Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 43 (2002), No. 1, 171-179.

A Remark on Mixed Curvature Measures for Sets with Positive Reach

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Abstract. The existence of mixed curvature measures of two sets in $\mathbb{R}^d$ with positive reach introduced in [6] is discussed. An example shows that the non-osculating condition from [6] does not ensure the locally bounded variation of the mixed curvature measures. Further, some sufficient conditions for the local boundedness of mixed curvature measures involving absolute curvature measures are presented.

For any two subsets $X, Y \subseteq \mathbb{R}^d$ with positive reach and $r, s \in \{0, 1, \dots, d-1\}$, $r+s \geq d$, the mixed curvature measures $C_{r,s}(X, Y; \cdot)$ have been defined in [6] (where a different notation, $\Psi_{r,s}(X, Y; \cdot)$, has been used) as integrals of certain $(2d-1)$-forms $\psi_{r,s}$ over the joint unit normal bundle

$$\operatorname{nor}(X, Y) = f(((\operatorname{nor} X \times \operatorname{nor} Y) \cap R) \times [0,1]),$$

where $\operatorname{nor} X, \operatorname{nor} Y$ are the unit normal bundles of X, Y, respectively,

$$R = \{(x, m, y, n) \in \mathbb{R}^{4d} : m + n \neq 0\}$$

and

$$f : (x, m, y, n, t) \mapsto \Big(x, y, \frac{\sin(1-t)\theta}{\sin\theta} m + \frac{\sin t\theta}{\sin\theta} n\Big),$$

$\theta \equiv \angle(m, n) \in [0, \pi]$. Then, a translative integral formula involving these mixed curvature measures was proved ([6, Theorem 1]) under the 'non-osculating assumption'

$$\mathcal{L}^d(\{z \in \mathbb{R}^d : \exists (x, m) \in \operatorname{nor} X, (x+z, -m) \in \operatorname{nor} Y\}) = 0. \tag{1}$$

[1]Supported by the Grant Agency of Czech Republic, Project No. 201/99/0269, and by MSM 113200007

It may, however, occur (as remarked by Joseph Fu in personal communication) that nor (X, Y) has not finite $\mathcal{H}^{2d-1}$ measure (since f is only locally Lipschitz) and the signed measures $C_{r,s}(X, Y; \cdot)$ may not be correctly defined (see Example 1 below). Therefore, it has been assumed additionally in [7] that

$$|\bar{C}|_{r,s}(X, Y; \cdot) \text{ is locally finite, } 0 \le r, s \le d-1, r+s \ge d, \tag{2}$$

where the (nonnegative) measures $|\bar{C}|_{r,s}(X, Y; \cdot)$ are defined below. Under this assumption, $C_{r,s}(X, Y; \cdot)$ is well defined for any admissible r, s and $|\bar{C}|_{r,s}(X, Y; \cdot)$ is the total variation measure of its projection $\bar{C}_{r,s}(X, Y; \cdot) = C_{r,s}(X, Y; \cdot \times \mathbb{S}^{d-1})$ (cf. [6, Theorem 2], [7, Theorem 4.2]).

The functionals $\bar{C}_{r,s}(X, Y; \cdot)$ were studied more extensively for convex bodies X, Y (or sets from the convex ring), see e.g. [9, 10]. Note that in [3], the notion 'mixed curvature measure' has been used for a different functional.

