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On a Certain Functional Identity in Prime Rings, II

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Abstract. Let $\mathcal{R}$ be a prime ring. It is shown that, under certain restrictions on $\mathrm{char}(\mathcal{R})$, $\mathcal{R}$ admits a functional identity $f(x)xf(x)\dots xf(x) = 0$, $x \in \mathcal{R}$, where $f : \mathcal{R} \to \mathcal{R}$ is a nonzero additive map, if and only if its central closure $\mathcal{S}$ contains an idempotent $e \neq 0, 1$ such that $e\mathcal{S}e = \mathcal{C}e$ where $\mathcal{C}$ is the extended centroid of $\mathcal{R}$.

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1. Introduction

By a functional identity on a ring $\mathcal{R}$ we mean, roughly speaking, an identical relation satisfied by elements in $\mathcal{R}$, which involves some maps of $\mathcal{R}$. The usual goal when treating a functional identity is to either describe the form of the maps appearing in the identity or, when this is not possible, to determine the structure of the ring admitting this identity. For a detailed account on the theory of functional identities and its applications we refer the reader to [4].

In contrast to thoroughly analyzed (see e.g. [1]) functional identities that involve expressions such as $f(x_1, \dots, x_{n-1})x_n$ or $x_n f(x_1, \dots, x_{n-1})$ (here f is a map of $\mathcal{R} \times \dots \times \mathcal{R}$ into $\mathcal{R}$), more general functional identities consisting of expressions as $g(x_1, \dots, x_{i-1})x_i h(x_{i+1}, \dots, x_n)$ are rather undiscovered. It seems that up till now the only work devoted to identities of such

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type is [5]. Its main result states that if $\mathcal{R}$ is a prime ring with $\mathrm{char}(\mathcal{R}) \neq 2$, then there exist nonzero additive maps $f, g : \mathcal{R} \to \mathcal{R}$ satisfying $f(x)xg(x) = 0$ for all $x \in \mathcal{R}$ if and only if the central closure $\mathcal{S}$ of $\mathcal{R}$ contains an idempotent $e \neq 0, 1$ such that $e\mathcal{S}e = \mathcal{C}e$ (thus, $\mathcal{S}$ is a primitive ring with nonzero socle and the associated division ring is a field). In this paper we consider a similar functional identity in more variables, however, involving one map only. Our main result is

Theorem 1.1. *Let $\mathcal{R}$ be a prime ring with extended centroid $\mathcal{C}$ and central closure $\mathcal{S}$. Let n be a positive integer and suppose that* $\mathrm{char}(\mathcal{R}) = 0$ *or* $\mathrm{char}(\mathcal{R}) > 4n - 2$. *Then there exists a nonzero additive map $f : \mathcal{R} \to \mathcal{R}$ satisfying*

$$(f(x)x)^n f(x) = 0 \quad \text{for all } x \in \mathcal{R} \tag{1}$$

if and only if $\mathcal{S}$ contains an idempotent $e \neq 0, 1$ such that $e\mathcal{S}e = \mathcal{C}e$.

We remark that the condition (1) may be regarded as a generalization of the condition appearing in the classical result, usually in the literature quoted as Levitzki's theorem (see e.g. [7, Lemma 1.1]). Namely, a possible way to state this result is the following one: If $\mathcal{R}$ is a (semi)prime ring and $a \in \mathcal{R}$ is such that $(ax)^n a = 0$ for all $x \in \mathcal{R}$ and some fixed n, then $a = 0$.

At several places the proof of Theorem 1.1 is similar to the one in [5], and moreover, some result from [5] will be used in the proof. Nevertheless, the proof in the present paper is considerably shorter and less complicated. This gives some hope that eventually one shall be able to investigate some more general functional identities.

