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Finitely Generated Weakly Reductive Commutative Semigroups*

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Abstract. We give characterizations of the property of being weakly reductive for finitely generated commutative semigroups. As a consequence of these characterizations we obtain an algorithm for determining from a presentation of a finitely generated commutative semigroup whether it is weakly reductive. Furthermore, we present some connections between cancellative finitely generated commutative semigroups, Archimedean weakly reductive finitely generated commutative semigroups and $\mathcal{N}$-semigroups.

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Introduction

Weakly reductive semigroups have been widely studied in semigroup theory. Since their set of translational hulls form an ideal extension of the semigroup, they play an important rôle in ideal extension theory (see [6]). Our principal aim in this paper is to give an algorithm for determining from a presentation of a finitely generated commutative semigroup whether it is weakly reductive. This algorithm is presented in Section 2. In Section 1 we introduce some basics concepts and results needed for Section 2. Finally, in Section 3 we prove that if we impose to a finitely generated semigroup the condition of being Archimedean, then we obtain a cancellative semigroup. These results and the concept of $\mathcal{N}$-semigroups allow us to give a structure theorem for Archimedean weakly reductive finitely generated commutative semigroups.

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Since all semigroups appearing in this work are commutative, in this work the concept "weakly reductive" is equivalent to "reductive". In the sequel we shall omit the adjective commutative. All the computations performed in the examples of this paper have been accomplished using `SINGULAR` (see [3]). We have used the correspondence between binomial ideals and congruences (see [5]). This correspondence allows the use of Gröbner basis for solving the word problem in finitely generated monoids. The reason to use `SINGULAR` instead of any other symbolic computation package is due to the fact that when we started using it we observed that the implementation of Buchberger's algorithm was fast enough for our purposes. This led us to implement some of our algorithms in this language and this is the reason why we use it in this work.

1. Preliminaries

Let $(S,+)$ be a semigroup generated by $\{s_1,\dots,s_p\}$ and let $I=\mathbb{N}^p\backslash\{0\}$ where $\mathbb{N}$ denotes the set of nonnegative integers. Let $\varphi: I \longrightarrow S$ be the semigroup homomorphism defined by $\varphi(x_1,\dots,x_p)=\sum_{i=1}^p x_i s_i$ and σ be the kernel congruence of φ ($x\ \sigma\ y$ if and only if $\varphi(x)=\varphi(y)$). We know that S is isomorphic to the quotient semigroup I/σ. It is known (see for instance [9]) that the congruence σ is finitely generated. Thus there exists $\rho=\{(\alpha_1,\beta_1),\dots,(\alpha_t,\beta_t)\}\subseteq I\times I$ such that σ is the least congruence on I containing ρ.

Let $a\in I$, denote $[a]=\{b\in I \mid a\ \sigma\ b\}$ the σ-class of a. Let $\preceq$ be a linear admissible order on $\mathbb{N}^p$ (a total order compatible with the componentwise sum on $\mathbb{N}^p$ and which contains the usual order of $\mathbb{N}^p$). Since $[a]$ is a nonempty subset of $\mathbb{N}^p$, by Dickson's Lemma, the set of minimal elements of $[a]$ under the usual order of $\mathbb{N}^p$ is finite and therefore there exists the minimum of $[a]$ with respect to $\preceq$. So, we can define the map

$$\mu: I\longrightarrow I,\ \mu(x)=\mathrm{minimum}_{\preceq}[x].$$

We shall refer to this map as the function *minimum* associated to σ with respect to $\preceq$. A system of generators $\rho=\{(\alpha_1,\beta_1),\dots,(\alpha_t,\beta_t)\}$ for a congruence σ is *reduced* with respect to $\preceq$ if it satisfies the following conditions:

(1) $\beta_i\prec\alpha_i$ for all $i\in\{1,\dots,t\}$,

(2) $\alpha_i-\alpha_j\notin\mathbb{N}^p$ if $i\neq j$,

(3) $\beta_i-\alpha_j\notin\mathbb{N}^p$ for all $i,j\in\{1,\dots,t\}$.