Let $\kappa_i \equiv \kappa_i(x, m)$ ($\lambda_i \equiv \lambda_i(y, n)$) be the (generalized) principal curvatures and $a_i \equiv a_i(x, m)$ ($b_i \equiv b_i(y, n)$) the corresponding (generalized) principal directions of X (Y) defined at $\mathcal{H}^{d-1}$-almost all $(x, m) \in \operatorname{nor} X$ ($(y, n) \in \operatorname{nor} Y$, respectively). We set for any bounded Borel subset $A \subseteq \mathbb{R}^{2d}$

$$|\bar{C}|_{r,s}(X, Y; A) = \int\limits_{(\operatorname{nor} X \times \operatorname{nor} Y) \cap R} \mathbf{1}_A(x, y) F_{r,s}(\theta) |W_{r,s}^{X,Y}(x, m, y, n)| \mathcal{H}^{2d-2}(d(x, m, y, n)), \tag{3}$$

where

$$F_{r,s}(\theta) = \mathcal{O}_{2d-1-r-s}^{-1} \frac{\theta}{\sin\theta} \int_0^1 \left(\frac{\sin(1-t)\theta}{\sin\theta}\right)^{d-1-r} \left(\frac{\sin t\theta}{\sin\theta}\right)^{d-1-s} dt,$$

$$W_{r,s}^{X,Y} = \sum_{|I|=r} \sum_{|J|=s} \frac{\prod_{i \notin I} \kappa_i \prod_{j \notin J} \lambda_j \left[\bigwedge_{i \in I} a_i, \bigwedge_{j \in J} b_j\right]^2}{\prod_{i=1}^{d-1} \sqrt{1+\kappa_i^2} \prod_{j=1}^{d-1} \sqrt{1+\lambda_j^2}}$$

(the summation is taken over all subsets $I, J \subseteq \{1, \dots, d-1\}$ of given cardinality), $\mathcal{O}_k$ is the k-dimensional measure of the unit sphere in $\mathbb{R}^{k+1}$ and $\left[\bigwedge_{i \in I} a_i, \bigwedge_{j \in J} b_j\right]$ is the Jacobian of the orthogonal projection of the linear subspace spanned by $\{a_i : i \in I\}$ onto the orthogonal complement of that spanned by $\{b_j : j \in J\}$. Note that the measure $\bar{C}_{r,s}(X, Y; A)$ can be represented as in (3) with the difference that the absolute value in the integrand is missing (see [7, Theorem 4.2]).

Example 1. *There exists a compact set $X \subseteq \mathbb{R}^4$ with positive reach and $m \in \mathbb{S}^3$ such that*

$$\mathcal{L}^1(\{x \cdot m : (x, m) \in \operatorname{nor} X \text{ or } (x, -m) \in \operatorname{nor} X\}) = 0 \tag{4}$$

and the positive part of the mixed curvature measure $\bar{C}_{1,3}(X, m^\perp; \cdot)$ is infinite on a compact set.

Remark 1. (4) is the particular form of (1) for X and $Y = m^\perp$ (cf. [5, Assumption (3.1)]).

Proof. Let $m \in \mathbb{S}^3$ be given and let $Z \subseteq \mathbb{R}^4$ be a four-dimensional cube of edge length $a > 0$ and two facets F_1, F_2 perpendicular to m. Let there be in F_1 a disjoint family of three-dimensional balls C_i of radii $r_i > 0$, $i = 1, 2, \ldots$. Note that we have $\sum_i r_i^3 < \infty$. Then, for each i, a unit four-dimensional ball B_i is placed so that its centre lies outside of Z and $B_i \cap F_1 = C_i$. The compact set $X = Z \setminus \bigcup_i (\operatorname{int} B_i)$ has reach greater than 3/4. Let $a_i \equiv a_i(x,n)$ be the (generalized) principal directions and $\kappa_i \equiv \kappa_i(x,n)$ the corresponding (generalized) principal curvatures of X at (x,n), $i = 1,2,3$. Using [5, Theorem 3.1] (or, equivalently, applying directly [6, Theorem 2]), one gets the formula

$$\begin{aligned}&\bar{C}_{1,3}(X, m^\perp; A \times B)\\ &= \frac{1}{4\pi} \int\limits_{\operatorname{nor} X} \mathbf{1}_A(x) \frac{\kappa_1\kappa_2(a_3\cdot m)^2 + \kappa_1\kappa_3(a_2\cdot m)^2 + \kappa_2\kappa_3(a_1\cdot m)^2}{|\sin\angle(m,n)|^3\sqrt{1+\kappa_1^2}\sqrt{1+\kappa_2^2}\sqrt{1+\kappa_3^2}} \mathcal{H}^3(d(x,n))\end{aligned}$$

