Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 43 (2002), No. 1, 9-26.

Picard Groups of Deligne-Lusztig Varieties – with a View toward Higher Codimensions

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Abstract. For a Deligne-Lusztig variety $\bar{X}(w)$ arising from one of the classical (possibly twisted) groups, we show that the Picard group of $\bar{X}(w)$ is generated by the finitely many Deligne-Lusztig subvarieties of $\bar{X}(w)$. It is conjectured that this more generally should hold in any codimension, and a good deal of supporting evidence for this claim is presented.

MSC 2000: 14M15, 14C25
Keywords: algebraic geometry, algebraic groups, Deligne-Lusztig varieties, Picard (and Chow) groups

Contents

1. Deligne-Lusztig varieties: definitions and preliminaries

Let (G, F) be a connected reductive algebraic group over an algebraically closed field k of positive characteristic p, equipped with an $\mathbb{F}_q$-structure coming from a Frobenius morphism $F : G \to G$. Let $L : G \to G$ be the corresponding Lang map taking an element $g \in G$ to

$g^{-1}F(g)$. By the Lang-Steinberg Theorem [1, Theorem 16.3] this morphism of varieties is surjective with finite fibres. From this result it follows that, by conjugacy of Borel subgroups, there exists an F-stable Borel subgroup B. Let $\pi : G \to G/B := X$ denote the quotient. There are then (with a slight abuse of notation) natural endomorphisms $F : W \to W$ and $F : X \to X$ of the Weyl group of G and the variety X of Borel subgroups of G. Let W be generated by the simple reflections $s_1, \dots, s_n$ and let $l(\cdot)$ be the length function with respect to these generators.

For an algebraic variety Y we let $\mathrm{A}_i(Y)$ denote the Chow group of cycles of dimension i modulo rational equivalence. We write $\mathrm{A}_i(Y)_{\mathbb{Q}}$ for $\mathrm{A}_i(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$. When Y is non-singular we shall write $\mathrm{CH}^*(Y)$ for the Chow ring of Y. A general reference for these notions is [6].

If A is an abelian group we shall for a prime ℓ denote by $A_{\ell'}$ the sub-group of A consisting of elements of order not divisible by ℓ.

Definition 1. *Fix an element w in the Weyl group W, and let $w = s_{i_1} \cdot \ldots \cdot s_{i_r}$ be a reduced expression of w. Call w a Coxeter element if there in this expression occurs exactly one s_i from each of the orbits of F on $\{s_1, \dots, s_n\}$. Denote by δ the order of F on this set.*

1. *The Deligne-Lusztig variety$X(w)$ is defined as the image of $L^{-1}(B\dot{w}B)$ in G/B. That is,*
$$X(w) = \pi(L^{-1}(B\dot{w}B)).$$
2. *Define the closed subvariety of X^{r+1}*
$$\bar{X}(s_{i_1}, \dots, s_{i_r}) = \big\{(g_0B, \dots, g_rB) \in X^{r+1} : \\ g_k^{-1}g_{k+1} \in B \cup Bs_{i_{k+1}}B \text{ for } 0 \le k < r,\ g_r^{-1}F(g_0) \in B\big\}.$$
In those cases where there is a unique product $s_{i_1} \cdot \ldots \cdot s_{i_r}$ such that $s_{i_1} \cdot \ldots \cdot s_{i_r} = w$ we shall write $\bar{X}(w)$ for the variety $\bar{X}(s_{i_1}, \dots, s_{i_r})$.
For any subset $\{s_{j_1}, \dots, s_{j_m}\} \subset \{s_{i_1}, \dots, s_{i_r}\}$, $\bar{X}(s_{j_1}, \dots, s_{j_m})$ defines in a natural way a closed subvariety of $\bar{X}(s_{i_1}, \dots, s_{i_r})$. In particular there are divisors
$$D_j = \bar{X}(s_{i_1}, \dots, \hat{s}_{i_j}, \dots, s_{i_r});\ j = 1, \dots, r.$$
3. *When G is semi-simple with connected Dynkin diagram $\mathcal{D}$ (with numbering of nodes and their associated simple reflections as in e.g.* [14, p. 58]*), there is a (unique) natural choice of Coxeter element: let $w = s_1 \cdot s_2 \cdot \ldots \cdot s_r$ with r maximal (under the condition that s_r is not in the F-orbit of any of the previous s_i, $i < r$; in* [15, p. 106] *the various r are listed). When choosing this particular Coxeter element, we shall refer to $X(w)$ (or $\bar{X}(w)$) as being of standard type.*
4. *Say that $\bar{X}(w)$ is of classical type if w is a Coxeter element for one of the classical groups: A_n, ${}^2\mathrm{A}_{2n}$, ${}^2\mathrm{A}_{2n+1}$, B_n, C_n, D_n or ${}^2\mathrm{D}_n$.*
5. *For $w_1, w_2 \in W$ we shall say that w_1 and w_2 are F-conjugate if there exists $w' \in W$ such that $w_2 = w'w_1F(w')^{-1}$. We note that w and $F(w)$ are F-conjugate for any $w \in W$ (take w' equal to w^{-1}).*

Since the morphism L is flat, it is open, hence $\overline{L^{-1}(B\dot{w}B)} = L^{-1}\big(\overline{B\dot{w}B}\big)$. So $X(w)$ is nonsingular of dimension n and the closure of $X(w)$ in X is given by the disjoint union
$$\overline{X(w)} = \bigcup_{w' \le w} X(w'),$$

where as usual $\leq$ is the Bruhat order in W. This closure is usually singular whenever the Schubert variety $X_w = \overline{B\dot{w}B}/B$ is. But since the open subset

$$\left\{(g_0B, \dots, g_rB) \in X^{r+1} : g_k^{-1}g_{k+1} \in Bs_{i_{k+1}}B,\ 0 \leq k < r,\ g_r^{-1}F(g_0) \in B\right\}$$

of the smooth projective variety $\bar{X}(w)$ maps isomorphically onto $X(w)$ under projection to the first factor [5, 9.10], we have a good compactification of $X(w)$. In fact the complement of $X(w)$ in $\bar{X}(w)$, which is easily seen to be the union of the divisors D_j defined above, is a divisor with normal crossings [5, 9.11].

If w is a Coxeter element, then $X(w)$ and $\bar{X}(w)$ are irreducible [15, Proposition (4.8)] and, in fact, $\overline{X(w)}$ is isomorphic to $\bar{X}(w)$, hence non-singular (see [12, Chapter 2]).

Remark 1. Suppose $\bar{X}(w)$ is of type A_n. Let $w' \leq w$. Then each irreducible component of $\bar{X}(w')$ is a product of Deligne-Lusztig varieties also of type A_n. For example: In $\bar{X}(s_1s_2s_3)$, the divisors D_1 and D_3 are disjoint unions of components of type A_2 and D_2 is a disjoint union of components of type $\mathrm{A}_1 \times \mathrm{A}_1$.

Similarly, when $\bar{X}(w)$ is of type ${}^2\mathrm{A}_n$, the divisor D_i is a disjoint union of Deligne-Lusztig varieties of type $\mathrm{A}_{i-1} \times {}^2\mathrm{A}_{n-i}$. The same remarks apply to any other Deligne-Lusztig variety of classical type. That is, if $\bar{X}(w)$ is of classical type, then so are the irreducible components of the divisors D_i (or, more generally, of any Deligne-Lusztig subvariety of $\bar{X}(w)$).

Remark 2. Groups G^F arising as the fixed-points of a Frobenius morphism acting on a reductive, connected linear algebraic group are called *finite groups of Lie type*. It was the search for a unified description of the representation theory of these groups that led Deligne and Lusztig to the construction of Deligne-Lusztig varieties [5]. (G^F acts on $X(w)$ as a group of automorphisms inducing an action on the ℓ-adic cohomology vector spaces of $\bar{X}(w)$. See also [8].)