If ρ is a reduced system of generators, then we define the map $\mathrm{NF}_\rho: I\longrightarrow I$ recurrently by:

(i) If $x-\alpha_i\notin\mathbb{N}^p$ for all $i\in\{1,\dots,t\}$, then $\mathrm{NF}_\rho(x)=x$.

(ii) If $x-\alpha_{i+1}\in\mathbb{N}^p$ and $x-\alpha_k\notin\mathbb{N}^p$ for all $k\in\{1,\dots,i\}$, then

$$\mathrm{NF}_\rho(x)=\mathrm{NF}_\rho(x-\alpha_{i+1}+\beta_{i+1}).$$

A reduced system of generators ρ for σ is a *canonical system* of generators of σ with respect to a linear admissible order $\preceq$ if $\mathrm{NF}_\rho(x)=\mu(x)$ for all $x\in I$.

From any system of generators of σ, using any linear admissible order $\preceq$ on $\mathbb{N}^p$ and applying the Knuth-Bendix's critical pair completion algorithm (see [7]), we obtain a canonical system of generators of σ with respect to $\preceq$. Furthermore, if $\rho=\{(\alpha_1,\beta_1),\dots,(\alpha_t,\beta_t)\}$ is the canonical system of generators obtained, then $\mathrm{Im}(\mu) = \{x \in I \mid x - \alpha_i \notin \mathbb{N}^p$ for all $i \in \{1, \dots, t\}\}$ and $(\alpha, \beta) \in \sigma$ if and only if $\mathrm{NF}_\rho(\alpha) = \mathrm{NF}_\rho(\beta)$.

Let $x \in I$, $\preceq$ be a linear order on $\mathbb{N}^p$ fulfilling that for all $z \in \mathbb{N}^p$ the set $\{y \in \mathbb{N}^p \mid y \preceq z\}$ is finite (for instance the total degree order of $\mathbb{N}^p$) and $\rho = \{(\alpha_1, \beta_1), \dots, (\alpha_t, \beta_t)\}$ be a canonical system of generators for σ with respect to a linear admissible order $\preceq$. We finish this section presenting an algorithm for computing the set

$$A_x = \{y \in I \mid y \; \sigma \; x, \; y \preceq x\}.$$

Algorithm 1. *The input is an element x of I. The output is the set A_x.*

(1) *Compute $\mu(x)$ (note that $\mu(x) = \mathrm{NF}_\rho(x)$).*

(2) $A_x = \{\mu(x)\}$.

(3) $B = \{\mu(x)\}$.

(4) *While $B \neq \emptyset$ do*

Choose $u \in B$.
$B := (B \backslash \{u\}) \cup \{u - \beta_j + \alpha_j \mid u - \beta_j \in \mathbb{N}^p, \; u - \beta_j + \alpha_j \preceq x\}$.
$A_x := A_x \cup \{u - \beta_j + \alpha_j \mid u - \beta_j \in \mathbb{N}^p, \; u - \beta_j + \alpha_j \preceq x\}$.

(5) *Return A_x.*

Note that if $x \in I$ and $\{y \in \mathbb{N}^p \mid y \preceq x\}$ is not finite, then the algorithm does not terminate. For this reason in the sequel we shall use only this kind of linear order on $\mathbb{N}^p$. Examples of linear orders satisfying this condition are the total degree order and the prime order. The lexicographical order does not fulfill this condition.

Some references for this section are [10], [11] and [12].

2. An algorithm for determining if a finitely generated semigroup is weakly reductive

A semigroup $(S, +)$ is *weakly reductive* if $x + s = y + s$ for all $s \in S$ implies that $x = y$.

Let $(S, +)$ be as in the preliminaries and let $\rho = \{(\alpha_1, \beta_1), \dots, (\alpha_t, \beta_t)\}$ be a canonical system of generators of σ with respect to a linear admissible order $\preceq$ on $\mathbb{N}^p$ such that the condition imposed in Algorithm 1 holds (for instance we can use the total degree order). Our goal is to give an algorithm for determining from ρ whether the semigroup I/σ (and thus S) is weakly reductive.

Denote by e_i the element of $\mathbb{N}^p$ whose i-th coordinate equals to 1 and the rest of coordinates equal to 0.