valid for any bounded Borel subsets $A \subseteq \mathbb{R}^4$ and $B \subseteq m^\perp$ with $\mathcal{L}^3(B) = 1$. If $x \in A_i = \operatorname{int} Z \cap \partial B_i$ (A_i is the surface of the spherical hole digged by B_i in X) then $\kappa_1 = \kappa_2 = \kappa_3 = -1$ and the principal directions can be chosen so that $a_1, a_2 \perp m$ and $a_3 \cdot m = \sin\angle(m,n)$. We thus obtain

$$\bar{C}_{1,3}(X, m^\perp; A_i \times B) = \frac{1}{8\sqrt{2}\pi} \int_{\operatorname{nor} X} \mathbf{1}_{A_i}(x) \frac{1}{|\sin\angle(m,n)|} \mathcal{H}^3(d(x,n)).$$

Consider the projection $\psi : (x,n) \mapsto m\cdot n$ restricted to $\operatorname{nor} X \cap (A_i \times \mathbb{S}^3)$, with Jacobian $J_1\psi(x,n) = \frac{1}{\sqrt{2}} \sin\angle(m,n)$. Applying the co-area formula, we get

$$\begin{aligned}\bar{C}_{1,3}(X, m^\perp; A_i \times B) &= \frac{1}{8\pi}\int_{\sqrt{1-r_i^2}}^1 \frac{1}{1-u^2}\mathcal{H}^2(\psi^{-1}u)\,du\\ &= \frac{1}{8\pi}\int_{\sqrt{1-r_i^2}}^1 \frac{1}{1-u^2} 8\pi(1-u^2)\,du\\ &= 1 - \sqrt{1-r_i^2}.\end{aligned}$$

Thus, choosing the r_i's so that $\sum_i r_i^2 = \infty$ (to see that this is possible, consider a partition of the cube Z into infinitely many parallel slices and fill a constant volume proportion of each slice by balls of diameters equal to the slice height), the positive part of $\bar{C}_{1,3}(X, m^\perp; X \times B \times \mathbb{S}^3)$ will be greater or equal to

$$\sum_i \bar{C}^+_{1,3}(X, m^\perp; A_i \times B) = \sum_i (1 - \sqrt{1-r_i^2}) = \infty.$$

To see that (1) is satisfied, note that the set $\{x\cdot m : (x,m) \in \operatorname{nor} X \text{ or } (x,-m) \in \operatorname{nor} X\} \setminus (F_1 \cup F_2)$ is countable. □

Remark 2. The construction from the proof can be easily adapted to $\mathbb{R}^5$, yielding a set $\tilde{X}$ with positive reach and $m \in \mathbb{S}^4$ with locally unbounded *negative* part of $\bar{C}_{1,4}(\tilde{X}, m^\perp; \cdot)$. Embedding then the set X from the example into a hyperplane in $\mathbb{R}^5$ containing m, we get the positive part of $\bar{C}_{1,4}(X, m^\perp; \cdot)$ locally unbounded (note that the intersection of X with a translate of $m^\perp$ is the same set, whether in $\mathbb{R}^4$, or embedded in $\mathbb{R}^5$, and its zeroth curvature measure - which is obviously connected with the mixed curvature measures in question - does not depend on the embedding space). Thus, the union of disjoint translates of X and $\tilde{X}$ presents an example of a set with positive reach in $\mathbb{R}^5$ which has locally unbounded positive and negative parts of the mixed curvature measure with the hyperplane $m^\perp$.