More recently, the study of Deligne-Lusztig varieties has been motivated by the fact that they have many rational points over their field of definition, making them well-suited for constructing long error-correcting codes (cf. [13] and the references in that paper).

Definition 2. *Introduce the following notation:*

$$\mathfrak{I} = \left\{ i \ : \begin{array}{c} \textit{some connected component of the Dynkin} \\ \textit{diagram corresponding to } D_i \textit{ occurs as a subgraph} \\ \textit{of the Dynkin diagram corresponding to } D_1 \end{array} \right\}.$$

Remark 3. The motivation for defining $\mathfrak{I}$ is the following: Suppose the subgraph of the Dynkin diagram defined by a boundary divisor D consists of the components $\mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_3$ (since we only 'remove' δ nodes we can only cut $\mathfrak{D}$ into 3 pieces, at the most). Now, if e.g. $\mathfrak{D}_2$ is a subgraph of the Dynkin diagram defined by D_1, this means geometrically that D is the direct product of the Deligne-Lusztig subvariety $D \cap D_1$ of D_1, with the other Deligne-Lusztig varieties corresponding to the diagrams $\mathfrak{D}_1$ and $\mathfrak{D}_3$. So, in particular, if D_1 is contracted to points, then also D_i drops in dimension for all $i \in \mathfrak{I}$.

Some examples of how the index set $\mathfrak{I}$ looks like, are listed in Table 1.

Lemma 1. *Assume G is semi-simple with connected Dynkin diagram, not of type ${}^3\mathrm{D}_4$. Suppose w and w' are two different Coxeter elements in W. Let $\bar{X}(w)$ and $\bar{X}(w')$ be the corresponding Deligne-Lusztig varieties. Then $\mathrm{A}_i(\bar{X}(w))_{p'} \simeq \mathrm{A}_i(\bar{X}(w'))_{p'}$ for all i.*

type of $\bar{X}(w)$	$\mathfrak{I}$ $(n \geq 2)$
A_n	$\{1, 2, \dots, n-1\}$
${}^2\mathrm{A}_{2n-1}$	$\{1, 2, \dots, n-1\}$
${}^2\mathrm{A}_{2n}$	$\{1, 2, \dots, n-1\}$
${}^2\mathrm{D}_n$	$\{1, 2, \dots, n-3\}$

Table 1. The index set $\mathfrak{I}$ for some standard Deligne-Lusztig varieties.

Proof. Let us first consider the case where $w' = F(w)$. Since the automorphism $F^{\delta} : \bar{X}(w) \to \bar{X}(w)$ induces multiplication by a power of q on $\mathrm{A}_i(\bar{X}(w))$ [6, Example 1.7.4], each of the homomorphisms in the composite (we have $\delta = 2$ since $F(w) = w' \neq w$)

$$\mathrm{A}_i(\bar{X}(w)) \xrightarrow{F_*} \mathrm{A}_i(\bar{X}(w')) \xrightarrow{F_*} \mathrm{A}_i(\bar{X}(w))$$

must be isomorphisms away from elements of order divisible by p.

By [15, (1.8) Lemma], the only other cases we need to consider are those where w is on the form $w = w_1 w_2$ and then $w' = w_2 F(w_1)$. The proof now follows the lines of the proof of [5, Theorem 1.6, case 1]:

For any $P = (g_0 B, g_1 B, \dots, g_{l(w_1)} B, \dots, F(g_0) B) \in \bar{X}(w)$ we have that

$$g_k^{-1} g_{k+1} \in B \cup B s_{i_{k+1}} B \text{ for } 0 \leq k < l(w_1)$$
$$g_k^{-1} g_{k+1} \in B \cup B s_{i_{k+1}} B \text{ for } l(w_1) \leq k < l(w), \quad \text{with } g_{l(w)} = F(g_0).$$

Hence assigning

$$\sigma(P) := (g_{l(w_1)} B, \dots, F(g_0) B, F(g_1) B, \dots, F(g_{l(w_1)}) B) \in \bar{X}(w')$$

defines a morphism $\sigma : \bar{X}(w) \to \bar{X}(w')$. In exactly the same way, we get a morphism $\tau : \bar{X}(w') \to \bar{X}(F(w))$. It follows that $F = \tau \circ \sigma$. Arguing as in the special case, it follows that $\tau_* : \mathrm{A}_i\big(\bar{X}(w')\big)_{p'} \to \mathrm{A}_i\big(\bar{X}(F(w))\big)_{p'}$ must be surjective. The assertion now follows by symmetry. $\square$

Remark 4. Since Lusztig has shown [15] that Deligne-Lusztig varieties coming from F-conjugate Coxeter elements have the same number of rational points [15, (1.10) Proposition], hence the same Zeta-function and Betti-numbers, the above lemma is only a natural parallel.

2. Picard groups of Deligne-Lusztig varieties of classical type

In this section we will examine the ${}^2\mathrm{A}_n$, B_n, C_n, D_n and ${}^2\mathrm{D}_n$ cases. (We shall postpone the description of the A_n-case to the next section.)

First we give (following [5, (2.1)] and [16]) an explicit description of the linear algebraic groups and their F-structures. To this end, let V be an N-dimensional vector space ($N \geq 2$) over k equipped with a Frobenius morphism $F_V : V \to V$. Assume furthermore that V comes equipped with a form of one of the following kinds:

(O): Let $char(k) \neq 2$ and let $Q : V \to k$ be a non-singular quadratic form defined over $\mathbb{F}_q$. That is, $Q(F_V(x)) = Q(x)^q$ for any $x \in V$. Define the inner product

$$\langle x, y \rangle_{\mathrm{O}} = Q(x+y) - Q(x) - Q(y)$$

on V. For N even, we will distinguish between the cases where Q is *split* and *non-split* (Q is split if F_V leaves stable some subspace $V' \subseteq V$ satisfying that $V' \subseteq V'^{\perp}$ and $Q|_{V'} = 0$ and that V' is maximal with property).
To be able to do explicit calculations, we fix a standard basis for V and let $Q(x)$ be defined as follows (with respect to the chosen basis):

$$Q(x) = \begin{cases} \sum_{i=1}^n x_i x_{i+n} & N = 2n \\ x_N^2 + \sum_{i=1}^n x_i x_{i+n} & N = 2n+1. \end{cases}$$

With this choice, F_V acts as follows:

$$F_V(x) = \begin{cases} (x_{n+1}^q, \ldots, x_N^q, x_1^q, \ldots, x_n^q) & N = 2n \\ (x_1^q, \ldots, x_N^q) & N = 2n+1. \end{cases}$$

(Sp): Assume N is even, $N = 2n$. Let $\langle\ ,\ \rangle_{\mathrm{Sp}} : V \times V \to k$ be a non-singular symplectic form defined over $\mathbb{F}_q$, that is, $\langle F_V(x), F_V(y)\rangle_{\mathrm{Sp}} = \langle x, y\rangle_{\mathrm{Sp}}^q$ for any $x, y \in V$.
In the chosen basis, F_V takes $(x_1, \ldots, x_N)$ to $(x_1^q, \ldots, x_N^q)$ and we may write the form as

$$\langle x, y\rangle_{\mathrm{Sp}} = \sum_{i=1}^n x_i y_{i+n} - x_{i+n} y_i.$$

(U): Here our base field is $\mathbb{F}_{q^2}$, that is, of square order. Let $\langle\ ,\ \rangle_{\mathrm{U}} : V \times V \to k$ be a non-singular sesquilinear form with respect to the automorphism $\lambda \mapsto \lambda^q$ of $\mathbb{F}_{q^2}$. That is, $\langle \lambda x, y\rangle_{\mathrm{U}} = \lambda\langle x, y\rangle_{\mathrm{U}}$ and $\langle x, \lambda y\rangle_{\mathrm{U}} = \lambda^q\langle x, y\rangle_{\mathrm{U}}$ for $x, y \in V, \lambda \in k$. Furthermore assume that

$$\langle F_V(x), y\rangle_{\mathrm{U}} = \langle y, x\rangle_{\mathrm{U}}^q$$

for $x, y \in V$.
In the chosen basis F_V takes $(x_1, \ldots, x_N)$ to $(x_1^{q^2}, \ldots, x_N^{q^2})$ and we may write the form as

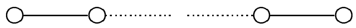
$$\langle x, y\rangle_{\mathrm{U}} = \begin{cases} \sum_{i=1}^n x_i y_{i+n}^q + x_{i+n} y_i^q & N = 2m \\ x_m y_m^q + \sum_{i=1}^n x_i y_{i+n}^q + x_{i+n} y_i^q & N = 2m-1 \end{cases}$$

In the following we shall omit the subscripts indicating whether the form is symplectic, orthogonal or unitary when we wish to speak of any of these types of forms.

