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To the Isotropic Generalization of Wallace Lines

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Abstract. The Wallace lines of a triangle in the affine-metric plane over $\mathbb{R}$ were studied by O. Giering [3]. This paper deals with the isotropic or galilean case [5] – which is not included in [3]. Essential means is the δ-footpoint definition of J. Lang [1].
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1. Let A, B, C be an admissible triangle [4, p.22] of the isotropic plane $I_2(\mathbb{R})$. We can select affine x, y-coordinates such that

$$A = (0,0),\ B = (a,0),\ C = (\mu b, b) \quad \text{with} \quad a, b, \mu \in \mathbb{R} \quad \text{and} \quad ab\mu(\mu b - a) \neq 0. \tag{1}$$

The absolute point is supposed to have the homogeneous coordinates $0:0:1$. Then, the equation for the isotropic *circumcircle* κ of ABC is

$$\kappa(x,y) \equiv y - Rx(x-a) = 0 \quad \text{with} \quad R\mu(\mu b - a) = 1. \tag{2}$$

For any $\delta \in \mathbb{R} \setminus \{0\}$ J. Lang (see [1, p.5]) defines the isotropic *δ-footpoint* $F(\delta)$ of the point X on a non-isotropic straight line g of $I_2(\mathbb{R})$ as follows:

$$\text{(a) for } X \notin g:\ F(\delta) \in g \text{ and } (X \vee F(\delta), g) = \delta, \quad \text{(b) for } X \in g:\ F(\delta) = X. \tag{3}$$

Here and in the following the symbol (h, g) means the *isotropic angle* of the non-isotropic straight lines h and g (see [4, p.17]).

J. Lang proved that for each $\delta \in \mathbb{R} \setminus \{0\}$ the δ-footpoints of a point X on the three lines determined by the sides of ABC are collinear, if and only if X is a point of the circumcircle κ.

For $X \in \kappa$ and $\delta \in \mathbb{R} \setminus \{0\}$ we call the connection line of the δ-footpoints of X on the lines determined by the sides of ABC the isotropic *Wallace line* $\omega(X, \delta)$ of X with respect to the angle δ.

If $X = (\xi, \eta) \in \kappa$, we get from (1) and (3) the analytical representation of $\omega(X, \delta)$ as

$$y = R(\mu b - \xi - \delta/R)(x - \xi - \eta/\delta). \tag{4}$$

2. The equation of the parabola $\pi(X)$ inscribed in ABC with $X = (\xi, \eta) \in \kappa \setminus \{A, B, C\}$ as isotropic *focal point* (see [4, p.74]) is

$$[y - \eta - R(\mu b - \xi)(x - \xi)]^2 - 4R\eta(\mu b - \xi)(x - \xi) = 0. \tag{5}$$

We call $\pi(X)$ the *Wallace parabola* of the point $X \in \kappa \setminus \{A, B, C\}$. $\pi(X)$ is at the same time the δ-envelope of the isotropic Wallace lines (cf. [1] and also [4, p.78f]).

Using (4), (5) and the *axis* $a(X)$ of the Wallace parabola $\pi(X)$ we see that

$$(\omega(X, \delta), a(X))) = \delta. \tag{6}$$

So we obtain as a supplement to [1] an analogous result as in the euclidean case (cf. [2, p.158]).

Theorem 1. *For an admissible triangle ABC of the isotropic plane $I_2(\mathbb{R})$ let $X \neq A, B, C$ be a point of the circumcircle κ of ABC and denote $\omega(X, \delta)$ the isotropic Wallace line to the angle $\delta \in \mathbb{R} \setminus \{0\}$. Then $\omega(X, \delta)$ is a tangent of the Wallace parabola $\pi(X)$ and intersects the axis $a(X)$ of $\pi(X)$ with the angle δ. The point of contact of $\omega(X, \delta)$ and $\pi(X)$ is the δ-footpoint of the isotropic focal point X of $\pi(X)$ on $\omega(X, \delta)$.*

The proof of the last statement is obtained by considering (5) and the representation

$$x_F = \xi + R\eta(\mu b - \xi)/\delta^2, \quad y_F = R(\mu b - \xi - \delta/R)(x_F - \xi - \eta/\delta)$$

of the δ-footpoint (x_F, y_F) of $X = (\xi, \eta) \in \kappa$ on $\omega(X, \delta)$.

3. In the euclidean situation the Wallace lines of a triangle ABC envelop a hypocycloid curve of Steiner. By a short calculation we find, that the envelope of the isotropic Wallace lines $\omega(X, \delta)$ is a rational divergent parabola of third order (see [4, p.182]) with the parametric equation

$$\begin{aligned} x(\xi) &= [1 + R(\xi - a)/\delta]\xi - [1 + R(2\xi - a)/\delta][(1/\mu - \delta)/R - (\xi - a)] \\ y(\xi) &= -R[1 + R(2\xi - a)/\delta][(1/\mu - \delta)/R - (\xi - a)]^2. \end{aligned} \tag{7}$$

Because of

$$\frac{dx}{d\xi} = 2[3R\xi/\delta - R(\mu b + a)/\delta + 2], \quad \frac{dy}{d\xi} = R[(1/\mu - \delta)/R - (\xi - a)]\frac{dx}{d\xi}$$