2. Proof

The rest of the paper is basically devoted to the proof of Theorem 1.1 only. It has already been shown that the existence of an idempotent $e \neq 0, 1$ in $\mathcal{S}$ such that $e\mathcal{S}e = \mathcal{C}e$ yields the existence of a nonzero additive map $f : \mathcal{R} \to \mathcal{R}$ satisfying $f(x)xf(x) = 0$ for all $x \in \mathcal{R}$ [5, p. 3766]. Therefore, we only need to prove the only if part of Theorem 1.1. This proof is broken up into the series of steps. The first one is the reduction of our problem to the case when $\mathcal{R}$ satisfies a generalized polynomial identity (GPI). The reader is referred to the book of Beidar et al. [2] for the basic terminology and results of the theory of rings with GPIs. The fundamental theorem of this theory, due to Martindale, states that a prime ring $\mathcal{R}$ satisfies a GPI (shortly, $\mathcal{R}$ is a GPI ring) if and only if its central closure $\mathcal{S}$ contains an idempotent e such that $e\mathcal{S}e$ is a finite dimensional division algebra over the extended centroid $\mathcal{C}$ [2, Theorem 6.1.6]. The proof of the next lemma rests heavily on a recent work of the second author [6]. More precisely, we shall use the following result which is a special case of [6, Theorems 2.6 and 2.7].

Proposition 2.1. *Let $\mathcal{R}$ be a prime ring. Suppose there exist maps $g_i : \mathcal{R}^{n-1} = \mathcal{R} \times \ldots \times \mathcal{R} \to \mathcal{R}$, $i = 1, \ldots, n$, and a nonzero element $a \in \mathcal{R}$ such that one of the following two identities*

$$\sum_{i=1}^{n} g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) x_i a = 0 \quad \text{for all } x_1, \ldots, x_n \in \mathcal{R},$$

$$\sum_{i=1}^{n} ax_i g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = 0 \quad \textit{for all } x_1, \ldots, x_n \in \mathcal{R}$$

holds true. Then either each $g_i = 0$ *or* $\mathcal{R}$ *is a GPI ring.*

Lemma 2.2. *Let* $\mathcal{R}$ *be a prime ring. Suppose there exists a nonzero additive map* $f : \mathcal{R} \to \mathcal{R}$ *satisfying* (1). *If* $\operatorname{char}(\mathcal{R}) = 0$ *is or* $\operatorname{char}(\mathcal{R}) > 4n - 2$, *then* $\mathcal{R}$ *is a GPI ring.*

Proof. Assume on the contrary that $\mathcal{R}$ is not a GPI ring. Also, we may assume that n is the smallest positive integer for which there is a nonzero map satisyfing the functional identity of type (1).

It will be useful to use (1) in its linearized form, that is,

$$\sum_{\sigma \in \mathcal{S}_{2n+1}} f(x_{\sigma(1)}) x_{\sigma(2)} f(x_{\sigma(3)}) \ldots x_{\sigma(2n)} f(x_{\sigma(2n+1)}) = 0 \tag{2}$$

for all $x_1, \ldots, x_{2n+1} \in \mathcal{R}$.

Claim 1. *There exist* $a, b \in \mathcal{R}$ *such that* $af(y) \neq 0$ *and* $f(y)b \neq 0$ *for some* $y \in \mathcal{R}$ *and* $af(x)b = 0$ *for all* $x \in \mathcal{R}$.

Proof. Set $x_1 = \ldots = x_{2n} = x$, $x_{2n+1} = y$ in (2) and multiply the identity obtained from the right by $(xf(x))^n$. Applying the assumption on $\operatorname{char}(\mathcal{R})$ it follows that $(f(x)x)^n f(y)(xf(x))^n = 0$ for all $x, y \in \mathcal{R}$. Therefore, it suffices to show that there is $x \in \mathcal{R}$ such that $(f(x)x)^n f(y) \neq 0$ and $f(y)(xf(x))^n \neq 0$ for some $y \in \mathcal{R}$. If this were not true we would have $f(y)(xf(x))^n z (f(x)x)^n f(y) = 0$ for all $x, y, z \in \mathcal{R}$. Fixing $y \in \mathcal{R}$ such that $f(y) \neq 0$ and linearizing the last identity we get