We may now give the explicit description of the classical linear algebraic groups with their Frobenius morphism $F : G \to G$. For later use we define in each of the non-SL cases an integer $a_0(V)$, depending on V and $\langle\ ,\ \rangle$. Furthermore, if $W \subseteq V$ is an F_V-stable subspace of V, it inherits the form $\langle\ ,\ \rangle$ and it then also makes sense to speak of $a_0(W)$.[1] If $\mathbb{P}(W) = E \subseteq \mathbb{P}(V)$ we shall also write $a_0(E)$ for $a_0(W)$. For clarity of notation we set $a_0(W) = 0$ whenever $\dim(W) \leq 1$.

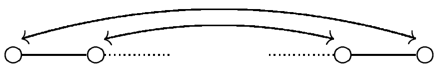
[1] In the symplectic case we must, strictly speaking, assume that the dimension of the subspace is even. To include the odd-dimensional case as well, we set (in the symplectic case): $a_0(W) = a_0(W')$, where $W' \subseteq V$ contains W and is minimal of even dimension.

(SL): We have $G = \mathrm{SL}_N(k) = \{g \in \mathrm{GL}_N(k) : \det(g) = 1\}$. The Frobenius morphism F acts on G by raising each entry of the matrix g to the q'th power, that is, $F(g) = g \circ F_V$. The corresponding Dynkin diagram is

A_{N-1} 

($N-1$ nodes, numbered from left to right).

(U): We have $G = \mathrm{SL}_N(k)$. Let $F' : G \to G$ be defined by $\langle F'(g)x, gy\rangle_{\mathrm{U}} = \langle x, y\rangle_{\mathrm{U}}$ for any $x, y \in V$. For any $g \in G$ we have $F'^2(g) = g \circ F_V$. This gives G an $\mathbb{F}_q$-rational structure. The corresponding Dynkin diagram is

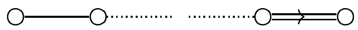
${}^2\mathrm{A}_{N-1}$ 

($N-1$ nodes, numbered from left to right). Define $a_0(V)$ by $N = 2(a_0(V)+1)$ for N even, and by $N = 2a_0(V)+1$ for N odd.

(O), $\boldsymbol{N = 2n+1}$: We have

$$\begin{aligned} G &= \mathrm{SO}_N(k) \\ &= \{g \in \mathrm{GL}_N(k) : \langle g(x), g(y)\rangle_{\mathrm{O}} = \langle x, y\rangle_{\mathrm{O}} \text{ for any } x, y \in V\}. \end{aligned}$$

Let F act on G by the rule: $F(g)F_V(x) = F_V(gx)$. The corresponding Dynkin diagram is

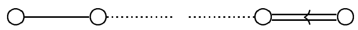
B_n 

(n nodes, numbered from left to right, $n \geq 2$). Set $a_0(V) = n$.

(Sp), $\boldsymbol{N = 2n}$: We have

$$\begin{aligned} G &= \mathrm{Sp}_n(k) \\ &= \{g \in \mathrm{GL}_N(k) : \langle g(x), g(y)\rangle_{\mathrm{Sp}} = \langle x, y\rangle_{\mathrm{Sp}} \text{ for any } x, y \in V\}. \end{aligned}$$

Let F act on G by the rule: $F(g)F_V(x) = F_V(gx)$. The corresponding Dynkin diagram is

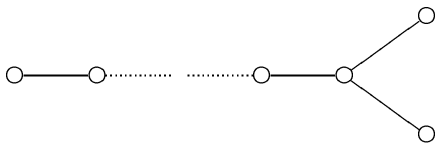
C_n 

(n nodes, numbered from left to right, $n \geq 3$). Set $a_0(V) = n$.

(O), $\boldsymbol{N = 2n}$, $\boldsymbol{Q}$ split: We have

$$\begin{aligned} G &= \mathrm{SO}_N(k) \\ &= \{g \in \mathrm{SL}_N(k) : \langle g(x), g(y)\rangle_{\mathrm{O}} = \langle x, y\rangle_{\mathrm{O}} \text{ for any } x, y \in V\}. \end{aligned}$$

Let F act on G by the rule: $F(g)F_V(x) = F_V(gx)$. The corresponding Dynkin diagram is

D_n 

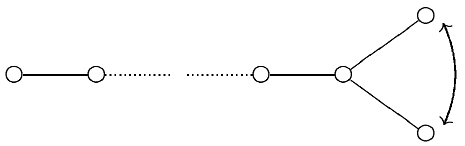
(n nodes, numbered from left to right (the two right-most being numbered top-down), $n \geq 4$). Set $a_0(V) = n-1$.

(O), $N = 2n$, Q non-split: We have

$$\begin{aligned} G &= \mathrm{SO}_N(k) \\ &= \{g \in \mathrm{GL}_N(k) : \langle g(x), g(y)\rangle_{\mathrm{O}} = \langle x, y\rangle_{\mathrm{O}} \text{ for any } x, y \in V\}. \end{aligned}$$

Let F act on G by the rule: $F(g)F_V(x) = F_V(gx)$. The corresponding Dynkin diagram is

${}^2\mathrm{D}_n$



(n nodes, numbered from left to right (the two right-most being numbered top-down), $n \geq 4$). Set $a_0(V) = n$.

Lemma 2. *Let $\bar{X}(w)$ be a standard Deligne-Lusztig variety. Let P be the parabolic subgroup generated by B together with the double cosets $Bs_2B, Bs_3B, \ldots, Bs_nB$. Then the map*

$$\pi : (G/B)^{l(w)+1} \to G/P$$

(projection to the first factor, followed by the quotient map) sends the divisor $D_1 \subseteq \bar{X}(w)$ to the points $G^F.P$. Hence, by Remark 3, all divisors D_i, $i \in \mathfrak{I}$ are mapped to subvarieties of codimension at least 2.

Proof. Since $\bar{X}(w)$ may be described as

$$\begin{aligned} \bar{X}(w) = \{&(g_0B, \ldots, g_rB) \in (G/B)^{r+1} : \\ &g_r^{-1}F(g_0) \in B;\ g_i^{-1}g_{i+1} \in \overline{Bs_{i+1}B},\ i = 0, 1, \ldots, r-1\}, \end{aligned} \tag{1}$$

it follows that D_1 consists of those $(g_0B, \ldots, g_rB) \in \bar{X}(w)$ such that $g_0^{-1}g_1 \in B$. But then

$$g_0^{-1}F(g_0) = (g_0^{-1}g_1)(g_1^{-1}g_2)\ldots(g_{r-1}^{-1}g_r)(g_r^{-1}F(g_0))$$

is a product of elements from P. Hence D_1 is mapped into the (finitely many) points gP of G/P satisfying $g^{-1}F(g) \in P$. □

To avoid confusion, let us recapitulate [9, p. 119] the following:

Definition 3. *A closed subscheme Y of $\mathbb{P}^N$ of codimension d is called an ideal-theoretic (or strict) complete intersection if Y is the scheme-theoretic intersection of d hyper-surfaces $H_1, \ldots, H_d$ in $\mathbb{P}^N$. In algebraic terms, if we let the hyper-surfaces be defined by the homogeneous polynomials $f_1, \ldots, f_d$, then $Y = \mathrm{Proj}(k[X_0, \ldots, X_N]/I)$ with $I = (f_1, \ldots, f_d)$.*

A closed subset $Y \subset \mathbb{P}^N$ is said to be a set-theoretic complete intersection if it is the support of an ideal-theoretic complete intersection.

