lemma 3 comes from the tensegrity configuration in Figure 2. The conditions $c_1 \leq c2$, $c_1^2 + s^2 \leq c_2^2$ are needed to insure that tensegrity configuration in Figure 2 is convex. The proof of Lemma 3 can be found also in Rado et al.'s original paper [6]. These first three lemmas were the only lemmas that Rado used in his proof. The next lemma, which Rado never used, is also very useful. This next lemma helped Bezdek and Connelly [2] improve the ratio c/s.

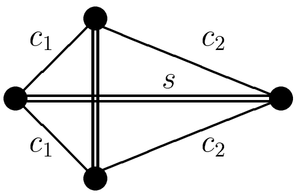

Figure 2. Tensegrity of Lemma 3

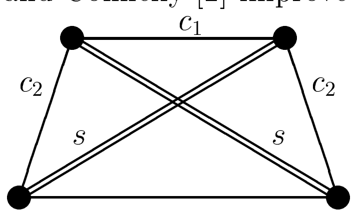

Figure 3. Tensegrity of Lemma 4

Lemma 4. *Let c_1, c_2, s be positive real number such that $c_1 + c_2 \geq s$, $c_1 \leq s$, where c_1 , c_2 are cable lengths and s is strut length, $n \geq 2$, $f\colon E^n \to E^m$ is a function. Then*

$$strut2 = (c_1, c_2, s) = \frac{s^2 - c_2^2}{c_1}$$

is an implied strut length.

The inequality conditions in Lemma 4 are needed to insure a convex quadriateral tensegrity in Figure 3. The proof of this lemma can be found in Bezdek and Connelly's paper [1], page 190. Note that the functions in Lemma 3 and Lemma 4 are continuous and monotone. These facts will help in the proof of our theorem in next section.

3. The proof of Theorem 1 in dimension 2

Theorem 2. $(0, \frac{\sqrt{5}-1}{2}) \in X_2$

In order to prove Theorem 2, we need to the following lemma.

Lemma 5. $[1/3, 1/2] \subset X_2$.

Proof of Lemma 5:
Step 1: We will find combinations coming from Lemmas 1–4 to map the interval [1/3,1/2] into the interval (0,1/3]. For a clearer picture, you can see the graphic version of this proof, the index numbers in the graphic version correspond to the index number in each case below.

In each case, the first line indicates the original interval or the domain of the function in the second line. The second line is the function that maps the interval in the first line. The last line is the image of the function or where the interval in the first line is mapped.

Our goal is to map the whole interval [1/3,1/2] into the *good interval* (0,1/3]. In each case, when we can find the right function that maps the set $A \subset [1/3, 1/2]$ into the good interval, we then are able to expand the *good interval* by adding the set A to the *good interval.* We will reach our goal when the *good interval* covers [1/3,1/2].

1. c_0 :=(1/3,0.353];
   $cable2(c_0,2^*c_0,1)$;
   (0,0.3261033180]
2. c_0 :=[0.34,0.3637];
   $cable2(c_0,c_0,1\text{-}c_0)$;
   $[0.1637070554, 0.3524671190]$
3. c_0 :=[0.3624,0.368]
   $cable2(c_0,cable2(c_0,2^*c_0,1),strut2(c_0,\ c_0,1\text{-}c_0))$
   $[0.03107088670, 0.3590937282]$
4. c_0 :=[0.3661, 0.3762];
   $cable2(c_0,c_0,strut2(c_0,c_0,1\text{-}c_0))$;
   $[0.03214314234, 0.3645962792]$

5. c_0 :=[0.3756,0.3782]
$cable2(c_0,cable2(c_0,c_0,1\text{-}c_0),strut2(c_0,\ c_0,strut2(c_0,c_0,1\text{-}c_0)))$;
$[0.03144332680, 0.3701045814]$

6. c_0 :=[0.378,0.3881]
$cable2(c_0,2{*}c_0,strut2(2{*}c_0,c_0,1))$;
$[0.02238863998, 0.3718300122]$