$$\sum_{\sigma \in \mathcal{S}_{4n}} f(y) x_{\sigma(1)} f(x_{\sigma(2)}) \ldots f(x_{\sigma(2n)}) z f(x_{\sigma(2n+1)}) \ldots f(x_{\sigma(4n-1)}) x_{\sigma(4n)} f(y) = 0$$

for all $x_1, \ldots, x_{4n} \in \mathcal{R}$. Now applying Proposition 2.1 twice, each time with $f(y)$ playing the role of a, however, once appearing on the left and once on the right, it follows that

$$\sum_{\sigma \in \mathcal{S}_{4n-2}} f(x_{\sigma(1)}) \ldots f(x_{\sigma(2n-1)}) z f(x_{\sigma(2n)}) \ldots f(x_{\sigma(4n-2)}) = 0$$

for all $x_1, \ldots, x_{4n-2} \in \mathcal{R}$. Again using the assumption on $\operatorname{char}(\mathcal{R})$, this time on the whole, it follows that $(f(x)x)^{n-1} f(x) z (f(x)x)^{n-1} f(x) = 0$ for all $x, z \in \mathcal{R}$. The primeness of $\mathcal{R}$ yields $(f(x)x)^{n-1} f(x) = 0$ for all $x \in \mathcal{R}$. However, this contradicts our assumption and so Claim 1 is proved.

Claim 2. $af(bx) = 0$ *for all* $x \in \mathcal{R}$.

Proof. Setting $x_1 = \ldots = x_{n+1} = bx$, $x_{n+2} = \ldots = x_{2n+1} = ya$ in (2), and then multiplying from the left by a, we get, using $af(\mathcal{R})b = 0$, that $af(bx)(yaf(bx))^n = 0$ for all $x, y \in \mathcal{R}$. The Levitzki's theorem mentioned in the introduction then yields $af(bx) = 0$.

Claim 3. $f(bx)b = 0$ *for all* $x \in \mathcal{R}$.

Proof. We want to show that the map $x \mapsto f(bx)b$ is zero. Suppose this were not true. Then we would have, according to our assumption, $([f(bx)b]x)^{n-1} f(bx) \neq 0$ for some $x \in \mathcal{R}$. Set

$x_1 = \ldots = x_{2n-1} = bx$ and $x_{2n} = x_{2n+1} = y$ in (2) and then multiply from the left by a. Using Claims 1 and 2 we see that the identity so obtained reduces to $af(y)y(f(bx)bx)^{n-1}f(bx) = 0$ for all $y \in \mathcal{R}$. Linearizing and then using Proposition 2.1 it follows that $af(y) = 0$ for all $y \in \mathcal{R}$. However, this contradicts Claim 1.

Claim 4. *$f(bxa) = 0$ for all $x \in \mathcal{R}$.*

Proof. Claims 2 and 3 imply, in particular, that $f(bxa)bxa = bxaf(bxa) = 0$ for all $x \in \mathcal{R}$. Therefore, setting $x_1 = \ldots = x_{n+1} = bxa$, $x_{n+2} = \ldots = x_{2n+1} = y$ in (2) we arrive at $(f(bxa)y)^n f(bxa) = 0$ for all $x, y \in \mathcal{R}$. Levitzki's theorem yields $f(bxa) = 0$.

Claim 5. *$af(y)yf(y)b = 0$ for all $y \in \mathcal{R}$.*

Proof. Since the assumption on char$(\mathcal{R})$ will not be used anymore, there is no loss of generality in assuming that n is an odd number > 1 (otherwise we multiply (1) from the left by $f(x)x$). So, let $n = 2k+1$, $k \geq 1$. Set $x_1 = \ldots = x_k = bxa$ and $x_{k+1} = \ldots = x_{4k+3} = y$ in (2) and then multiply from the left by a and from the right by bx. Using $f(b\mathcal{R}a) = 0$ and $af(\mathcal{R})b = 0$ we see that the identity obtained reduces to $((af(y)yf(y)b)x)^{k+1} = 0$ for all $x, y \in \mathcal{R}$. Levitzki's theorem therefore implies the desired conclusion.