Theorem 3. *Let $\bar{X}(w)$ be a standard Deligne-Lusztig variety of type ${}^2\mathrm{A}_n$, B_n, C_n, D_n or ${}^2\mathrm{D}_n$. Assume char$(k) \neq 2$ in the orthogonal cases. Let P be as in Lemma 2 and let*

$$\pi : (G/B)^{l(w)+1} \to G/P$$

be the projection. Denote by L^e the e-dimensional linear subspace of $\mathbb{P}^{N-1}$ obtained by setting the $N - 1 - e$ last coordinates equal to zero.

	${}^2\mathrm{A}_{2(m-1)}$	${}^2\mathrm{A}_{2m-1}$	B_n	C_n	D_n	${}^2\mathrm{D}_n$
$\dim(\mathbb{P}(V)) = N-1$	$2(m-1)$	$2m-1$	$2n$	$2n-1$	$2n-1$	$2n-1$
$\dim(\bar{X}(w)) = \dim(Z)$	$m-1$	m	n	n	n	$n-1$
$a_0(V)$	$m-1$	$m-1$	n	n	$n-1$	n
#equations defining Z	$m-1$	$m-1$	n	$n-1$	$n-1$	n
form defining H_0	$\sum_j X_j^{q+1}$	$\sum_j X_j^{q+1}$	$\sum_j X_j^2$	none	$\sum_j X_j^2$	$\sum_j X_j^2$

Table 2. Data relating to Deligne-Lusztig varieties of classical type. The condition $\langle x, x\rangle = 0$ is always true in the symplectic case, whence the difference in the C_n-case between $a_0(V)$ and the number of defining equations. We see that in all cases, Z has the 'correct' codimension in $\mathbb{P}(V)$. The equations for the hypersurfaces (5) can be transformed to an (equivalent) diagonal form via a projective transformation (possibly with coefficients in a larger field). This allows us to use the common expression $\sum_j X_j^{q^{i\delta+1}+1} = 0$ for all hypersurfaces H_i $(i > 0)$ and those given in the table for H_0.

1. *The image $Z = \pi(\bar{X}(w))$ is a normal, strict complete intersection. In the unitary and orthogonal cases the singular locus of Z, Z_{sing}, consists of the finitely many G^F-translates of the closed subscheme $Z \cap L^{a_0(V)-1}$. Hence*
$$\mathit{codim}(Z_{sing}, Z) = N + 1 - 2a_0(V) + a_0(L^{a_0(V)-1}). \tag{2}$$
In the symplectic case Z_{sing} consists of the G^F-translates of the closed subscheme $Z \cap L^{a_0(V)-2}$, and the formula (2) becomes
$$\mathrm{codim}(Z_{\mathrm{sing}}, Z) = 2 + a_0(L^{a_0(V)-2}). \tag{3}$$
2. *For $\mathit{codim}(Z_{sing}, Z) \geq 4$, $\mathrm{Pic}(Z) = \mathbb{Z}$ and consequently*
$$\begin{aligned}\mathrm{Pic}(\bar{X}(w)) = \mathbb{Z}[\pi^*H] \oplus \mathbb{Z}[\{[V] : V \textit{ component of } D_1\}]\\ \oplus\, j_*\,\mathrm{A}_{l(w)-1}\big(\cup_{i\in\mathfrak{I}-\{1\}} D_i\big)\end{aligned} \tag{4}$$
where H is the hyperplane section of Z and j is the obvious inclusion.
3. *For any Coxeter element w' we have*
$$\mathrm{Pic}(\bar{X}(w'))_{p'} \simeq \mathrm{Pic}(\bar{X}(w))_{p'}.$$

Proof. First we will handle the non-${}^2\mathrm{D}_n$ case. From Lemma 2 it follows that π contracts the divisor D_1 mapping it to the $\mathbb{F}_{q^\delta}$-rational points of $G/P \subseteq \mathbb{P}(V) \simeq \mathbb{P}^{N-1}$ (this inclusion is an equality in the non-orthogonal cases). Consider the hypersurfaces in $\mathbb{P}^{N-1}$:
$$H_i = \{(x_1 : x_2 : \cdots : x_N) \in \mathbb{P}^{N-1} : \langle x, F_V^i(x)\rangle = 0\} \tag{5}$$
where $i = 0, 1, \ldots, a_0(V) - 1$ (with $a_0(V)$ defined as above) and
$$H_0 = \{(x_1 : x_2 : \cdots : x_N) \in \mathbb{P}^{N-1} : Q(x) = 0\} \simeq G/P$$
in the orthogonal cases. Note that in the C_n-case, $H_0 = \mathbb{P}(V)$ since $\langle\ ,\ \rangle_{\mathrm{Sp}}$ is alternating.

Lusztig shows [16, p. 444–445] (see also [19]) that Z equals the support of the scheme-theoretic complete intersection $Z' = \cap_{i=0}^{a_0(V)-1} H_i$, with $X(w)$ mapping isomorphically onto the open subset $\langle x, F_V^{a_0(V)}(x)\rangle \neq 0$ of Z. We claim that Z' and Z are equal *as schemes*; that is, if we let $f_i \in k[X_1, \ldots, X_N]$ denote the form defining the hypersurface H_i (see Table 2), then the ideal $(f_0, \ldots, f_{a_0(V)-1})$ is prime. Indeed, Z' is a complete intersection and is therefore Cohen-Macaulay. So the problem amounts to showing that Z' is regular in codimension 1 (by Serre's Criterion for normality [10, Proposition II.8.23]). So suppose $P = (x_1 : x_2 : \cdots : x_N) \in Z'$ is a singular point. This means that the rank of the Jacobian $\left(\frac{\partial f_i}{\partial X_j}\right)$ is not maximal in the point P.

Let us interpret what this means in the unitary case: In that case $P = (x_1 : x_2 : \cdots : x_N) \in Z'$ is singular if and only if

$$\operatorname{rank}\begin{pmatrix} x_1^q & x_1^{q^{\delta+1}} & \cdots & \cdots & x_1^{q^{(a_0(V)-1)\delta+1}} \\ x_2^q & x_2^{q^{\delta+1}} & \cdots & \cdots & x_2^{q^{(a_0(V)-1)\delta+1}} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ x_N^q & x_N^{q^{\delta+1}} & \cdots & \cdots & x_N^{q^{(a_0(V)-1)\delta+1}} \end{pmatrix} < a_0(V). \tag{6}$$

In other words, $P = (x_1 : x_2 : \cdots : x_N) \in Z'$ is a singular point only if the iterates of $(x_1^q : x_2^q : \cdots : x_N^q)$ under F_V are contained in an F_V-stable linear subspace of V, of dimension $a_0(V) - 1$ over k. Hence the singular locus of Z' is contained in the union (in $\mathbb{P}^{N-1}$) of all F_V-stable linear subspaces of (projective) dimension $a_0(V) - 1$. One such is $L^{a_0(V)-1}$, and all others are conjugated to this one under the action of G^F.