Lemma 2. *The following statements are equivalent.*

(1) *The semigroup I/σ is weakly reductive.*

(2) *If $x, y \in I$ and $x + e_i \; \sigma \; y + e_i$ for all $i \in \{1, \dots, p\}$, then $x \; \sigma \; y$.*

Proof. (1) implies (2). If $x + e_i \; \sigma \; y + e_i$ for all $i \in \{1, \dots, p\}$, then $[x] + [e_i] = [x + e_i] = [y + e_i] = [y] + [e_i]$ for all $i \in \{1, \dots, p\}$. Hence, $[x] + [z] = [y] + [z]$ for all $[z] \in I/\sigma$. Using that I/σ is weakly reductive, we deduce that $[x] = [y]$ and therefore $x \; \sigma \; y$.

(2) implies (1). If $[x]+[z]=[y]+[z]$ for all $[z]\in I/\sigma$, then $x+e_i\ \sigma\ y+e_i$ for all $i\in\{1,\ldots,p\}$. Hence $x\ \sigma\ y$ and thus $[x]=[y]$. $\square$

Using Lemma 2 we can assert that if I/σ is not weakly reductive, then there exist $a,b\in I$ such that $(a+e_i,b+e_i)\in\sigma$ for all $i\in\{1,\ldots,p\}$ and $(a,b)\notin\sigma$. Since $(a,b)\notin\sigma$, we have that $(\mu(a),\mu(b))\notin\sigma$ (recall that μ is the function minimum with respect to $\preceq$). Furthermore, $(\mu(a)+e_i,\mu(b)+e_i)\in\sigma$ for all $i\in\{1,\ldots,p\}$ because $(a+e_i,b+e_i)\in\sigma$. Since $(\mu(a),\mu(b))\notin\sigma$, we obtain that $\mu(a)\neq\mu(b)$. Without loss of generality we can assume that $\mu(b)\prec\mu(a)$ and therefore $\mu(b)+e_i\prec\mu(a)+e_i$. Thus, $\mu(a)+e_i\neq\mu(a+e_i)$ and therefore $\mu(a)+e_i\notin\mathrm{Im}(\mu)$. Note also that $\mu(a)\in\mathrm{Im}(\mu)$.

We can now state the following result.

Lemma 3. *The semigroup I/σ is not weakly reductive if and only if there exists $(x,y)\in I\times I$ such that the following conditions hold:*

(1) $(x,y)\notin\sigma$,

(2) $x\in\mathrm{Im}(\mu)$,

(3) $x+e_i\notin\mathrm{Im}(\mu)$ *for all* $i\in\{1,\ldots,p\}$,

(4) $(x+e_i,y+e_i)\in\sigma$ *for all* $i\in\{1,\ldots,p\}$,

(5) $y\prec x$.

Shortly, we shall show that there are only finitely many of elements of I satisfying conditions (2) and (3) of Lemma 3.

Let $x=(x_1,\ldots,x_p),y=(y_1,\ldots,y_p)\in\mathbb{N}^p$. Denote by

$$x\vee y=(\max\{x_1,y_1\},\ldots,\max\{x_p,y_p\})$$

(here $\max(a,b)$ stands for the maximum of $\{a,b\}$). Given $x\in\mathbb{N}^p$ we denote by $(x)_i$ its i-th coordinate.

Lemma 4. *If $x\in\mathrm{Im}(\mu)$ and $x+e_i\notin\mathrm{Im}(\mu)$ for all $i\in\{1,\ldots,p\}$, then there exists $\{\alpha_{l_1},\ldots,\alpha_{l_p}\}\subseteq\{\alpha_1,\ldots,\alpha_t\}$ such that the following conditions hold:*

(1) *for all $j\neq k$ in $\{1,\ldots,p\}$ we have $(\alpha_{l_k})_j<(\alpha_{l_j})_j$,*

(2) $x=((\alpha_{l_1})_1-1,\ldots,(\alpha_{l_p})_p-1)$.