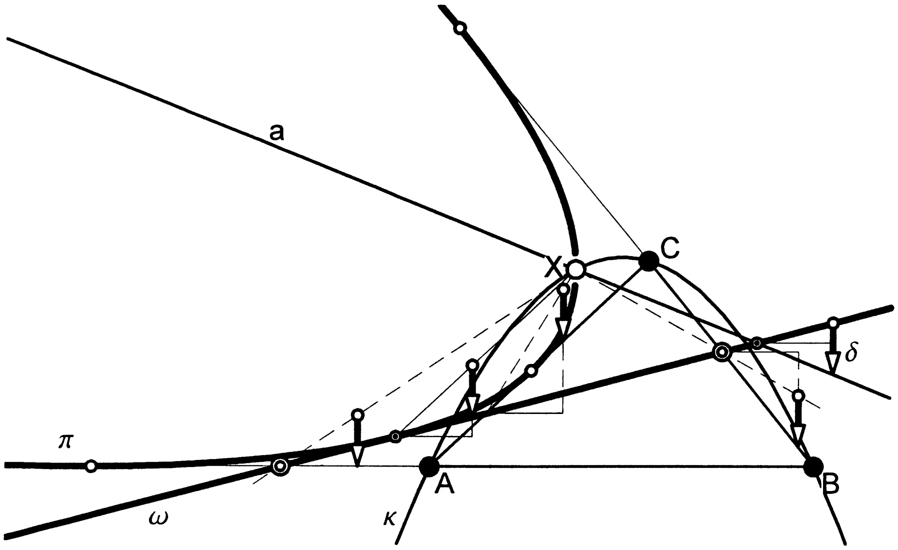

Figure 1

the point S defined by

$$S = (x(\xi_0), y(\xi_0)) \quad \text{with} \quad 3\xi_0 := 1/\mu R + 2a - 2\delta/R \tag{8}$$

is the *singular point* of the envelope (7).

A short calculation shows that (7) and (8) imply

$$y(\xi) - y(\xi_0) - R[(1/\mu - \delta)/R - (\xi_0 - a)](x(\xi) - x(\xi_0)) = -2R^2(\xi - \xi_0)^3/\delta.$$

This means that the singular point S of the envelope (7) is a cusp. One verifies that the tangent of the envelope (7) at the point S is the Wallace line $\omega(Y, \delta)$ of the point $Y := (\xi_0, \eta_0) \in \kappa$.

To determine the point Y we make use of the *centroid line* σ of ABC. This straight line was introduced in isotropic triangle geometry by K. Strubecker [6]. Using (1) and the abbreviation in (2) we find that

$$3y = (aR + 2/\mu)x - Ra^2 - a/\mu \tag{9}$$

is the equation of σ. Thus (5) leads to the *angle relation*

$$(\sigma, a(X)) = \frac{2}{3}\delta + R(\xi_0 - \xi), \quad X = (\xi, \eta) \in \kappa \setminus \{A, B, C\}. \tag{10}$$

The relation (10) and the lines determined by the sides of ABC as tangents determine the Wallace parabola $\pi(Y)$. So Y on $\pi(Y)$ is determined as the isotropic focal point and hence $\omega(Y, \delta)$ by Theorem 1.

Theorem 2. *The Wallace lines $\omega(X, \delta)$ of an admissible triangle ABC of the isotropic plane $I_2(\mathbb{R})$ envelop a parabola of Neil. The angle $(\omega(Y, \delta), \sigma)$ of the cusp tangent $\omega(Y, \delta)$ with the centroid line σ of ABC is $\delta/3$.*

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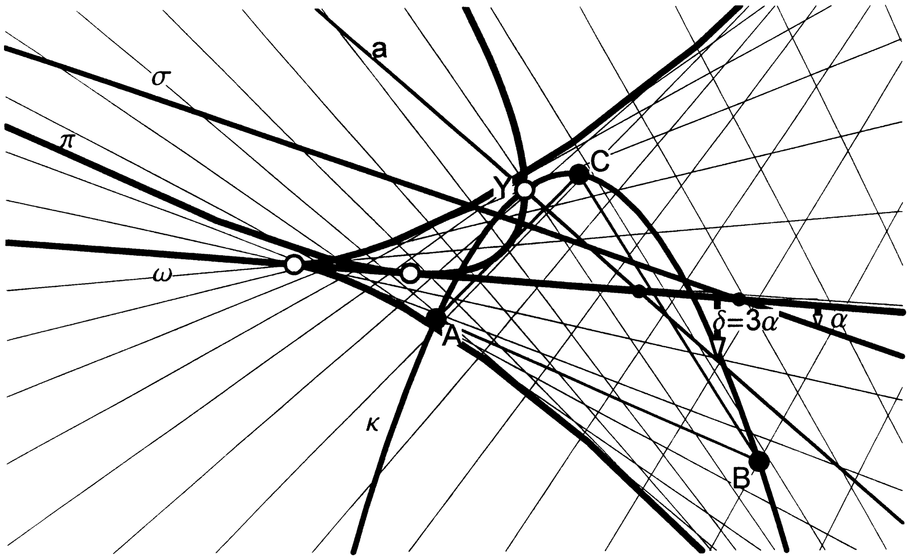

Figure 2

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