Conversely, if $P \in Z'$ is contained in an F_V-stable subspace of dimension $a_0(V) - 1$ or less, it follows that P is a singular point on Z'. So, as the elements of G^F act on Z' as automorphisms, we have

$$Z'_{\text{sing}} = \bigcup_{g \in G^F} (g.L^{a_0(V)-1}) \cap Z' = \bigcup_{g \in G^F} g.(L^{a_0(V)-1} \cap Z').$$

Now, scheme-theoretically,

$$\begin{aligned} L^{a_0(V)-1} \cap Z' = \Big\{ x \in \mathbb{P}^{N-1} : \sum_{j=1}^{a_0(V)} x_j^{q^{i\delta+1}+1} = 0\,;\ i = 0, 1, \ldots, a_0(V) - 1\,; \\ x_{a_0(V)+1} = x_{a_0(V)+2} = \ldots = x_N = 0 \Big\}. \end{aligned}$$

So $L^{a_0(V)-1} \cap Z'$ is the image of the natural embedding into $\mathbb{P}^{N-1}$ of the closed subscheme

$$\Big\{ x \in \mathbb{P}^{a_0(V)-1} : \sum_{j=1}^{a_0(V)} x_j^{q^{i\delta+1}+1} = 0\,;\ i = 0, 1, \ldots, a_0(V) - 1 \Big\}.$$

But this is nothing but the boundary divisor on the complete intersection in $\mathbb{P}^{a_0(V)-1}$ of the same type as Z', of lower dimension (about the half of that of Z'). More precisely, by [16, 3. Lemma], this scheme is a complete intersection (normal by induction) of codimension $a_0(L^{a_0(V)-1})$ in $\mathbb{P}^{a_0(V)-1}$ (and the boundary divisor on it, cut out by the many extra equations, is then of codimension $a_0(L^{a_0(V)-1}) + 1$).

Similarly, the codimension of Z' in $\mathbb{P}^{N-1}$ is $a_0(V)$. Hence

$$\begin{aligned}\operatorname{codim}(Z'_{\text{sing}}, Z') &= ((N-1) - a_0(V)) - ((a_0(V) - 1) - (a_0(L^{a_0(V)-1}) + 1)) \\ &= N + 1 - 2a_0(V) + a_0(L^{a_0(V)-1})\end{aligned}$$

In the orthogonal cases similar arguments apply since any one of the hyper-surfaces H_i $(i > 0)$ intersects the quadric hyper-surface $H_0 \simeq G/P$ transversely, whence the codimensions are left unchanged.

In the symplectic case there is one equation less defining Z (cf. Table 2). So the singular locus is contained in the G^F-translates of the closed subscheme $Z \cap L^{a_0(V)-2}$. One then calculates the singular locus as above. (The more explicit formula comes from using Table 2, and similar expressions can of course be extracted for the unitary and orthogonal cases.)

In conclusion, it follows that Z' is regular in codimension one (the singularities being of codimension at least one plus half the dimension of Z') and therefore Z and Z' are equal also as schemes. This also shows that Z is normal [10, Proposition II.8.23]. Assertion 1 of the theorem is now proved.

As for the claim 2, it follows from Corollary 14 that, under the assumption $\operatorname{codim}(Z_{\text{sing}}, Z) \geq 4$,

$$\operatorname{Pic}(Z) = \mathrm{A}_{l(w)-1}(Z - Z_{\text{sing}}) = \mathrm{A}_{l(w)-1}(Z).$$

As $l(w) \geq 3$, we have $\operatorname{Pic}(Z) = \mathbb{Z}$, by the Lefschetz theorem for Picard groups [7, Exposé XII, Corollaire 3.7]. The formula (4) then follows from Lemma 11 and Lemma 12.

The ${}^2\mathrm{D}_n$ case is quite similar. Again we have a birational morphism

$$\pi : \bar{X}(w) \cup \bar{X}(F(w)) \to Z$$

contracting the divisor $D_1 \cup F(D_1)$ to points and mapping the locally closed subvariety $X(w) \cup X(F(w))$ of X isomorphically onto an open subset U of the complete intersection Z. Arguing as above one arrives at the conclusion that, under the given assumptions, the Picard group of Z equals the class group of Z. Then, by symmetry, it follows from [6, 1.3.1 (c)] that (4) also holds in this case.

Finally, the last assertion follows from Lemma 1. □

Corollary 4. *Under the same assumptions as in the above theorem, we have*

$$\operatorname{Pic}(X(w)) = \mathrm{A}_{l(w)-1}(X(w)) = \mathbb{Z}/m\mathbb{Z} \qquad ; \; m = q^{a_0(V)\delta+1} + 1.$$

Proof. As we noted in the proof of the theorem, $X(w)$ (or $X(w) \cup X(F(w))$ in the ${}^2\mathrm{D}_n$ case) is isomorphic to the complement U (in Z) of the hypersurface $H_{a_0(V)}$. We also saw that, under the given assumptions, Z is locally factorial, hence the localisation sequence [6, Proposition 1.8] may be applied to the pair $H_{a_0(V)}$ and Z, yielding an exact sequence

$$\begin{tikzcd} \mathrm{A}_{l(w)-1}(H_{a_0(V)}) \arrow[r, "\alpha"] & \operatorname{Pic}(Z) \arrow[r] \arrow[d, equal] & \mathrm{A}_{l(w)-1}(U) \arrow[r] & 0 \\ & \mathbb{Z}[H] & & \end{tikzcd}$$

As $[H_{a_0(V)}]$ maps to $m[H]$ under α, the assertion follows. □

Example 1. Consider $X(w)$, (and $\bar{X}(w)$, Z) as above. Suppose $l(w)$ is large enough to make the assumptions in the theorem be satisfied (that is, the singularities of Z are of codimension 4 or more). For the sake of concreteness, assume $X(w)$ is standard of type ${}^2\mathrm{A}_n$. For any subset I of $\{1,\dots,n\}$ one may form the parabolic group P_I, with unipotent radical U_I, and then study $X_I(w)$ — that is, the part of $X(w)$ coming from Borel subgroups contained in P_I (see [15, (1.23)]). From [15, Corollary (2.10)] it follows that we have a decomposition

$$\mathrm{A}_k(X(w)/U_I^F) = \bigoplus_{i+j=k} \mathrm{A}_i(X_I(w)) \otimes \mathrm{A}_j(\mathbb{A}^1 - \{0\}).$$

By choosing I to correspond to 'the last $l(w)-1$ orbits' we get that $X_I(w)$ is a standard Deligne-Lusztig variety of type ${}^2\mathrm{A}_{n-1}$ (of dimension $l(w)-1$). Using this recursively we get $\mathrm{A}_{l(w)-1}(X(w))_{\mathbb{Q}} = 0$ also in the ${}^2\mathrm{A}_3$, ${}^2\mathrm{A}_4, \dots, {}^2\mathrm{A}_{n-1}$ cases: Indeed,

$$0 = \mathrm{A}_{l(w)-1}(X(w))_{\mathbb{Q}} \supseteq \mathrm{A}_{l(w)-1}(X(w)/U_i^F)_{\mathbb{Q}} \simeq \mathrm{A}_{l(w)-2}(X_I(w))_{\mathbb{Q}}.$$

Similarly, we can describe the Picard group of each the low-dimensional Deligne-Lusztig varieties in the B_2, C_2 and ${}^*\mathrm{D}_2$ cases as a quotient of the Picard group of some Deligne-Lusztig variety of sufficiently high enough dimension. It follows that, in either case, $\mathrm{A}_{l(w)-1}(\bar{X}(w))_{\mathbb{Q}}$ is generated by the classes of the components of the boundary divisors D_1 and D_2.

From these remarks it more generally follows that to prove the vanishing of $\mathrm{A}_i(X(w))_{\mathbb{Q}}$ for a given standard Deligne-Lusztig variety $X(w)$ of classical type, it is sufficient to prove it for just one standard Deligne-Lusztig variety (of the same type, of course) of higher dimension.