7. c_0 :=(0.5,0.536]
$cable2(c_0,c_0,1)$;
$(0, 0.3862434466]$

8. c_0 :=[0.5347,0.5383]
$cable2(c_0,cable2(c_0,cable2(c_0,2{*}c_0,1),\ strut2(c_0,c_0,1)),strut2(c_0,c_0,1))$;
$[0.02684482072, 0.3878249358]$

9. c_0 :=[0.549,0.5615]
$cable2(c_0,cable2(c_0,c_0,1),1)$;
$[0.06951931530, 0.3871562036]$

10. c_0 :=[0.561,0.5673] $cable2(cable2(c_0,c_0,1),cable2(c_0,c_0,1),1)$;
$[0.1885099456, 0.3863529992]$

11. c_0 :=[0.388,0.3914]
$cable2(c_0,2{*}c_0,1)$;
$[0.5490615054, 0.5663250798]$

12. c_0 :=[0.3911,0.393]
$cable2(c_0,c_0,strut2(c_0,c_0,1\text{-}c_0))$;
$[0.5492808514, 0.5668190916]$

13. c_0 :=[0.3929,0.3975]
$cable2(c_0,cable2(c_0,2{*}c_0,strut2(c_0,\ 2{*}c_0,1)),1)$
$[0.03983024480, 0.3821907320]$

14. c_0 :=[0.394,0.402]
$cable2(c_0,c_0,1\text{-}c_0)$;
$[0.5036943518, 0.5249790472]$

15. c_0 :=[0.3987,0.413]
$cable2(c_0,cable2(c_0,2{*}c_0,1),1)$
$[0.03183759412, 0.3926189032]$

16. c_0 :=[0.4083,0.4174]
$cable2(c_0,c_0,strut2(c_0,2{*}c_0,1))$;
$[0.03182726504, 0.4117628914]$

17. c_0 :=[0.415,0.433]
$cable2(c_0,cable2(c_0,c_0,1\text{-}c_0),1)$;
$[0.08583238782, 0.4169431664]$

18. c_0 :=[0.43,0.44]
    $cable2(c_0,cable2(c_0,cable2(c_0,2{*}c_0,1),1),1)$;
    $[0.02900380664, 0.4203027528]$
19. c_0 :=[0.438,0.4421]
    $cable2(c_0,cable2(c_0,cable2(c_0,cable2(c_0,\ 2{*}c_0,1),strut2(2{*}c_0,c_0,1)),1),1)$;
    $[0.08994337774, 0.4369543904]$
20. c_0 :=[0.5378,0.5465]
    $cable2(c_0,c_0,1)$;
    $[0.3961254346, 0.4411904350]$
21. c_0 := [0.4945,0.5]
    $cable2(c_0,cable2(c_0,cable2(c_0,c_0,\ strut2(2{*}c_0,c_0,1)),1),1)$
    $[0.02193974476, 0.4142651334]$
22. c_0 :=[0.442,0.449]
    $cable2(c_0,cable2(c_0,c_0,1\text{-}c_0),1)$;
    $[0.5054072868, 0.5639905626]$
23. c_0 := [0.448,0.4708]
    $cable2(c_0,c_0,strut2(2{*}c_0,c_0,1))$;
    $[0.08381268640, 0.4481168696]$
24. c_0 :=[0.4674,0.473233]
    $cable2(cable2(c_0,c_0,strut2(2{*}c_0,c_0,1)),\ cable2(c_0,c_0,strut2(2{*}c_0,c_0,1)),\ strut2(2{*}c_0,c_0,1))$;
    $[0.0229503464, 0.4707850134]$
25. c_0 :=[0.47323,0.4766]
    $cable2(c_0,c_0,strut2(c_0,cable2(c_0,\ c_0,1\text{-}c_0),1))$;
    $[0.4951843826, 0.5665053568]$
26. c_0 :=[0.4758,0.4853]
    $cable2(c_0,c_0,strut2(2{*}c_0,c_0,1))$;
    $[0.4946067364, 0.5671743442]$
27. c_0 := [0.4791,0.4932]
    $cable2(c_0,cable2(c_0,c_0,strut2(2{*}c_0,\ c_0,1)),1)$
    $[0.02300919816, 0.4851621654]$
28. c_0 := [0.492,0.495]
    $cable2(c_0,cable2(c_0,c_0,strut2(c_0,cable2(c_0,c_0,\ 1\text{-}c_0),1)),$
    $strut2(c_0,cable2(c_0,c_0,strut2(2{*}c_0,c_0,\ 1)),1))$;
    $[0.1537565895, 0.4833622674]$