Proof. First recall that $\mathrm{Im}(\mu)=\{x\in I\mid x-\alpha_i\notin\mathbb{N}^p$ for all $i\in\{1,\ldots,t\}\}$. Let $j\in\{1,\ldots,p\}$, since $x+e_j\notin\mathrm{Im}(\mu)$, then there exist $l_j\in\{1,\ldots,t\}$ and $d_j\in\mathbb{N}^p$ such that $x+e_j=\alpha_{l_j}+d_j$. So, we deduce that $\alpha_{l_j}-e_j\in\mathbb{N}^p$ because $x\in\mathrm{Im}(\mu)$. Thus, $x=(\alpha_{l_1}-e_1)+d_1=\ldots=(\alpha_{l_p}-e_p)+d_p$ and therefore $x=((\alpha_{l_1}-e_1)\vee\ldots\vee(\alpha_{l_p}-e_p))+y$ for some $y\in\mathbb{N}^p$. If $y\neq 0$, then there exists $j\in\{1,\ldots,p\}$ such that $y-e_j\in\mathbb{N}^p$ and therefore $x-\alpha_{l_j}\in\mathbb{N}^p$ which is absurd because $x\in\mathrm{Im}(\mu)$. Hence, we have that $x=(\alpha_{l_1}-e_i)\vee\ldots\vee(\alpha_{l_p}-e_p)$ and therefore $(x)_k\geq(\alpha_{l_j})_k$ for all $k\neq j$ and $(x)_j\geq(\alpha_{l_j})_j-1$.

Now we prove (1). If $(\alpha_{l_j})_j\leq(\alpha_{l_k})_j$ for some $k\neq j$, then

$$(\alpha_{l_j})_j-1<(\alpha_{l_k})_j\leq\mathrm{maximum}\{(\alpha_{l_1})_j,\ldots,(\alpha_{l_j})_j-1,\ldots,(\alpha_{l_p})_j\}=(x)_j$$

and therefore $(\alpha_{l_j})_j \leq (x)_j$. Hence $(\alpha_{l_j})_k \leq (x)_k$ for all $k \in \{1,\dots,p\}$. This implies that $x - \alpha_{l_j} \in \mathbb{N}^p$ which is in contradiction with $x \in \mathrm{Im}(\mu)$.

Finally for proving (2) we only have to take into account that $x = (\alpha_{l_1} - e_1) \vee \dots \vee (\alpha_{l_p} - e_p)$ and apply (1). □

Corollary 5. *Let $I = \mathbb{N}^p \backslash \{0\}$ and σ be a congruence on I which admits a canonical system of generators ρ with cardinality less than p. Then I/σ is weakly reductive.*

Proof. By Lemma 3 we know that if I/σ is not weakly reductive, then there exists $x \in \mathrm{Im}(\mu)$ such that $x + e_i \notin \mathrm{Im}(\mu)$ for all $i \in \{1,\dots,p\}$. But, using Lemma 4, since the set $\{\alpha_{l_1},\dots,\alpha_{l_p}\}$ has p elements, the cardinality of ρ is greater or equal than p. □

We complete this section with an algorithm for deciding whether a finitely generated semigroup is weakly reductive or not.

Algorithm 6. *The input is a canonical system of generators $\rho = \{(\alpha_1,\beta_1),\dots,(\alpha_t,\beta_t)\}$ for the congruence σ on $I = \mathbb{N}^p \backslash \{0\}$ with respect to a linear admissible order $\preceq$. The output is TRUE if the semigroup I/σ is weakly reductive and FALSE otherwise.*

(1) *Compute the set*

$$X = \{x \in I \mid x \in \mathrm{Im}(\mu),\ x + e_i \notin \mathrm{Im}(\mu) \text{ for all } \ i \in \{1,\dots,p\}\}$$

(note that Lemma 4 gives us an algorithmic method for computing X).

(2) *For every $x \in X$ and $i \in \{1,\dots,p\}$ compute the set*

$$C_x = \bigcap_{i=1}^{p} \{z - e_i \in \mathbb{N}^p \backslash \{0\} \mid z \in A_{x+e_i}\}$$

(we use Algorithm 1 for computing A_{x+e_i}).

(3) *For every $x \in X$ and $y \in C_x \backslash \{x\}$ check whether $(x,y) \in \sigma$. If for some pair $(x,y) \notin \sigma$, then return FALSE.*

(4) *Return TRUE.*

We see now some examples of this Algorithm.