We have thereby reduced the functional identity under consideration to the one treated in [5]. Applying [5, Lemma 2.4] we get that either $af(y) = 0$ or $f(y)b = 0$ for each $y \in \mathcal{R}$ which, however, contradicts Claim 1. The proof of the lemma is therefore complete.

We continue by treating our functional identity in a rather special setting, to which the general case will be reduced in the proof below.

Lemma 2.3. *Let $\mathcal{D}$ be a domain and Δ be its subring. Suppose there exists a nonzero additive map $f : M_m(\Delta) \to M_m(\mathcal{D})$, $m \geq 1$, and a positive integer n such that $(f(A)A)^n f(A) = 0$ for all $A \in M_m(\Delta)$. Then Δ is commutative.*

Proof. First of all, it is clear that $m > 1$. Let us assume that Δ is noncommutative. Our goal is to show that this contradicts the assumption $f \neq 0$.

Following [5] we write $f : M_m(\Delta) \to M_m(\mathcal{D})$ in the matrix form $f = (f_{ij})$, where $f_{ij} : M_m(\Delta) \to \mathcal{D}$ are additive maps. Moreover, each $f_{ij} : \mathcal{M}_m(\Delta) \to \mathcal{D}$ can be presented as $f_{ij}(A) = \sum_{k=1}^m \sum_{l=1}^m f_{ij}^{kl}(a_{kl})$, where $f_{ij}^{kl} : \Delta \to \mathcal{D}$ are additive maps (here a_{kl} denotes the entry of the matrix A). By $a_{ij}E_{ij}$ we denote the matrix whose entry in position (i,j) is a_{ij} and all other entries are zero. Letting $A = a_{ji}E_{ji}$ in $(f(A)A)^{n+1} = 0$ we get, by considering the position (i,i), that $(f_{ij}^{ji}(a_{ji})a_{ji})^{n+1} = 0$, which gives $f_{ij}^{ji} = 0$ for all i, j. Using this we see that letting $A = a_{ki}E_{ki} + a_{ji}E_{ji}$ in $(f(A)A)^{n+1} = 0$ and again considering the position (i,i), we get $(f_{ij}^{ki}(a_{ki})a_{ji} + f_{ik}^{ji}(a_{ji})a_{ki})^{n+1} = 0$, and hence $f_{ij}^{ki}(a_{ki})a_{ji} + f_{ik}^{ji}(a_{ji})a_{ki} = 0$. But then, since Δ is assumed to be noncommutative, [5, Lemma 2.5] yields $f_{ij}^{ki} = 0$ for all i, j, k. In a similar fashion, by considering $A = a_{ik}E_{ik} + a_{ij}E_{ij}$ in $(Af(A))^{n+1} = 0$, we see that $f_{ji}^{ik} = 0$ for all i, j, k. Finally, lettting $A = a_{kl}E_{kl} + a_{ji}E_{ji}$, $l \neq i$, in $(f(A)A)^{n+1} = 0$ we get by considering the position (i,i) that $(f_{ij}^{kl}(a_{kl})a_{ji})^{n+1} = 0$, which gives $f_{ij}^{kl} = 0$ for all i, j, k, l. But this means that $f = 0$. The lemma is thereby proved.

Proof of Theorem 1.1. As mentioned above, we only need to prove the only if part of the theorem. By Lemma 2.2 $\mathcal{R}$ is a GPI ring and so the central closure $\mathcal{S}$ of $\mathcal{R}$ contains an idempotent e such that $\mathcal{D} = e\mathcal{S}e$ is a finite dimensional division algebra over its center $\mathcal{C}e$, where $\mathcal{C}$ is the extended centroid of $\mathcal{R}$. Moreover, $e \neq 1$ for otherwise $\mathcal{R}$ would be a domain which is clearly impossible. Our goal is to show that $\mathcal{D}$ is actually 1-dimensional over $\mathcal{C}$, that is, that $\mathcal{D}$ is commutative.