Theorem 1. *If for any compact subset $K \subseteq \mathbb{R}^{4d}$*

$$\int_{K\cap(\operatorname{nor} X\times\operatorname{nor} Y)\times R} (\sin\angle(m,n))^{3-d}\,\mathcal{H}^{2d-2}(d(x,m,y,n)) < \infty,$$

then (2) holds. Consequently, (2) is satisfied automatically in dimensions $d \le 3$. Moreover, for any $X, Y \subseteq \mathbb{R}^d$ with positive reach, (2) is satisfied by $X, \rho Y$ for ϑ_d-almost all rotations $\rho \in \mathrm{SO}(d)$, where ϑ_d is the rotation invariant probability measure on $\mathrm{SO}(d)$.

Proof. Note that $F_{r,s}(\theta) \le (\sin\theta)^{-(2d-1-r-s)}$ for any $\theta \in [0,\pi]$ and

$$[\bigwedge_{i\in I} a_i, \bigwedge_{j\in J} b_j] \le \sin\angle(m,n)$$

for all $I, J \subseteq \{1,\dots,d-1\}$. Thus, the first statement follows directly from (3). The last statement follows from the first one by integration over $\mathrm{SO}(d)$ since

$$\int_{\mathrm{SO}(d)} (\sin\angle(m,\rho n))^{3-d}\vartheta_d(d\rho)$$

is a finite constant depending on d only (see [7, Proposition 4.6] for a more detailed explanation). □

In the sequel, let $G(d,k)$ denote the Grassmannian of k-dimensional linear subspaces in $\mathbb{R}^d$ endowed with the rotational invariant probability measure ν_k^d. For $X \subseteq \mathbb{R}^d$ with positive reach, $0 \le k \le d-1$ and $W \in G(d,k+1)$, we define the mapping

$$t_W : \operatorname{nor} X \cap (\mathbb{R}^d \times W) \to W \times W, \quad (x,m) \mapsto (p_W x, m),$$

where p_W is the orthogonal projection from $\mathbb{R}^d$ onto W. Further, let $\pi_W : (z,m) \mapsto z$ denote the first coordinate projection defined on the image of t_W. The image $T_W X = (\pi_W \circ t_W)(\operatorname{nor} X \cap (\mathbb{R}^d \times W))$ will be called the *tangential projection* of X onto W. Note that $T_W X$ is a closed subset of $\mathbb{R}^d$ since $\operatorname{nor} X$ is a closed set. We need the following fact (cf. [8, p. 230]). Let $R_W X$ denote the set of all points $z \in T_W X$ such that $(\pi_W \circ t_W)^{-1}\{z\}$ is either a singleton or a pair of points of the form $(x,m), (x,-m) \in \operatorname{nor} X$.

Lemma 1. *We have*

$$\nu_{k+1}^d(\{W \in G(d,k+1) : \mathcal{H}^k(T_W X \setminus R_W X) > 0\}) = 0.$$

In other words, for ν_{k+1}^d-almost all $W \in G(d,k+1)$ and $\mathcal{H}^k$-almost all $z \in T_W X$, if $\pi_W(t_W(x,m)) = \pi_W(t_W(y,n)) = z$ then $x = y$ and $m = \pm n$.

Proof. Consider the countably p-rectifiable set

$$G = \{(x, m, W) \in \operatorname{nor} X \times G(d, k+1) : m \in W\},$$

$$p = d - 1 + k(d - 1 - k) = k + (k+1)(d - 1 - k)$$

(G corresponds to the $(d-1-k)$th Grassmann bundle of X introduced in [8]). The points $(x, m) \in \operatorname{nor} X$ where $\operatorname{Tan}(\operatorname{nor} X, (x, m))$ is not a $(d-1)$-dimensional linear subspace have $\mathcal{H}^{d-1}$-measure zero, thus, their contribution to the tangential projections can be neglected since the corresponding set $G_0 \subseteq G$ has $\mathcal{H}^p$-measure zero. Let G_1, G_2 be the set of such points $(x, m, W) \in G$ where

$$\dim \operatorname{Tan}(\operatorname{nor} X, (x, m)) \cap (\mathbb{R}^d \times W) > k,$$

$$\dim \operatorname{Tan}(\operatorname{nor} X, (x, m)) \cap (\{0\} \times W) > 0,$$

respectively. The co-area formula applied to the projection $F : (x, m, W) \mapsto W$ defined on G yields

$$\int_{G(d,k+1)} \mathcal{H}^k(G_1 \cap F^{-1}(W)) \nu^d_{k+1}(dW) = \int_{G_1} J_{(k+1)(d-1-k)} F \, d\mathcal{H}^p.$$