Example 7. Let σ be the congruence on $I = \mathbb{N}\backslash\{0\}$ generated by $\rho = \{(4,2)\}$. It is easy to see that ρ is a canonical system of generators for σ with respect to the usual order of $\mathbb{N}$. Applying now Algorithm 6 we obtain:

(1) $X = \{3\}$.

(2) For $x = 3$ we have that $A_4 = \{z \in [4] \mid z \leq 4\} = \{2,4\}$ and $C_3 = \{1,3\}$.

(3) We have to check if the pair $(3,1)$ belongs to σ. Clearly $(3,1) \notin \sigma$.

Thus the algorithm returns FALSE and therefore the semigroup is not weakly reductive.

Example 8. Take now σ the congruence on $I = \mathbb{N}^2 \setminus \{0\}$ generated by

$$\rho = \{((7,7),(3,8)),((3,12),(5,7)),((5,11),(3,7)),((15,2),(11,3))\}.$$

It can be proved that ρ is a canonical system of generators of σ with respect to the total degree order of $\mathbb{N}^2$. We apply now Algorithm 6.

(1) We use Lemma 4 for computing X and thus we have that X is a subset of the set

$$\{(6,11),(6,10),(6,1),(2,6),(2,10),(2,1),\\ (4,6),(4,11),(4,1),(14,6),(14,11),(14,10)\}.$$

We eliminate from this set the elements z such that $z \notin \mathrm{Im}(\mu)$ and those such that $z + e_1 \in \mathrm{Im}(\mu)$ or $z + e_2 \in \mathrm{Im}(\mu)$. Hence

$$X = \{(6,10),(4,11),(14,6)\}.$$

(2) We compute now $C_{(6,10)}$, $C_{(4,11)}$ and $C_{(14,6)}$. For this purpose we apply Algorithm 1 to the elements $x + e_1$ and $x + e_2$ with $x \in X$.

- For $x = (6,10)$ we have that $A_{x+e_1} = \{(7,10),(3,11)\}$ and $A_{x+e_2} = \{(6,11),(4,7)\}$. Thus we obtain that $C_{(6,10)} = \{(6,10)\}$.
- For $x = (4,11)$ we have that $A_{x+e_1} = \{(5,11),(3,7)\}$ and $A_{x+e_2} = \{(4,12),(6,7)\}$. Thus we obtain that $C_{(4,11)} = \{(4,11)\}$.
- For $x = (14,6)$ we have that

$$A_{x+e_1} = \{(15,6),(3,9),(3,18),(5,13),(7,8),(9,12)\},\\ A_{x+e_2} = \{(14,7),(4,14),(6,9),(8,13),(10,8)\}.$$

Thus $C_{(14,6)} = \{(14,6),(4,13),(6,8),(8,12),(10,7)\}$.

(3) If we find two elements $x \in X$ and $y \in C_x \setminus \{x\}$ such that $(x,y) \notin \sigma$ then the algorithm returns FALSE. Take $x = (14,6)$ and $y = (6,8) \in C_{(14,6)}$. Clearly $(14,6),(6,8) \in \mathrm{Im}(\mu)$ and $(14,6) \neq (6,8)$. Hence $((14,6),(6,8)) \notin \sigma$ and the algorithm returns FALSE.

Since the Algorithm returns FALSE, the semigroup is not weakly reductive.

Example 9. Let us show now an example of a weakly reductive semigroup. Let σ be the congruence on $I = \mathbb{N}^3 \setminus \{0\}$ with canonical system of generators

$$\rho = \{((1,0,1),(0,2,0)),((2,1,0),(0,0,2)),\\ ((3,0,0),(0,1,1)),((1,3,0),(0,0,3)),(0,5,0),(0,0,4))\}.$$

It can be proved that ρ is a canonical system of generators of σ with respect to the total degree order of $\mathbb{N}^3$. Applying Algorithm 6, we obtain the following:

(1) Using Lemma 4 we obtain that X is a subset of the set

$$\{(1,2,0),(1,4,0),(2,0,0),(2,2,0),(2,4,0),(0,4,0)\}.$$

Now eliminating from this set the elements z such that $z \notin \mathrm{Im}(\mu)$ and those elements such that $z + e_1 \in \mathrm{Im}(\mu)$ or $z + e_2 \in \mathrm{Im}(\mu)$ or $z + e_3 \in \mathrm{Im}(\mu)$ we obtain that

$$X = \{(1,2,0),(2,0,0)\}.$$

(2) We compute now the sets C_x for each of the elements of X.
   - For $x=(1,2,0)$ we have that $A_{x+e_1}=\{(2,2,0),(0,1,2)\}$, $A_{x+e_2}=\{(1,3,0),(0,0,3)\}$ and $A_{x+e_3}=\{(1,2,1),(0,4,0)\}$. Thus we obtain that $C_{(1,2,0)}=\{(1,2,0)\}$.
   - For $x=(2,0,0)$ we have that $A_{x+e_1}=\{(3,0,0),(0,1,1)\}$, $A_{x+e_2}=\{(2,1,0),(0,1,2)\}$ and $A_{x+e_3}=\{(2,0,1),(1,2,1)\}$. Thus we obtain that $C_{(2,0,0)}=\{(2,0,0)\}$.

(3) Since for all $x\in X$ the set C_x contains only one element there is not any pair of elements x,y such that $x\in X$ and $y\in C_x$ with $(x,y)\notin\sigma$.

(4) Return TRUE.

Thus the algorithm returns TRUE and so the semigroup I/σ is weakly reductive.

3. Archimedean weakly reductive finitely generated semigroups

An element x of a semigroup $(S,+)$ is *Archimedean* if for any $y\in S$ there exist $k\in\mathbb{N}\backslash\{0\}$ and $z\in S$ such that $kx=y+z$. A semigroup is *Archimedean* if all its elements are Archimedean. A semigroup $(S,+)$ is *cancellative* if $a+c=b+c$ implies that $a=b$ for all $a,b,c\in S$.

Clearly every cancellative semigroup is weakly reductive. Our first aim in this section is to prove that the converse is also true when the semigroup is Archimedean and finitely generated.

Theorem 10. *Let $(S,+)$ be an Archimedean weakly reductive finitely generated semigroup. Then $(S,+)$ is a cancellative semigroup.*

Proof. Let $\{s_1,\dots,s_p\}$ be a system of generators of S. If S is not cancellative, then there exist $s,t,x\in S$ such that $s\neq t$ and $s+x=t+x$. Since S is Archimedean, for every $i\in\{1,\dots,p\}$ there exist $k_i\in\mathbb{N}\backslash\{0\}$ and $y_i\in S$ such that $k_is_i=x+y_i$. Thus $s+k_1s_1=t+k_1s_1,\dots,s+k_ps_p=t+k_ps_p$. Hence if $s+(a_1s_1+\ldots+a_ps_p)\neq t+(a_1s_1+\ldots+a_ps_p)$, then $(a_1,\dots,a_p)\leq(k_1,\dots,k_p)$ (where $\leq$ denotes the usual order of $\mathbb{N}^p$). Since the set of p-tuples less or equal that $(k_1,\dots,k_p)$ (with the usual order of $\mathbb{N}^p$) is finite and S is weakly reductive, the set

$$M=\text{maximals}_{\leq}\left\{\ (a_1,\dots,a_p)\in\mathbb{N}^p\backslash\{0\}\ \middle|\ \begin{array}{c} s+(a_1s_1+\ldots+a_ps_p)\neq\\ \neq t+(a_1s_1+\ldots+a_ps_p)\end{array}\ \right\}.$$

is finite and nonempty. If $(d_1,\dots,d_p)\in M$, then $s+d_1s_1+\ldots+d_ps_p\neq t+d_1s_1+\ldots+d_ps_p$ and using the maximality of $(d_1,\dots,d_p)$ we have that

$$(s+d_1s_1+\ldots+d_ps_p)+s_i=(t+d_1s_1+\ldots+d_ps_p)+s_i$$

for all $i\in\{1,\dots,p\}$. Using now that S is weakly reductive and applying Lemma 2, we obtain that $s+d_1s_1+\ldots+d_ps_p=t+d_1s_1+\ldots+d_ps_p$. But this contradicts the fact that $(d_1,\dots,d_p)\in M$. □

Our next goal is to give a description of the structure of Archimedean weakly reductive finitely generated semigroups. But before we need to introduce some concepts and results.