Let $\mathcal{H}$ be the socle of $\mathcal{S}$ and set $\mathcal{I} = \mathcal{H} \cap \mathcal{R}$. Since $\mathcal{H} \neq 0$ (in particular, $e \in \mathcal{H}$), it follows easily that $\mathcal{I}$ is a nonzero ideal of $\mathcal{R}$. Suppose $f(\mathcal{I}) = 0$. Pick $r \in \mathcal{R}$ such that $f(r) \neq 0$. Setting $x_1 = \ldots = x_n = x \in \mathcal{I}$ and $x_{n+1} = \ldots = x_{2n+1} = r$ in (2) we arrive at $(f(r)x)^n f(r) = 0$ for every $x \in \mathcal{I}$. But then Levitzki's theorem yields $f(r)\mathcal{I} = 0$ which in turn gives $f(r) = 0$, contrary to the assumption. Thus $f(\mathcal{I}) \neq 0$. This further implies that $f(x)x \neq 0$ for some $x \in \mathcal{I}$ by [3, Lemma 4.4]. According to Litoff's theorem [2, Theorem 4.3.11] there exists an idempotent $u \in \mathcal{H}$ such that $x, f(x)x \in \mathcal{B} = u\mathcal{S}u \cong M_m(\mathcal{D})$ for some $m \geq 1$. Set $\mathcal{R}' = \mathcal{R} \cap \mathcal{B}$ and define $g : \mathcal{R}' \to \mathcal{B}$ by $g(y) = uf(y)u$. Since $f(x)x \neq 0$ and $x, f(x)x \in \mathcal{B}$, it follows that $g(x) \neq 0$. Further, $(g(y)y)^n g(y) = 0$ for all $y \in \mathcal{R}'$, because $uyu = y$. Using [2, Theorem 4.3.7 (iii) and (viii)] we see that $\mathcal{B} = u\mathcal{S}u = u\mathcal{Q}_s(\mathcal{R})u$, where $\mathcal{Q}_s(\mathcal{R})$ is the symmetric Martindale ring of quotients of $\mathcal{R}$, and so [2, Proposition 2.3.14] implies that $\mathcal{R}'$ is a prime ring and $\mathcal{B} = \mathcal{Q}_s(\mathcal{R}')$. As $\mathcal{B} \cong M_m(\mathcal{D})$, $\mathcal{B}$ is a PI ring and so $\mathcal{R}'$ is a PI ring too. By Posner's Theorem [2, Theorem 6.1.11] $\mathcal{B}$ is the classical ring of quotients of $\mathcal{R}'$ and $\mathcal{R}'$ is a two sided Goldie ring. By the Faith-Utumi Theorem [8, Theorem 3.2.6] $\mathcal{R}'$ contains a subring $\mathcal{A}$ of the form $M_m(\Delta)$, where Δ is a domain whose classical ring of quotients is isomorphic $\mathcal{D}$. So we have $M_m(\Delta) \cong \mathcal{A} \subseteq \mathcal{R}' \subseteq \mathcal{B} \cong M_m(\mathcal{D})$. According to Lemma 2.3 we have that either $g(\mathcal{A}) = 0$ or Δ is commutative. Arguing as above when we showed that $f(\mathcal{I}) \neq 0$, we see that $g(\mathcal{A}) = 0$ implies $(g(z)y)^n g(z) = 0$ for all $z \in \mathcal{R}'$ and $y \in \mathcal{A}$. Now Levitzki's theorem is not really applicable, but using a sharper result [2, Theorem 6.6.2] we get $g(z) = 0$ for all $z \in \mathcal{R}'$. However, $g(x) \neq 0$. Therefore, the only possibility is that Δ is commutative. But then $\mathcal{D}$ is commutative too.

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