Since the Jacobian $J_{(k+1)(d-1-k)}F$ vanishes on G_1 ($\ker DF(x, m, W)$ contains all vectors $(v, w, 0)$ such that $(v, w) \in \operatorname{Tan}(\operatorname{nor} X, (x, m))$ and $w \in W$ and has thus dimension at least $k+1$), the last integral vanishes and the contribution of points from G_1 to the tangential projections can be neglected. Consider further the projection $\bar{f} : (x, m, W) \mapsto (p_W x, W)$ defined again on G. We have $J_p \bar{f} = 0$ on G_2 and, consequently, $\mathcal{H}^p(\bar{f}(G_2)) = 0$ by the area formula and the contribution of points from G_2 to the tangential projections can be neglected as well.

Finally, we consider the tangential projection of points from $G_3 = G \setminus (G_0 \cup G_1 \cup G_2)$. Denote

$$\begin{aligned} Z \;=\; & \{(x, m, y, n, W) \in \operatorname{nor} X \times \operatorname{nor} X \times G(d, k+1) : \\ & (x, m, W) \in G_3, (y, n, W) \in G_3, 0 \neq x - y \perp W\}. \end{aligned}$$

In fact, Z consists of pairs of 'regular' points from the unit normal bundle of X which have different first coordinates and whose first coordinates of the tangential projections coincide. (Note that we need not consider pairs with $x = y$ and m, n linearly independent since both of these points fall into G_2 with the corresponding subspace W.) We shall show that $\mathcal{H}^p(Z) = 0$. Consequently, the image of Z under the Lipschitz mapping $\tilde{f} : (x, m, y, n, W) \mapsto (p_W x, W)$ has again $\mathcal{H}^p$-measure zero and the assertion follows then when applying the co-area theorem for the projection on the second coordinate on the set $\tilde{f}(Z)$.

Note that the points of Z are zero points of the Lipschitz mapping

$$h : (x, m, y, n, W) \mapsto (W \wedge m, W \wedge n, W^\perp \wedge (x - y)).$$

Choose any $0 < \varepsilon < \operatorname{reach} X$ and let Z_ε denote the image of Z under the bi-Lipschitz mapping

$$\phi_\varepsilon : (x, m, y, n, W) \mapsto (x + \varepsilon m, y + \varepsilon n, W).$$

Z_ε is a subset of the $\mathcal{C}^1$ manifold $\partial X_\varepsilon \times \partial X_\varepsilon \times G(d, k+1)$ of dimension

$$q = 2d - 2 + (k+1)(d-1-k)$$

(here X_ε denotes the ε-parallel body to X) and the Lipschitz mapping $h \circ \phi_\varepsilon^{-1}$ vanishes on Z_ε. Further, for any $(x, m, y, n, W) \in Z$, $h \circ \phi_\varepsilon^{-1}$ is differentiable at $z = \phi_\varepsilon(x, m, y, n, W)$ and

$$\operatorname{rank} D(h \circ \phi_\varepsilon^{-1})(z) \geq 2d - 1 - k,$$

since $\operatorname{im} D(h \circ \phi_\varepsilon^{-1})(z)$ contains the vectors

$$\begin{array}{ll}
(W \wedge v, 0, 0), & v \in W^\perp, \\
(0, W \wedge v, 0), & v \in W^\perp, \\
((W \cap u^\perp) \wedge (x-y) \wedge m, (W \cap u^\perp) \wedge (x-y) \wedge n, W^\perp \wedge u), & u \in W.
\end{array}$$