Remark 5. From the proof of the theorem we also get that, for standard Deligne-Lusztig varieties of classical type, $X(w)$ is the complement (in Z) of the ample divisor $H_{a_0(V)}$. Hence we get (using [9, Proposition II.2.1]) a much simpler proof of the affinity of $X(w)$ than the one given in [15].[2]

Example 2 (${}^2\mathrm{A}_3$ case)**.** In this case $P = \langle B, Bs_2B, Bs_3B\rangle$ and $\mathfrak{I} = \{1\}$. Consider the projection $\pi : (G/B)^3 \to G/B \to G/P \simeq \mathbb{P}^3$. We have $\pi(\bar{X}(w)) = Z(f)$, $f = X^{q+1} + Y^{q+1} + Z^{q+1}$. D_1 is the union of $(q^2+1)(q^3+1)$ lines and $D_2 = \bigcup_{g\in M} g.V$ where V is the component of D_2 containing eB and M is a set of representatives of $G^F/\overline{Bs_1s_3B}^F$. We have $\#M = (q^3+1)(q+1)$. A set of representatives could for example be:

$$\begin{aligned} M = eB/B \cup (Bs_2B)^F/B \cup (Bs_1s_2s_3B \cup Bs_3s_2s_1B)^F/B \\ \cup\, (Bs_1s_2s_3s_2B \cup Bs_3s_2s_1s_2B)^F/B \end{aligned}$$

(there are $1+q+q^3+q^4$ elements here). Under the projection $G/B \to G/P$, $(Bs_2B)^F/B$ is mapped to eP. The second contribution is mapped to q different points and the last to q^2 other points. Hence, M is mapped to $1+q+q^2$ points.

Let us now take $q = 2$. Then $\bar{X}(w)$ is the blow-up of the non-singular Fermat cubic surface

$$Z: \qquad X_0^3 + X_1^3 + X_2^3 + X_3^3 = 0$$

in its $(q^3+1)(q^2+1) = 45$ $\mathbb{F}_{q^2}$-rational points. Being a non-singular cubic surface in $\mathbb{P}^3$, Z is the projective plane blown up in 6 points in general position [10, Section V.4], hence $\bar{X}(w)$ is the blow-up of the projective plane in 51 points. So $\mathrm{A}_1(\bar{X}(w)) = \bigoplus_{i=1}^{51} \mathbb{Z}[E_i] \oplus \mathbb{Z}[H]$ where H

[2] The restriction that $\bar{X}(w)$ be of standard type can be omitted, observing that the morphisms σ and τ of Lemma 1 are finite, whence affine [10, Exercise II.5.17 (b)].

is the pull-back of a line in $\mathbb{P}^2$ and the E_i are the exceptional divisors, cf. [6, Example 8.3.10]. In [18] the Betti numbers of $\bar{X}(w)$ have been determined; we find

$$b_2 = \dim_{\mathbb{Q}_l}\left(\mathrm{H}^2(\bar{X}(w)_{\text{ét}}, \mathbb{Q}_l)\right)^{\mathrm{Gal}(k/\mathbb{F}_q)} = q^5 + 2q^3 + q + 2 = 52$$

and this is also the pole-order of the zeta function $Z(\bar{X}(w), t)$ of $\bar{X}(w)$, at $t = q^{-i}$. So all the (etale) cohomology is algebraic as predicted by Tate [21] and Soulé [20]. We also note that for $q = 2$, D_2 has 27 components — these are the proper transforms of the 27 lines on S (cf. [10, Section V.4]).

Remark 6. The author expects that further studies of the morphism $\pi : \bar{X}(w) \to Z$ would make it possible to reduce the set of generators given in Theorem 3 to a basis for $\mathrm{Pic}(\bar{X}(w))$.

3. Higher codimensions

Based on the above results and other examples (see [12]), we boldly claim:

Conjecture 1. *Let $X(w)$ be a Deligne-Lusztig variety. Then the Abelian group $\mathrm{A}_i(X(w))$ has rank zero for $i < l(w)$.*

Below we shall exhibit further evidence for this conjecture. As there are (at present) no results generalizing the Lefschetz theorem for Picard groups to higher codimensions, we need to find a different approach in order to obtain a general description of the Chow groups of Deligne-Lusztig varieties. In this (and the next) section we present some results in this direction.

3.1. A straight-forward case

In the simplest case we may attack the problem directly.

Theorem 5. *Let $X(w)$ be a standard Deligne-Lusztig variety corresponding to the A_n case. Then*

$$\mathrm{A}_k(X(w)) = 0 \tag{7}$$

unless $k = l(w)$ (in which case $\mathrm{A}_k(X(w)) = \mathbb{Z}$). Furthermore, for any variety Y, we have $\mathrm{A}_*(X(w) \times Y) \simeq A_*(X(w)) \otimes \mathrm{A}_*(Y) = \mathrm{A}_*(Y)$.

Proof. In this case, $X(w)$ is identified with the open subset one gets when removing all $\mathbb{F}_q$-rational hyper-planes in $\mathbb{P}^{l(w)}$ (see [5, 2.2]). That is,

$$X(w) = \mathbb{P}^{l(w)} \setminus \bigcup_{P \in \mathbb{P}^{l(w)^*}} D_P = \bigcap_{P \in \mathbb{P}^{l(w)^*}} (\mathbb{P}^{l(w)} \setminus D_P).$$

Since the latter intersection is an open subset of $\mathbb{P}^{l(w)} \setminus \{X_0 = 0\} \simeq \mathbb{A}^{l(w)}$, the assertion follows from [6, Proposition 1.8] and the fact that the Chow groups of affine space vanish in positive codimension [6, p. 23].

For the last assertion we consider the commutative diagram:

$$\begin{tikzcd} \mathrm{A}_*(D\times Y) \arrow[r, "p"] & \mathrm{A}_*(\mathbb{P}^{l(w)}\times Y) \arrow[r, "q"] & \mathrm{A}_*(X(w)\times Y) \arrow[r] & 0 \\ \mathrm{A}_*(D)\otimes \mathrm{A}_*(Y) \arrow[r, "\bar{p}"] \arrow[u, "\varphi_1"] & \mathrm{A}_*(\mathbb{P}^{l(w)})\otimes \mathrm{A}_*(Y) \arrow[r, "\bar{q}"] \arrow[u, "\varphi_2"] & \mathrm{A}_*(X(w))\otimes \mathrm{A}_*(Y) \arrow[r] \arrow[u, "\varphi_3"] & 0. \end{tikzcd}$$

By [6, Example 8.3.7], φ_2 is an isomorphism. Since D is a union of hyper-planes (each isomorphic to $\mathbb{P}^{l(w)-1}$) we conclude that φ_1 also is an isomorphism. Since q, φ_2 and $\bar{q}$ are surjective, commutativity of the diagram forces φ_3 to be surjective as well. Suppose $\varphi_3(\beta) = 0$. Choose $\gamma \in \mathrm{A}_*(\mathbb{P}^{l(w)}) \otimes \mathrm{A}_*(Y)$ such that $\bar{q}(\gamma) = \beta$. Then $q\varphi_2(\gamma) = 0$, hence $\varphi_2(\gamma) = p(\delta)$ for some $\delta \in \mathrm{A}_*(D \times Y)$. But then $\bar{p}\varphi_1^{-1}(\delta) = \varphi_2^{-1}p(\delta) = \gamma$, hence $\beta = \bar{q}\bar{p}\varphi_1^{-1}(\delta)$ and $\beta = 0$. $\square$

Corollary 6. *Let $\bar{X}(w)$ a Deligne-Lusztig variety of type A_n, with w a Coxeter element. Let $j : D \to \bar{X}(w)$ be the inclusion of the boundary divisors. Then $\mathrm{A}_{l(w)}(\bar{X}(w)) = \mathbb{Z}$ and $\mathrm{A}_k(\bar{X}(w))_{p'} = j_* \mathrm{A}_k(D)_{p'}$ for $k < l(w)$.*

Proof. By Lemma 1 we may assume $\bar{X}(w)$ is of standard type. Then it follows from Remark 1 and Theorem 5 that $\bar{X}(w)$ has a stratification satisfying Lemma 10. $\square$

3.2. The G^F-invariant part

Let $\bar{X}(w)$ be of standard type. In [12, Section 1.6] it was noted that there is a (finite) subgroup U^F of G^F acting on $X(w)$, with quotient $X(w)/U^F$ isomorphic to an open subset of a torus. Since the Chow groups of affine space vanish in positive codimension [6, p. 23] the same is true for tori and therefore also for the quotient variety $X(w)/U^F$ [6, Proposition 1.8]. Since there is a finite surjective morphism $X(w)/U^F \to X(w)/G^F$ (inducing a surjection in Chow groups with rational coefficients) it follows that the G^F-invariant Chow groups of $X(w)$ satisfy

$$\mathrm{A}_i(X(w))_{\mathbb{Q}}^{G^F} = 0 \qquad \text{for } i < l(w) \tag{8}$$

(see [6, Example 1.7.6]). So the conjecture stated above holds at least for the G^F-invariant part.