Step 2: After proving that $[1/3, 1/2]$ can be mapped into $(0, 1/3]$ but using only tensegrity functions in Lemma 1 to 4 ,we will show that for all ϵ bigger than zero, there exists c' an implied cable in the interval $(0,\varepsilon)$ by induction in order to complete the proof for the Lemma 5. This part of the proof is essentially a restatement of the corresponding lemma in Rado et al. [6].

Induction Statement $P(n)$: For $n \geq 2$, $c =$ cable length , $s =$ strut length, where $c \in (\frac{s}{n+1}, \frac{s}{n})$ then there exists an implied cable length $x \in (0, \frac{s}{n+1})$.

1. Basic case $n = 2 : P(2)$ is true by assumption.
2. Assume $P(n-1)$ is true (induction hypothesis).
3. Prove that P(n) is true:
   $c \in (\frac{s}{n+1}, \frac{s}{n})$, for there exists a strut $s - c \in (s(1-\frac{1}{n}), s(1-\frac{1}{n+1}))$.
   But $\frac{c}{s-c} = \frac{s+c-s}{s-c} = \frac{s}{s-c} - 1 \in (\frac{1}{1-\frac{1}{n+1}} - 1, \frac{1}{1-\frac{1}{n}} - 1) = (\frac{n+1}{n+1-1} - 1, \frac{n}{n-1} - 1) = (\frac{1}{n}, \frac{1}{n-1})$.
   So $c \in (\frac{s-c}{n}, \frac{s-c}{n-1})$.
   From induction hypothesis $P(n-1)$ hence
   $c' \in (0, \frac{s-c}{n}) \subset (0, s\frac{(1-\frac{1}{n+1})}{n}) = (0, \frac{s}{n+1})$. □

Now we can move on and complete the proof of Theorem 2 by using the same techniques that Bezdek and Connelly used. First, let $\tau = \frac{\sqrt{5}+1}{2}$.

Lemma 6. *Let* $g \colon E^1 \to E^1$ *be defined by* $g(t) = t^2 - 1$, *with domain the interval* $[1, 2]$, *and range the interval* $[0, 3]$. *Then for any to* $t_0 \in (\tau, 2]$ *some finite iterate of to by* g *lies in the interval* $[2, 3]$.

It is easy to see that g is strictly monotone increasing and continuous throughout its domain. The unique fixed point of the function g is obviously the solution of the quadratic equation $t = t^2 - 1$ which is the golden ratio τ. The function $g^{-1}(t) = \sqrt{t+1}$ will send the interval [2,3] to [$\sqrt{3}$,2] then send [$\sqrt{3}$,2] to [$\sqrt{\sqrt{3}+1}, \sqrt{3}$] and repeat this process infinitely many times. Finally, the union of all these intervals will be (τ,3]. The completed proof of Lemma 6 can be found in Bezdek and Connelly's paper [2].

Proof of Theorem 2 (from Bezdek and Connelly's paper[2]).

Suppose that $0 < c < s$ and c/s is the element of $(0, \tau^{-1})$ and let f be the function on E^n to E^n. Without loss of generality, we will assume that $c = 1$.

Case 1: If c/s is less than 1/2 then Lemma 5 shows that f is a congruence.

Case 2: If c/s is between 1/2 and τ^{-1} then s is element of interval (τ,2]. By Lemma 6, an infinite iteration of the tensegrity function *strut*2, in Lemma 4, eventually maps s to the interval [2,3]. Hence s is an implied strut. Therefore, f is a tensegrity function. □

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