Using [1, Lemma 6.1], we infer that Z_ε is countably $(p-1)$-rectifiable, since $p - 1 = q - (2d - 1 - k)$. Hence, $\mathcal{H}^p(Z_\varepsilon) = 0$ and, consequently, also $\mathcal{H}^p(Z) = 0$, which completes the proof. □

Remark 3. Using the notation from the proof of Lemma 1, we have that

$$\bar{f}(G) = \{(z, W) : z \in T_W X, W \in G(d, k+1)\}$$

is countably p-rectifiable and $\mathcal{H}^p$-mesurable. Consequently, applying [2, §3.2.22] for the projection $(z, W) \mapsto W$, we get that $T_W X$ is $(\mathcal{H}^k, k)$-rectifiable and $\mathcal{H}^k$-measurable for $\mathcal{H}^{(k+1)(d-1-k)}$-almost all $W \in G(d, k+1)$ and the function $W \mapsto \mathcal{H}^k(T_W X)$ is measurable.

The *absolute curvature measure* of X *of order* k can be defined by

$$\begin{aligned}
&\int g(x,m) C_k^{\mathrm{abs}}(X; d(x,m)) \\
&\quad = c_{d,k} \int_{G(d,k+1)} \int g \circ t_W^{-1}(z,m) C_k(T_W X; d(z,m)) \nu_{k+1}^d(dW),
\end{aligned} \tag{5}$$

where the measure $C_k(T_W X; \cdot)$ on $T_W X \times \mathbb{S}^k$ is defined by

$$C_k(T_W X; \cdot) = \int_{T_W X} \mathcal{H}^0(\cdot \cap \pi_W^{-1}\{z\}) \mathcal{H}^k(dz)$$

and

$$c_{d,k} = \binom{d-1}{k} \frac{\mathcal{H}^{(k+1)(d-1-k)}(G(d, k+1))}{\mathcal{H}^{d-1-k}(\mathbb{S}^{d-1-k}) \mathcal{H}^{k(d-1-k)}(G(d-1, k))}.$$

Note that $t_W^{-1}(z, m)$ is correctly defined for almost all (z, m) by Lemma 1.

Remark 4. This definition is equivalent to the first equality in [8, Equation (3)]. The second equality in [8, Equation (3)] holds as well if X is d-dimensional in the sense that there is no $(x, m) \in \operatorname{nor} X$ with $(x, -m) \in \operatorname{nor} X$. (For lower dimensional sets, the image points of the mapping f introduced in [8] have typically two pre-images.) [8, Equation (4)] holds in general since it has been derived from the first equality.

We shall say that a set $X \subseteq \mathbb{R}^d$ with positive reach has *locally bounded tangential projections* if for any compact subset $K \subseteq \mathbb{R}^d$ and $0 \le k \le d-2$,

$$\sup_{W \in G(d,k+1)} \mathcal{H}^k(T_W X \cap p_W K) < \infty. \tag{6}$$

Note that

$$\begin{aligned} C_{k-1}(T_W X, p_W K \times \mathbb{S}^k) &= \int_{T_W X \cap p_W K} \mathcal{H}^0(\pi_W^{-1}\{z\}) \mathcal{H}^k(dz) \\ &\le 2\mathcal{H}^k(p_W K \cap T_W X) \end{aligned}$$

for almost all $W \in G(d,k+1)$, since $\mathcal{H}^0(\pi_W^{-1}\{z\}) \le 2$ for $\mathcal{H}^k$-almost all $z \in T_W X$ by Lemma 1.

We present now a sufficient condition for (2). As an immediate consequence one obtains that any two convex sets satisfy (2) (cf. [7, Proposition 4.5]).