4. Relating the Chow groups of $\bar{X}(w)$ to those of G/B

Chow groups of flag varieties were first described in Chevalley's manuscript [2] (unpublished until recently) and later in [3, 4]. We recall the following facts:

1. The action of G induced on $\mathrm{A}_*(X)$ is trivial.
2. $\{[X_w] : w \in W\}$ is a basis of $\mathrm{A}_*(X)$ with $[X_w] \in \mathrm{A}_{l(w)}(X)$. Setting $Y_w = X_{w_0 w}$ we get that $\{[Y_w] : w \in W\}$ is a basis of $\mathrm{CH}^*(X)$; $[Y_w] \in \mathrm{CH}^{l(w)}(X)$. These bases are dual, in the sense that

$$[X_w] \cdot [Y_{w'}] = [X_w \cap w_0 Y_{w'}] = \begin{cases} [\{\dot{w}B\}] & w = w' \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

3. $\mathrm{CH}^*(X)$ is generated in degree 1: any Schubert variety X_w is a component in an iterated intersection of Schubert varieties of codimension 1.
4. The intersection pairing
$$\mathrm{CH}^1(X) \times \mathrm{CH}^k(X) \to \mathrm{CH}^{k-1}(X)$$
is given in terms of the Cartan matrix (A_{ij}) of G: let $\lambda_i \in X_0$ be the fundamental weight corresponding to the root α_i. These are given in terms of a base-change under the Cartan matrix (and are listed in e.g. [14, p. 69]). Then, for $w \in W$ and $s_i \in S$,
$$[Y_{s_i}] \cdot [Y_w] = \sum_{\{\beta \in \Phi^+ : l(ws_\beta) = l(w)+1\}} \langle \lambda_i, \beta^\vee \rangle [Y_{ws_\beta}]. \tag{10}$$

Proposition 7. *The cycles* $\{[\overline{X(w)}] : w \in W\}$ *do* also *form a basis for* $\mathrm{A}_*(X)_{\mathbb{Q}}$.

Proof. Since the cardinality of the set in each degree is correct (being the same as that of Schubert varieties), we only need to prove that the cycles are linearly independent in $\mathrm{A}_*(X)_{\mathbb{Q}}$. Like in the proof of the corresponding statement for Schubert varieties, it will suffice to find a set of $\mathbb{Q}$-dual elements [3]. To this end, let $\dot{w}_0$ denote a representative of the longest element in W and let $w' \in W$ be arbitrary. Set $\overline{Y(w')} = \pi(L^{-1}(\dot{w}_0 \overline{B\dot{w}'B}))$. Set-theoretically we have
$$\begin{aligned}\overline{X(w)} \cap \overline{Y(w')} &= \pi(L^{-1}(\overline{B\dot{w}B})) \cap \pi(L^{-1}(\dot{w}_0\overline{B\dot{w}'B})) \\ &= \pi(L^{-1}(\overline{B\dot{w}B} \cap \dot{w}_0\overline{B\dot{w}'B})).\end{aligned}$$
Since $\overline{B\dot{w}B} = \pi^{-1}(X_w)$ (similarly for w') it follows from the properties of Schubert varieties that
$$\overline{X(w)} \cap \overline{Y(w')} = \begin{cases} \pi(L^{-1}(\dot{w}_0)) & w' = w_0 w \\ \emptyset & \text{otherwise.} \end{cases} \tag{11}$$
Since the intersection is proper when non-empty, we see that we have the wanted $\mathbb{Q}$-dual basis (X is projective). As $F(w_0) = w_0$, it follows that $L(w_0 g) = L(g)$ for all $g \in G$. Hence $\overline{X(w)} \cap \overline{Y(w)} = X(e)$. ☐

Corollary 8. *Let $\bar{X}(w)$ be a Deligne-Lusztig variety and let $\bar{X}(w_1)$, $\bar{X}(w_2)$ be two different Deligne-Lusztig subvarieties of $\bar{X}(w)$. Then $\bar{X}(w_1)$ and $\bar{X}(w_2)$ are linearly independent in* $\mathrm{A}_*(\bar{X}(w))$ *(similarly in $\overline{X(w)}$).*

Proof. If $\bar{X}(w_1)$ and $\bar{X}(w_2)$ are linearly dependent, then so are $\overline{X(w_1)}$ and $\overline{X(w_2)}$ [6, Theorem 1.4]. Pushing this equivalence forward to $\mathrm{A}_*(X)_{\mathbb{Q}}$ allows us to use Proposition 7. ☐

Corollary 9. *Let w_0 denote the longest element in W. For $k < l(w_0)$ we have* $\mathrm{A}_k(X(w_0))_{\mathbb{Q}} = 0$*. More generally, for all k, n such that $k < n \le l(w_0)$, we have that*
$$\mathrm{A}_k\big(\cup_{l(w) \ge n} X(w)\big)_{\mathbb{Q}} = 0. \tag{12}$$

Proof. From Proposition 7 it follows that in the short exact sequence [6, Proposition 1.8] of finite-dimensional $\mathbb{Q}$-vector spaces,
$$\bigoplus_{i=1}^{N} \mathrm{A}_k\big(\overline{X(w_0 s_i)}\big)_{\mathbb{Q}} \xrightarrow{\varphi} \mathrm{A}_k(X)_{\mathbb{Q}} \to \mathrm{A}_k(X(w_0))_{\mathbb{Q}} \to 0$$

φ has to be surjective. The first assertion then follows. For the last assertion we may argue similarly, using the exact sequence

$$\bigoplus_{l(w)=k} \mathrm{A}_k\big(\overline{X(w)}\big)_{\mathbb{Q}} \xrightarrow{\varphi} \mathrm{A}_k(X)_{\mathbb{Q}} \to \mathrm{A}_k\big(\cup_{l(w)>k} X(w)\big)_{\mathbb{Q}} \to 0$$

plus the fact that the union $\cup_{l(w)\geq n} X(w)$ is open in $\cup_{l(w)>k} X(w)$. □

Remark 7. From the above, Deligne-Lusztig varieties and Schubert varieties seem quite similar: they are defined in almost the same way; they constitute a basis for the rational Chow groups of G/B; and, conjecturally, they both have a good cell-decomposition (compare Lemma 10 below) for calculating their respective (rational) Chow groups.

However, in some other respects, Deligne-Lusztig varieties behave rather differently from Schubert varieties. For example, it is by now well-known [17] that Schubert varieties are Frobenius split (in the sense of [17]). But from the description given in Section 2 it follows rather easily (see [12, Section 4.1]) that Deligne-Lusztig varieties in most cases cannot be Frobenius split.

It is also worth mentioning that whereas the inverse canonical divisor $K_{X_w}^{-1}$ is effective for all Schubert varieties, there exists [11] a whole family of Deligne-Lusztig varieties $\bar{X}(w)$ such that $K_{\bar{X}(w)}$ is ample.