An element x of a semigroup $(S,+)$ is *idempotent* if $x+x=x$. The following result is well known (see for instance [4] or [1]).

Proposition 11. *An Archimedean cancellative semigroup has an idempotent element if and only if it is a group.*

Using the definition of Petrich in [8], an *$\mathcal{N}$-semigroup* is an Archimedean cancellative semigroup without idempotent elements. The following proposition, proved by Tamura in [13], characterizes all the $\mathcal{N}$-semigroups.

Proposition 12. *Let $(G, \oplus)$ be a group and $I : G \times G \longrightarrow \mathbb{N}$ be a map satisfying the following conditions:*

(T.1) $I(g_1, g_2) = I(g_2, g_1)$ *for all* $g_1, g_2 \in G$.

(T.2) $I(g_1, g_2) + I(g_1 \oplus g_2, g_3) = I(g_2, g_3) + I(g_1, g_2 \oplus g_3)$ *for all* $g_1, g_2, g_3 \in G$.

(T.3) $I(0, g) = 1$ *for all* $g \in G$ *(where* 0 *denotes the identity element of* G*).*

(T.4) *For all* $g \in G$ *there exists* $k \in \mathbb{N} \backslash \{0\}$ *such that* $I(g, kg) > 0$.

On the set $\mathbb{N} \times G$ *define the following operation*

$$(a_1, g_1) +_I (a_2, g_2) = (a_1 + a_2 + I(g_1, g_2), g_1 \oplus g_2).$$

Then $(\mathbb{N} \times G, +_I)$ *is an* $\mathcal{N}$*-semigroup and every* $\mathcal{N}$*-semigroup is isomorphic to a semigroup of this form.*

Let $(H, +)$ be an abelian group and $h \in H$. A group H is *periodic* if for all $h \in H$ there exists $k \in \mathbb{N} \backslash \{0\}$ such that $kh = 0$. It can be proved easily that every finite group is periodic.

The following result was proved by Chrislock and appears in [2].

Proposition 13. *Let the hypothesis be as in Proposition* 12*. Then the semigroup* $(\mathbb{N} \times G, +_I)$ *is finitely generated if and only if* G *is finite.*

As a consequence of the above results we obtain the following result.

Theorem 14. *The pair* $(S, +)$ *is an Archimedean weakly reductive finitely generated semigroup if and only if* S *is a finitely generated group or a finitely generated* $\mathcal{N}$*-semigroup. Furthermore, under these hypotheses the following statements are fulfilled:*

(1) S *is a group if and only if* S *has an idempotent element,*

(2) S *is an* $\mathcal{N}$*-semigroup if and only if* S *is isomorphic to a semigroup of the form* $(\mathbb{N} \times G, +_I)$ *with* G *a finite group and* $I : G \times G \longrightarrow \mathbb{N}$ *satisfying* (T.1), (T.2) *and* (T.3).

Proof. As a consequence of Theorem 10, we know that $(S, +)$ is an Archimedean weakly reductive finitely generated semigroup if and only if $(S, +)$ is an Archimedean cancellative finitely generated semigroup. By the definition of $\mathcal{N}$-semigroup and using Proposition 11 we deduce that $(S, +)$ is a finitely generated group or a finitely generated $\mathcal{N}$-semigroup depending on the existence of idempotent elements in S.

In the case S is an $\mathcal{N}$-semigroup, by Proposition 12, there exists a semigroup isomorphic to S of the form $(\mathbb{N} \times G, +_I)$ with I satisfying (T.1), (T.2), (T.3) and (T.4). Since S is finitely generated, $\mathbb{N} \times G$ is also finitely generated and by Proposition 13, the group G is finite.

Conversely, assume now that S is isomorphic to a semigroup of the form $(\mathbb{N} \times G, +_I)$ with G a finite group and $I : G \times G \longrightarrow \mathbb{N}$ satisfying (T.1), (T.2) and (T.3). Since every finite group is periodic, condition (T.4) is fulfilled and therefore S is an $\mathcal{N}$-semigroup. $\square$

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