Lemma 2. *Let $X, Y \subseteq \mathbb{R}^d$ have positive reach and assume that $X, \alpha Y$ fulfil* (1) *for each $0 < \alpha \le 1$ and that for any $0 \le k \le d-1$ and bounded Borel subsets $A, B \subseteq \mathbb{R}^d$,*

$$\sup_{\alpha \in \mathcal{I}} \int_A C_k^{\mathrm{abs}}(X \cap (\alpha Y + z); B \times \mathbb{S}^{d-1}) \mathcal{L}^d(dz) < \infty$$

for some subset $\mathcal{I} \subseteq (0,\infty)$ of cardinality $d+1$. Then X, Y satisfy (2).

Remark 5. Note that, due to the positive homogeneity of mixed curvature measures (see [6, Proposition 1]), if X, Y satisfy (2) then also $X, \alpha Y$ satisfy (2) for any $\alpha > 0$.

Proof. We shall assume without loss of generality that X, Y are compact. Then we have by the assumption

$$c(k) := \sup_{\alpha \in \mathcal{I}} \int_{\mathbb{R}^d} C_k^{\mathrm{abs}}(X \cap \alpha(Y+z); \mathbb{R}^d \times \mathbb{S}^{d-1}) \mathcal{L}^d(dz) < \infty$$

for any $0 \le k \le d-1$. Since the absolute curvature measures majorize the total variations of the curvature measures (see [8]), we have for any measurable function h on $\mathbb{R}^{2d} \times \mathbb{S}^{d-1}$ with $|h| \le 1$

$$\left| \int h(z,x,u)\, C_k(X \cap (\alpha Y + z); d(x,u)) \mathcal{L}^d(dz) \right| \le c(k).$$

Assume, on the contrary, that (2) does not hold, and let s_0 be the least natural number such that $|C|_{r_0,s_0}(X,Y;\cdot)$ is unbounded for some $d - s_0 \le r_0 \le d-1$. Let $A \subseteq \mathbb{R}^{2d} \times \mathbb{S}^{d-1}$ be a Borel set which supports an unbounded (positive or negative) part of $|C|_{r_0,s_0}(X,Y;\cdot)$. Given $\varepsilon > 0$, denote

$$A_\varepsilon = A \cap f\Big(((\operatorname{nor} X \times \operatorname{nor} Y) \cap R^\varepsilon) \times [0,1]\Big),$$

where $R^\varepsilon = \{(x,m,y,n) \in (\mathbb{R}^d)^4 : \angle(m,n) \le \pi - \varepsilon\}$. Then all the mixed curvature measures are locally bounded on A^ε (cf. [7, Definition 4.1]) and the translative integral formula [6, Theorem 1] can be applied for the function $h^\varepsilon(z,x,u) = \mathbf{1}_{A^\varepsilon}(x, x-z, u)$ and $k = r_0 + s_0 - d$, yielding

$$\int h^\varepsilon(z,x,u)\, C_k(X \cap (Y+z); d(x,u)) \mathcal{L}^d(dz) = \sum_{s=k}^{d+k} C_{d+k-s,s}(X,Y;A^\varepsilon).$$

Replacing Y with αY and taking into account the positive homogeneity of mixed curvature measures, we get

$$\int h_\alpha^\varepsilon(z,x,u)\, C_k(X\cap(\alpha Y+z);d(x,u))\mathcal{L}^d(dz)=\sum_{s=k}^{d+k}\alpha^s C_{d+k-s,s}(X,Y;A^\varepsilon)$$

for the function $h_\alpha^\varepsilon(z,x,u)=\mathbf{1}_{A^\varepsilon}(x,\alpha^{-1}(x-z),u)$ with $|h_\alpha^\varepsilon|\le 1$. The left hand side is bounded by $c(k)$ for any $\varepsilon>0$ and $\alpha\in\mathcal{I}$ by the assumption. Since any polynomial in $\mathbb{R}^d$ is uniquely determined by its values at any $d+1$ different points, all the coefficients of the polynomial in α on the right hand side of the last equation have to be uniformly bounded in ε as well. This contradicts the fact that

$$\lim_{\varepsilon\to 0_+}|C_{r_0,s_0}(X,Y;A^\varepsilon)|=|C_{r_0,s_0}(X,Y;A)|=\infty. \qquad \square$$

Finally, we present a sufficient condition for a set with positive reach to satisfy (1) and (2) with any affine subspace of $\mathbb{R}^d$. Under this condition, the translative formulae from [5] can be applied.