Acknowledgments. The author wishes to thank H.A. Nielsen for helpful discussions relating to the contents of the below appendix. Discussions with J.F. Thomsen have also helped the author to keep on track.

This paper is based on Chapter 3 of the author's Ph.D.-thesis *"The geometry of Deligne-Lusztig varieties; Higher-dimensional AG codes"* [12]. The author would like to take this opportunity to thank his thesis advisor Johan P. Hansen for numerous enjoyable and stimulating conversations.

Valuable comments and suggestions made by the referee are gratefully acknowledged.

Appendix A. Auxiliary lemmas

Lemma 10. *Let X be an algebraic scheme (not necessarily irreducible) with a stratification*

$$X_0 \subset X_1 \subset \cdots \subset X_n = X \quad ;\ X_i \text{ closed subschemes of pure dimension } i$$

such that $\mathrm{A}_k(X_i - X_{i-1}) = 0$ for $k \neq i$. Then for all $k \leq n$ we have surjections

$$\mathrm{A}_k(X_k) \to \mathrm{A}_k(X) \to 0. \tag{13}$$

Proof. For $k = n$ the assertion is trivial, and from [6, Proposition 1.9] we have the exact sequence

$$\mathrm{A}_k(X_{n-1}) \to \mathrm{A}_k(X_n) \to \mathrm{A}_k(X_n - X_{n-1}) \to 0 \tag{14}$$

hence surjections $\mathrm{A}_k(X_{n-1}) \to \mathrm{A}_k(X_n) \to 0$ for all $k < n$. By induction we may assume we have surjections $\mathrm{A}_k(X_k) \to \mathrm{A}_k(X_{n-1}) \to 0$ for all $k < n-1$. Now compose these surjections. □

Remark 8. Of course, the lemma also holds for Chow groups with rational coefficients.

Lemma 11. *Let $V_1, \dots, V_m$ be prime divisors on a non-singular projective variety X; $\dim X \geq 2$. Assume that the V_i are contracted to distinct points $P_1, \dots, P_m$ under a morphism $\pi : X \to Y$ where $\dim X = \dim Y$, Y is projective and $\pi^{-1}(P_i) = V_i$. Then the V_i are independent in* $\mathrm{Pic}(X)$. *Hence, for any (non-zero) $L \in \mathrm{Pic}(Y)$, the classes $\pi^*L, V_1, \dots, V_m$ in* $\mathrm{Pic}(X)$ *are linearly independent too.*

Proof. A non-trivial dependence relation $0 = \sum_i n_i[V_i]$, $n_i \in \mathbb{Z} \setminus \{0\}$, will imply $[V_i]^2 = 0$ (as a cycle in $\mathrm{A}^2(X)$) for any i. We shall see that this cannot be the case.

Let V be any of the V_i's and let $P = \pi(V)$. Since Y is projective we may choose a very ample (Cartier) divisor H on Y. Choose furthermore effective divisors H_0, H_1 linearly equivalent to H such that P is in H_0 but not in H_1. On an open neighborhood[3] of P, the map π looks like Figure 1.

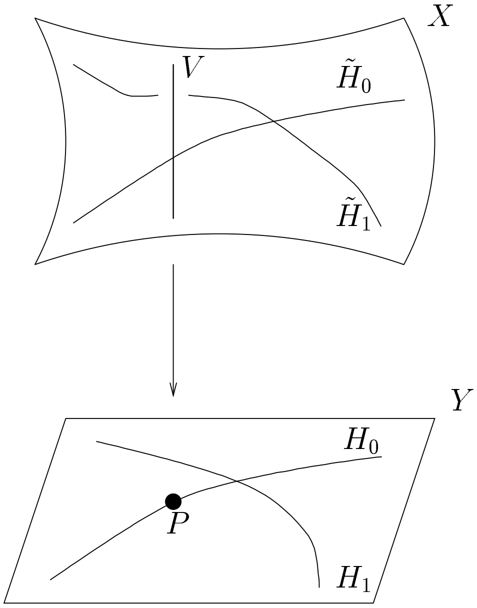


Figure 1. The blow-down of the divisor V

Let $m_P(H_0)$ denote the multiplicity of H_0 at P. By choice of H_0, $m_P(H_0) > 0$. Since π^*H_1 does not intersect V,

$$0 = [\pi^*H_1] \cdot [V] = \pi^*[H_0] \cdot [V] = \big([\tilde{H}_0] + m_P(H_0)[V]\big) \cdot [V].$$

Hence $[V]^2$ is a (negative) non-zero multiple of the proper (non-zero) intersection $[V] \cdot [\tilde{H}_0]$, a contradiction.

For the last assertion assume $d\pi^*[L] = \sum_i n_i[V_i]$. Then, by pushing down with π we get the relation $d\pi_*\pi^*[L] = \sum_i n_i \pi_*[V_i] = 0$. Since $\pi_*\pi^*[L]$ is a (non-zero) multiple of $[L]$ we must have $d = 0$ and, by the above, all $n_i = 0$. □

Lemma 12. *Let $f : X \to Y$ be a birational morphism of algebraic schemes with exceptional locus E, codim$(E, X) \geq 1$. Let $\alpha \in \mathrm{A}_*(X)$, $\alpha \not\subset E$. Then, if $f_*\alpha$ is zero in $\mathrm{A}_*(Y)$, so is α. That is, the kernel of $\pi_* : \mathrm{A}_*(X) \to \mathrm{A}_*(Y)$ is supported on E.*

[3]If the self-intersection is non-zero on an open subset of X, it cannot be zero in X.

Proof. Obvious (restrict to the open subset where f is an isomorphism and use [6, Proposition 1.8]). □

The following was conjectured by Samuel and proved by Grothendieck:

Theorem 13. (Samuel–Grothendieck [7, Corollaire 3.14, p. 132])
Let A be a Noetherian local ring that is a complete intersection. Assume A is factorial in codimension 3 (that is, A_P is factorial when localising in all primes P satisfying $\dim A_P \leq 3$). Then A is factorial.

Corollary 14. *Let X be a normal variety, such that the singular locus of X has codimension at least 4 (this property is sometimes being referred to as 'X is regular in codimension 3'). Assume furthermore that X is a strict complete intersection. Then X is locally factorial, hence $\mathrm{Pic}(X) = \mathrm{A}_{\dim X-1}(X)$.*

Proof. Let S be a local ring of X. Then S is a complete intersection ring. Let P be a prime in S such that $\dim S_P \leq 3$. Then S_P is a local ring in X of dimension at most 3, hence S_P is regular (whence factorial, by the Auslander-Buchsbaum theorem). Conclusion by Theorem 13 and [10, Section II.6]. □

Example 3. It is necessary to assume that the singularities only occur in codimension at least 4: For any field k of characteristic different from 2 the projective quadric hyper-surface $H:\ 0 = x_0^2 + x_1^2 + x_2^2 + x_3^2$ in $\mathbb{P}^4$ has the following properties [10, Exercise II.6.5]:

- H is normal; $H_{\text{sing}} = \{(0:0:0:0:1)\}$, that is, $\mathrm{codim}(H_{\text{sing}}, H) = 3$.
- $\mathrm{A}_2(H) = \mathrm{Cl}(H) = \mathbb{Z} \oplus \mathbb{Z}$.

Whereas, by the Lefschetz theorem for Picard groups, $\mathrm{Pic}(H) = \mathbb{Z}$.

Remark 9. For quadric hyper-surfaces of the type $x_0^2 + x_1^2 + \ldots + x_r^2$ in some $\mathbb{P}^n$ $(n \geq r)$, Corollary 14 is known as *Klein's theorem*, cf. [10, Exercise II.6.5 (d)].

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Received December 14, 1999; revised version September 5, 2000