Theorem 2. *Let $X\subseteq\mathbb{R}^d$ with positive reach have locally bounded tangential projections. Then, for any $0<j\le d$ and $L\in G(d,j)$, the pair X,L satisfies* (1) *and* (2).

Proof. First, we shall verify (1). We have to show that the Lebesgue measure of

$$N=\{z\in\mathbb{R}^d:\exists(x,m)\in\operatorname{nor}X, m\perp L, p_{L^\perp}x=p_{L^\perp}z\}$$

is zero. Note that $N=T_{L^\perp}X\oplus L$. The $\mathcal{H}^{d-j-1}$-measure of $T_{L^\perp}X$ is locally bounded by assumption. Hence, the $\mathcal{H}^{d-1}$-measure of N is locally bounded, which implies $\mathcal{L}^d(N)=0$. Note that (1) implies that $X\cap(L+z)$ has positive reach for $\mathcal{L}^{d-j}$-almost all shifts $z\in L^\perp$ (cf. [1, Theorem 4.10], [6, Proof of Theorem 1]).

To verify (2), we shall assume without loss of generality that X is compact. Due to Lemma 2, it suffices to show that for any $0\le k\le j-1$,

$$\int C_k^{\mathrm{abs}}(X\cap(L+z);\mathbb{R}^d\times\mathbb{S}^{d-1})\mathcal{L}^{d-j}(dz)<\infty.$$

Due to the considerations above, it is enough to integrate over the complement of $T_{L^\perp}X$ in $L^\perp$, hence we can neglect the points $(x,m)\in\operatorname{nor}X$ with $m\perp L$. Note that the total (absolute) curvature measures are independent of the dimension of the embedding space (cf. [8, Theorem 2]). Hence, we can treat the sets $X\cap(L+z)$ as subsets of the j-dimensional space L and we have by definition

$$\begin{aligned}&C_k^{\mathrm{abs}}(X\cap(L+z);L\times\mathbb{S}^{j-1})\\&\qquad\le\ 2c_{j,k}\int_{G(j,k+1)}\mathcal{H}^k(T_W(X\cap(L+z)))\nu_{k+1}^j(dW)\end{aligned}$$

for $\mathcal{L}^{d-j}$-almost all $z\in L^\perp$ (for the existence of the last integral, see Remark 3). Let $W\in G(j,k+1)$ be fixed and consider the subspace $\widetilde{W}=W+L^\perp\in G(d,d+k-j+1)$. For any $z\in L^\perp\setminus T_{L^\perp}X$ we have

$$T_W(X\cap(L+z))=T_{\widetilde{W}}X\cap p_{L^\perp}^{-1}\{z\}.$$

Applying [4, §7.7] or [2, §2.10.25] for the set $T_{\widetilde{W}}X$ and projection $z \mapsto p_{L^\perp}z$, we get

$$\int_{L^\perp} \mathcal{H}^k(T_W(X \cap (L+z)))\mathcal{L}^{d-j}(dz) \leq \text{const} \cdot \mathcal{H}^{d+k-j}(T_{\widetilde{W}}X),$$

where the last term is bounded uniformly in W by the assumption and the assertion follows. □

Remark 6. Theorems 1, 2 hold for locally finite unions of sets with positive reach (see [7]) as well. For these sets, an additional multiplicity term $i_X(x,m)i_Y(y,n)$ appears under the integral in (3), but both index functions i_X, i_Y are locally uniformly bounded.

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Received July 18, 2000; revised version November 13, 2000